The Euler-Maclaurin Formula and Sums of Powers

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Mathematicians have long been intrigued by the sum $1^m + 2^m + \cdots + n^m$ of the first n integers, where m is a nonnegative integer. The study of this sum of powers led Jakob Bernoulli to the discovery of Bernoulli numbers and Bernoulli polynomials. There are expressions for sums of powers in terms of Eulerian numbers and Stirling numbers [**5**, p. 199]. In addition, past articles in this MAGAZINE contain algorithms for producing a formula for the sum involving powers of $m + 1$ from that involving powers of m [**1, 4**]. (The algorithm in Bloom [**1**] is actually Bernoulli's method.)

This note involves a curious property concerning sums of integer powers, namely,

$$1^m + 2^m + \cdots + (m-1)^m < m^m, \text{ for } m \geq 1. \tag{1}$$

In other words, the sum of the $m - 1$ terms from 1^m to $(m-1)^m$ is always less than the single term m^m, regardless of how large m is. This inequality is not true for an arbitrary number of terms; $1^m + 2^m + \cdots + (n-1)^m$ is not necessarily less than n^m for all n, but the inequality is true when $n = m$.

Proving (1) is not too difficult. In fact, one proof is a nice first-semester calculus problem using left-hand Riemann sums to underestimate the integral $\int_0^m x^m \, dx$. Another establishes $(x+1)^m - x^m > x^m$ for $x < m$ via the binomial theorem; replacing x successively with $0, 1, 2, \ldots, m-1$ and summing yields (1).

There is a deeper question here, though. Dividing (1) by m^m produces the inequality

$$\left(\frac{1}{m}\right)^m + \left(\frac{2}{m}\right)^m + \cdots + \left(\frac{m-1}{m}\right)^m < 1. \tag{2}$$

Since this relation holds regardless of the value of m, a natural question to ask is this: What is the limiting value of the expression on the left of (2) as m approaches infinity? Our investigation of this value involves a useful tool in any mathematician's bag of tricks—one that is, unfortunately, not often taught in undergraduate courses—the Euler-Maclaurin formula for approximating a finite sum by an integral. Along the way we also prove (1) using Euler-Maclaurin, thus illustrating the use of the Euler-Maclaurin formula with remainder.

Rota calls Euler-Maclaurin "one of the most remarkable formulas of mathematics" [**6**, p. 11]. After all, it shows us how to trade a finite sum for an integral. It works much like Taylor's formula: The equation involves an infinite series that may be truncated at any point, leaving an error term that can be bounded.

The formula uses the very numbers discovered by Bernoulli during his investigations into the power sum, and the error term uses Bernoulli's polynomials. For example, the second-order formula with error term is given in *Concrete Mathematics* [**2**, p. 469]:

$$\sum_{j=0}^{n-1} f(j) = \int_0^n f(x)dx + \frac{B_1}{1!}\Big(f(n) - f(0)\Big) + \frac{B_2}{2!}\Big(f'(n) - f'(0)\Big)$$
$$+ (-1)^3 \frac{1}{2!}\int_0^n B_2(\{x\}) f''(x)dx, \tag{3}$$

where

- B_i is the ith Bernoulli number ($B_1 = -1/2$, $B_2 = 1/6$),
- $B_2(x)$ is the second Bernoulli polynomial: $x^2 - x + 1/6$,
- $\{x\} = x - \lfloor x \rfloor$, and
- f is twice-differentiable.

Since $\{x\}$ is the fractional part of x, the function $B_2(\{x\})$ in (3) is just the periodic extension of the parabola $B_2(x) = x^2 - x + 1/6$ from [0, 1] to the entire real number line. In other words, $B_2(\{x\})$ agrees with $B_2(x)$ on [0, 1] and is periodic with period 1.

Proving (3) involves nothing more complicated than integration by parts. A brief outline is as follows: Start with $(1/2)\int_0^1 (y^2 - y + 1/6)\, g''(y)\, dy$. Use integration by parts twice and solve for $g(0)$. Let $g(y) = f(y + j)$, and then substitute x for $y + j$ to find an expression for $f(j)$. Sum this expression as j varies from 0 to $n - 1$, noting that the terms involving $f'(j)$ and $f'(j + 1)$ telescope, while those involving $f(j + 1)$ are absorbed into the sum. This yields (3), since $B_2(\{y\}) = B_2(\{x\})$. The interested reader is invited to fill in the details.

The full Euler-Maclaurin formula with no remainder term (for infinitely differentiable f) is given in *Concrete Mathematics* [**2**, p. 471]:

$$\sum_{j=0}^{m-1} f(j) = \int_0^m f(x)dx + \sum_{k=1}^{\infty} \frac{B_k}{k!}\Big(f^{(k-1)}(m) - f^{(k-1)}(0)\Big). \tag{4}$$

Unfortunately, the infinite sum on the right-hand side often diverges. This formula can also be proved using integration by parts; Lampret, in fact, shows how to use parts to prove Euler-Maclaurin for arbitrary orders [**3**].

