Second resolution. I formulated Assumption (B) to reflect our ordinary experience that several coats of paint may be needed in order to cover a wall. But you may object that your paint is no ordinary paint—yours is a special, *mathematical* paint that's an opaque, continuous fluid. Any thickness of such paint, however small, is sufficient to hide the wall. Your paint, then, satisfies the following assumption:

(C) *Any thickness of paint will cover a surface.*

"Painting a surface" now means covering it with paint of positive thickness, the thickness possibly varying with the point being covered. Under this assumption, a sufficiently smooth surface of revolution will *always* be paintable (even if the volume inside is infinite). To prove that, we let $g(x)$ be a continuous, positive, decreasing function that approaches 0 as $x \to \infty$, and we consider the hollow tube obtained by revolving the region between the curves $g(x)$ and $g(x)(1 + 1/x^2)$ for $x \geq 1$ about the x-axis. Then we see from elementary calculus that this tube has finite volume, and so to paint the surface of revolution of $g(x)$ we simply fill our tube with paint. There is no paradox. We normally think of the boundary surface of a solid as being "smaller" than the solid itself, so there is nothing surprising about the case where our container has infinite volume yet can be covered with a finite volume of paint.

Mathematical models. Is there any reason to prefer one of our assumptions over the other? Or is there perhaps some other assumption more compelling than either of ours? Why do we need *any* assumption?

My answer to the third question is that the Gabriel's horn paradox is essentially an exercise in mathematical modeling; in making a model we first make assumptions. After all, the paradox offends our sensibilities not because of any internal logical inconsistency but because the conclusion contradicts our everyday experience. If we regard Assumptions (A), (B), and (C) as bases for models of the process of painting a surface, then we can view the paradox as simply pointing out that a certain prediction that follows from Assumption (A) does not agree with experience. We therefore discard (A) in the hope of replacing it by another, more realistic one.

Of the two candidates (B) and (C) offered here, (B) seems to me the more realistic. But both are unrealistic in that they assume that paint is continuous. Since real paint is composed of molecules, and since molecules have small but finite size, we cannot in reality fill Gabriel's horn with paint—and again there is no paradox.

References

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A Paradoxical Paint Pail

Mark Lynch (lynchmj@millsaps.edu), Millsaps College, Jackson, MS 39210

We are all familiar with Gabriel's horn from calculus [**1**], [**2**]. (See also the preceding Capsule.) It is the object obtained by rotating the graph of $y = 1/x$ for $x \geq 1$

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around the x-axis. We tell our students that since it has finite volume and infinite surface area, it can be filled with paint but cannot be painted. However, sometimes clever physics students point out that since Gabriel's horn is unbounded, it would take an infinite amount of time to fill it with paint and so we really can't fill it with paint either. In this note, we answer the critics' charge by constructing a bounded paint pail (hence, it can be filled in finite time) with infinite surface area.

Define a function $f(x)$ on [0, 1] as follows: $f(x) = 1$ if $x = 0$ or $x = 1/n$ for n a positive integer, and in the interval

$$\left(\frac{1}{n+1}, \frac{1}{n}\right),$$

let f be given by a "spike" of length $1/n$. See the figure. Finding expressions for $f(x)$ on the intervals is a good exercise for students.

