In other words, the speed is the product of a geometric quantity and the flow rate of the liquid. The speed can also be expressed as an explicit function of time, provided we can solve $V(t) = \phi(h)$ explicitly for h.

More variants of this problem can be obtained by defining the revolving function differently, by computing the volume with a different technique, etc. One favorite function of mine is $y = -1/\sqrt{x}$ because revolving that curve, e.g., for $0 < x \leq 1$, around the y-axis generates an infinite vessel of finite volume.

———∘———

Streaks and Generalized Fibonacci Sequences

Shahla Ahdout (shahla.ahdout@liu.edu), C.W. Post Campus of Long Island University, Brookville, NY 11548; Sheldon Rothman (sheldon.rothman@liu.edu), C.W. Post Campus of Long Island University, Brookville, NY 11548, and Helen Strassberg (strassberg@york.cuny.edu), York College, CUNY, Jamaica, NY 11451

While prolific rabbits may have been the inspiration for Fibonacci when he introduced his renowned sequence, mathematicians, both professional and amateur, find it in many other situations. In this note, we look at how it and a generalization arise in the number of strings of n events having k straight successes.

More formally, we call a sequence of n trials whose outcomes are either successes (S) or failures (F) an *n-string*, and k successes in a row we call a *k-streak*. (Note that streaks only refer to successes.) It turns out that Fibonacci's famous sequence of numbers f_n appears in a formula for the number of 2-streaks in n-strings. Furthermore, there are other Fibonacci-like sequences that appear in the counting of strings with k streaks.

We let $S(n, k)$ denote the number of n-strings that contain a k-streak and $F(n, k)$ the number of these in which the only k-streak occurs at the end. For example, there are eight 4-strings with a 2-streak, but in only two of these, *SFSS* and *FFSS*, is there no 2-streak until the end.

Note that for $n < k$, $S(n, k) = 0$, while for $n = k$, $S(n, k) = 1$. Table 1 shows some other values of $S(n, k)$.

Table 1. Numbers of strings with streaks

n	2	3	4	5	6	7
1	0	0	0	0	0	0
2	1	0	0	0	0	0
3	3	1	0	0	0	0
4	8	3	1	0	0	0
5	19	8	3	1	0	0
6	43	20	8	3	1	0
7	94	47	20	8	3	1
8	201	107	48	20	8	3
9	423	238	111	48	20	8

Focusing our attention on the first column, in looking for a pattern in these numbers—in particular, a relationship between consecutive entries—we find this:

$$S(1,2) = 2S(0,2) + 0$$
$$S(2,2) = 2S(1,2) + 1$$
$$S(3,2) = 2S(2,2) + 1$$
$$S(4,2) = 2S(3,2) + 2$$
$$S(5,2) = 2S(4,2) + 3$$

There's Fibonacci's famous sequence—or so it appears. If we begin with 0 instead of Fibonacci's 1, this sequence has $f_1 = 0$, $f_2 = 1$, and for $n > 2$, $f_n = f_{n-2} + f_{n-1}$. Thus, our table suggests this recursion formula:

$$S(n,2) = 2S(n-1,2) + f_n. \tag{1}$$

To get a feeling for why this is true, note that every sequence of length $n-1$ with a 2-streak gives rise to two n-strings with a 2-streak, by either success or failure on the next trial. What is left to count? Precisely those sequences in which the first $n-1$ trials do not include a 2-streak, but getting an S on the nth trial does produce one. But this is just our definition of $F(n,2)$, which strongly suggests that $F(n,2) = f_n$.

So what about the general case? Following the same reasoning as above, we find that

$$S(n,k) = 2S(n-1,k) + F(n,k), \tag{2}$$

where $2S(n-1,k)$ is the number of n-strings that have a k-streak in the first $n-1$ trials, and the remainder term $F(n,k)$ counts the remaining ways to get a k-streak in an n-string.