On to the proof of (1): We can easily verify the inequality for small values of m. In particular, for $m = 1$, we have $0 < 1 = 1^1$, and for $m = 2$, we have $1^2 = 1 < 4 = 2^2$. For $m \geq 3$, we turn to Euler-Maclaurin. Plugging $f(x) = x^m$ and $n = m$ into (3) yields

$$\sum_{j=1}^{m-1} j^m = \int_0^m x^m dx - \frac{1}{2}m^m + \frac{1}{12}m\, m^{m-1} - \frac{1}{2!}\int_0^m B_2(\{x\})m(m-1)x^{m-2}dx$$
$$= \frac{m^{m+1}}{m+1} - \frac{5}{12}m^m - \frac{1}{2}\int_0^m B_2(\{x\})m(m-1)x^{m-2}dx. \tag{5}$$

Now, let's deal with the error term. Completing the square on the parabola $B_2(x)$ gives us $B_2(x) = (x - 1/2)^2 - 1/12$. This tells us that the minimum value of $B_2(x)$ on [0, 1] is $-1/12$, occurring at $x = 1/2$, and the maximum value on [0, 1] is 1/6, occurring at the two endpoints $x = 0$ and $x = 1$. Since $B_2(\{x\})$ is the periodic extension of $B_2(x)$ from [0, 1] to the real number line, the minimum and maximum values of $B_2(\{x\})$ over the real numbers are $-1/12$ and 1/6, respectively (which, incidentally, occur infinitely often). This tells us that $-1/2B_2(\{x\}) \leq (-1/2)(-1/12) = 1/24$.

Therefore,

$$\frac{-1}{2}\int_0^m B_2(\{x\})m(m-1)x^{m-2}dx \le \frac{1}{24}\int_0^m m(m-1)x^{m-2}dx$$
$$= \frac{m}{24}m^{m-1} = \frac{1}{24}m^m.$$

Plugging back into (5) produces

$$\sum_{j=1}^{m-1} j^m \le \frac{m^{m+1}}{m+1} - \frac{5}{12}m^m + \frac{1}{24}m^m$$
$$< m^m - \frac{3}{8}m^m = \frac{5}{8}m^m.$$

This establishes the inequality (1), namely $1^m + 2^m + \cdots + (m-1)^m < m^m$, for all positive integers m, via the second-order Euler-Maclaurin formula with remainder.

We now move on to our main question—determining the limiting expression for

$$\left(\frac{1}{m}\right)^m + \left(\frac{2}{m}\right)^m + \cdots + \left(\frac{m-1}{m}\right)^m.$$

From our proof of (1), we know that the limit must be less than 5/8. To find the exact value we use the full Euler-Maclaurin formula (4). For fixed m and $f(x) = x^m$, we have

$$\sum_{j=1}^{m-1}\left(\frac{j}{m}\right)^m = \frac{1}{m^m}\sum_{j=0}^{m-1} j^m$$
$$= \frac{1}{m^m}\int_0^m x^m dx + \frac{1}{m^m}\sum_{k=1}^{\infty}\frac{B_k}{k!}\Big(f^{(k-1)}(m) - f^{(k-1)}(0)\Big)$$
$$= \frac{m}{m+1} + \frac{1}{m^m}\sum_{k=1}^{\infty}\frac{B_k}{k!}(f^{(k-1)}(m) - f^{(k-1)}(0)).$$

Since $f^{(k-1)}(m) - f^{(k-1)}(0)$ is nonzero only for $k \le m$, this yields

$$\sum_{j=1}^{m-1}\left(\frac{j}{m}\right)^m = \frac{m}{m+1} + \frac{1}{m^m}\sum_{k=1}^{m}\frac{B_k}{k!}f^{(k-1)}(m)$$
$$= \frac{m}{m+1} + \frac{1}{m^m}\sum_{k=1}^{m}\left[\frac{B_k}{k!}m^{m-k+1}\Big(m(m-1)\cdots(m-k+2)\Big)\right]$$
$$= \frac{m}{m+1} + \sum_{k=1}^{m}\left[\frac{B_k}{k!}m^{1-k}(m(m-1)\cdots(m-k+2))\right].$$

There are exactly $k-1$ factors in the expression $m(m-1)\cdots(m-k+2)$. Thus the resulting polynomial is m^{k-1} plus a polynomial of degree $k-2$. For our purposes, all that matters of the latter polynomial is its degree. We can therefore use "big-O" notation to express $m(m-1)\cdots(m-k+2)$ as $m^{k-1} + O(m^{k-2})$. Here, $O(m^{k-2})$ effectively means that the expression added to m^{k-1} is of order no larger than that of m^{k-2}. (For a more precise definition and a discussion of big-O notation, see *Concrete*