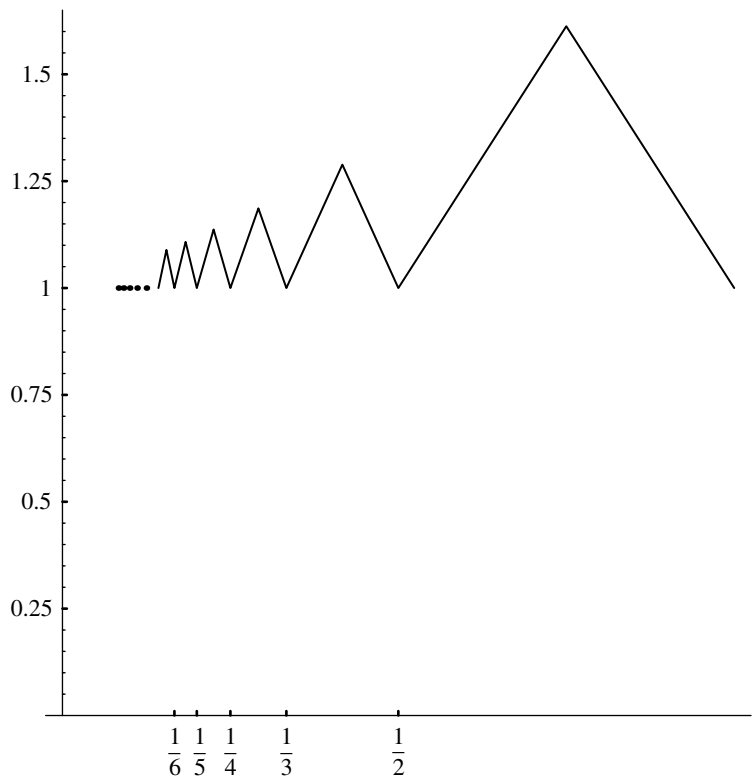

Figure 1. The spike function

Thus, $f(x)$ is a piecewise linear function, but with infinitely many pieces. Since the spike heights approach 0 as $n \to \infty$, f is continuous at 0 and hence on [0, 1]. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞, the graph has infinite arclength. Now, rotate the graph of f around the x-axis and place a unit disk on, say, the end at $x = 1$. The resulting paint pail has finite volume and infinite surface area.

Note that we could not construct such an example with a function g with a continuous derivative, for then the arclength $\int_0^1 (1 + (g'(x))^2)^{1/2}\, dx$ would be finite.

References

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Differentiate Early, Differentiate Often!

Robert Dawson (rdawson@cs.stmarys.ca), Saint Mary's University, Halifax, Nova Scotia, Canada B3H 3C3

In first-year calculus, constrained-optimization and related-rates word problems are two of the biggest stumbling blocks. In this note, I contrast the methods suggested in calculus textbooks for the solution of these two types of problems, and conclude that a different approach to constrained-optimization problems, similar to that widely used for related-rates problems, would be advantageous.

Let us first consider related-rates problems. Traditional textbooks (see, for instance, Adams [**1**, p. 235]; Edwards and Penney [**3**, p. 193]; Finney, Weir, and Giordano [**5**, p. 209]; Johnston and Mathews [**6**, p. 316]; Stewart [**8**, p. 258], and Strauss *et al.* [**9**, p. 158]) introduce these shortly after implicit differentiation. These texts all suggest that implicit differentiation of the equation relating the rates should be an early step in the solution of such a problem. Nonetheless, many students, faced with a related-rates problem, persistently avoid implicit differentiation by eliminating a variable. For instance:

Problem 1. *A ladder of length* 5 *m is sliding with one end on the ground and the other on a vertical wall. The end on the ground is sliding away from the wall at a constant rate of* 1 *m/sec. How fast is the end on the wall moving when it is* 4 *m off the ground?*

Solution A (standard). By the Pythagorean theorem, the distance x from the foot of the wall to the ladder and the height y of the top of the ladder are linked by the relation

$$x^2 + y^2 = 25; \tag{1}$$

differentiating implicitly with respect to t yields

$$x\,dx/dt + y\,dy/dt = 0. \tag{2}$$

We can now substitute the instantaneous value $y = 4$ into (1) to obtain $x = 3$; substituting these values and $dx/dt = 1$ into (2) we obtain $3 + 4dy/dt = 0$, so that $dy/dt = -3/4$ m/sec.

Solution B (avoiding implicit differentiation). Solving (1) for y, we obtain

$$y = \sqrt{25 - x^2}. \tag{3}$$

Differentiating with respect to x gives

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{25 - x^2}}$$

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