From the entries in the table for $S(n,3)$ we find that the sequence of "remainder terms" is 0, 0, 1, 1, 2, 4, 7, 13, 24, ... This sequence can be described by saying that the first three terms are 0, 0, 1, and then each successive term is found by adding together the three previous terms. This leads us to this definition (from [**3**]): the *Fibonacci sequence of order k* is the sequence that starts with $k-1$ 0s followed by a 1 and every term after that is the sum of the k previous terms.

Proposition. $F(n,k)$ is the nth term in the Fibonacci sequence of order k.

Proof. From what we observed earlier about $S(n,k)$ when $k \geq n$, we see that the theorem holds for these values. So we assume that $n > k$, and count the total number of strings where the last $k+1$ trials are $FSS\ldots S$ and the first $n-k-1$ trials have no k-streak. Such a string looks like this:

$$\underset{\substack{n-k-1 \text{ trials} \\ \text{with no } k\text{-streaks}}}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \underset{k+1 \text{ trials}}{FSS\ldots\ldots S}$$

Therefore, the number of strings of this type is

$$\begin{aligned} F(n,k) &= \text{the number of strings of length } n-k-1 \text{ having no } k\text{-streaks} \\ &= \text{the number of strings of length } n-k-1 \text{ minus the number of strings of} \\ &\quad\ \text{length } n-k-1 \text{ that have a } k\text{-streak.} \end{aligned}$$

Thus,

$$F(n,k) = 2^{n-k-1} - S(n-k-1,k). \tag{3}$$

To confirm that $F(n,k)$ is indeed the Fibonacci sequence of order k, we show that for $n > k$,

$$F(n,k) = F(n-1,k) + F(n-2,k) + \cdots + F(n-k,k). \tag{4}$$

We consider this in two cases.

Case 1: $k < n \leq 2k$. Then $n-k-1 < k$, so $S(n-k-1,k) = 0$, and hence $F(n,k) = 2^{n-k-1}$ by (3). Therefore, the sequence of remainder terms begins

$$\underbrace{0, 0, \ldots, 0}_{k-1 \text{ terms}}, 1, 2^0, 2^1, \ldots, 2^{k-1}.$$

It follows that for $k < n \leq 2k$, the sum of the k terms preceding the nth is 2^{n-k-1}, which establishes (4) in this case.

Case 2: $n > 2k$. We prove (4) by induction on n. For the base case, $n = 2k+1$, we see from (3) that $F(n,k) = 2^k - 1$, which is precisely the sum of the k terms from the $(k+1)$st to the $2k$th. Now assume that

$$F(n-1,k) = F(n-2,k) + F(n-3,k) + \cdots + F(n-k-1,k).$$

Substituting $n-k-1$ for n in (2) gives

$$S(n-k-1,k) = 2S(n-k-2,k) + F(n-k-1,k).$$

From (3) and some algebra we find that $F(n,k) = 2F(n-1,k) - F(n-k-1,k)$. Using the induction hypothesis for one $F(n-1,k)$, here, we deduce that $F(n,k) = F(n-1,k) + F(n-2,k) + \cdots + F(n-k,k)$, which completes the induction argument and therefore establishes the proposition. ■

In closing, we note that other occurrences of generalized Fibonacci sequences can be found in [**1**] and [**2**].Thus, the Fibonacci sequence and its generalizations continue to arise in unexpected situations, and interesting induction arguments are often used to establish properties of these sequences.

References

1. I. Flores, Direct calculation of k-generalized Fibonacci numbers, *Fibonacci Quart.* **5** (1967) 259–266.
2. E. P. Miles, Generalized Fibonacci numbers and associated matrices, *Amer. Math. Monthly*, **67** (1960) 745–752.
3. A. N. Philippou and A .A. Mufawi, Waiting for the kth consecutive success and the Fibonacci sequence of order k, *Fibonacci Quart.* **20** (1982) 28–32.

The Non-Attacking Queens Game

Hassan Noon (hnoon@bennington.edu) and Glen Van Brummelen (gvanbrum@bennington.edu), Bennington College, Bennington, VT 05201