Mathematics [**2**, p. 471].) Multiplying through by m^{1-k} then yields the expression $1 + O(1/m)$. Substituting back in, we have

$$\sum_{j=1}^{m-1}\left(\frac{j}{m}\right)^m = \frac{m}{m+1} + \sum_{k=1}^{m}\frac{B_k}{k!}\left[1 + O\left(\frac{1}{m}\right)\right].$$

Now we take the limit to get

$$\lim_{m\to\infty}\sum_{j=1}^{m-1}\left(\frac{j}{m}\right)^m = \lim_{m\to\infty}\left\{\frac{m}{m+1} + \sum_{k=1}^{m}\frac{B_k}{k!}\left[1 + O\left(\frac{1}{m}\right)\right]\right\}$$
$$= 1 + \sum_{k=1}^{\infty}\frac{B_k}{k!} + \lim_{m\to\infty}\left\{O\left(\frac{1}{m}\right)\sum_{k=1}^{m}\frac{B_k}{k!}\right\}.$$

The crucial question for both the second and third terms is the convergence of $\sum_{k=0}^{\infty} B_k/k!$. Fortunately, the infinite sum is a special case of the exponential generating function for the Bernoulli numbers,

$$\sum_{k=0}^{\infty} B_k\frac{x^k}{k!} = \frac{x}{e^x - 1},$$

valid for $|x| < 2\pi$ [**5**, p.147]. Therefore, $\sum_{k=1}^{m} B_k/k!$ is bounded by a constant, yielding

$$\lim_{m\to\infty}\left\{O\left(\frac{1}{m}\right)\sum_{k=1}^{m}\frac{B_k}{k!}\right\} = 0.$$

Since $B_0 = 1$, we have

$$\lim_{m\to\infty}\sum_{j=1}^{m-1}\left(\frac{j}{m}\right)^m = \sum_{k=0}^{\infty}\frac{B_k}{k!},$$

which gives us the simple limiting expression

$$\lim_{m\to\infty}\left[\left(\frac{1}{m}\right)^m + \left(\frac{2}{m}\right)^m + \cdots + \left(\frac{m-1}{m}\right)^m\right] = \frac{1}{e-1}.$$

Thus, in the limit, the sum $1^m + 2^m + \cdots + (m-1)^m$ will represent $(e-1)^{-1}$ (approximately 0.582) of m^m.

The interested reader may enjoy showing that the left-hand side of (2) actually increases to $1/(e-1)$. In addition, the excellent text *Concrete Mathematics* contains numerous further examples of the use of the Euler-Maclaurin summation formula [**2**, pp. 469–489].

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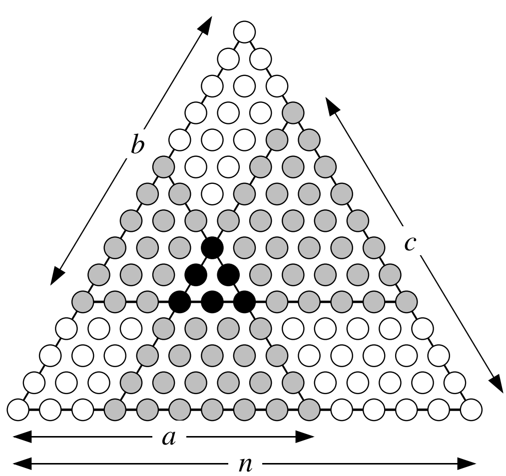
Proof Without Words: Inclusion-Exclusion for Triangular Numbers

THEOREM. Let $t_k = 1 + 2 + \cdots + k$, $t_0 = 0$. If $0 \le a, b, c \le n$ and $2n \le a + b + c$, then

$$t_a + t_b + t_c - t_{a+b-n} - t_{b+c-n} - t_{c+a-n} + t_{a+b+c-2n} = t_n$$

Proof.



NOTES:

1. If $0 \le a, b, c \le n$, $2n > a + b + c$, but $n \le \min(a + b, b + c, c + a)$, then the identity is $t_a + t_b + t_c - t_{a+b-n} - t_{b+c-n} - t_{c+a-n} + t_{2n-a-b-c-1} = t_n$, with a similar proof.
2. The following special cases are of interest:
   (a) If $(n; a, b, c) = (2k - j; k, k, k)$, then $3(t_k - t_j) = t_{2k-j} - t_{2j-k}$;
   (b) If $(n; a, b, c) = (a + b + c; 2a, 2b, 2c)$, then $t_{2a} + t_{2b} + t_{2c} = t_{a+b+c} + t_{a+b-c} + t_{a-b+c} + t_{-a+b+c}$;
   (c) If $(n; a, b, c) = (3k; 2k, 2k, 2k)$, then $3(t_{2k} - t_k) = t_{3k}$.

Illustrations for all of these cases can be found at the MAGAZINE website.

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