Thus,

$$F(n,k) = 2^{n-k-1} - S(n-k-1,k). \tag{3}$$

To confirm that $F(n,k)$ is indeed the Fibonacci sequence of order k, we show that for $n > k$,

$$F(n,k) = F(n-1,k) + F(n-2,k) + \cdots + F(n-k,k). \tag{4}$$

We consider this in two cases.

Case 1: $k < n \leq 2k$. Then $n-k-1 < k$, so $S(n-k-1,k) = 0$, and hence $F(n,k) = 2^{n-k-1}$ by (3). Therefore, the sequence of remainder terms begins

$$\underbrace{0, 0, \ldots, 0}_{k-1 \text{ terms}}, 1, 2^0, 2^1, \ldots, 2^{k-1}.$$

It follows that for $k < n \leq 2k$, the sum of the k terms preceding the nth is 2^{n-k-1}, which establishes (4) in this case.

Case 2: $n > 2k$. We prove (4) by induction on n. For the base case, $n = 2k+1$, we see from (3) that $F(n,k) = 2^k - 1$, which is precisely the sum of the k terms from the $(k+1)$st to the $2k$th. Now assume that

$$F(n-1,k) = F(n-2,k) + F(n-3,k) + \cdots + F(n-k-1,k).$$

Substituting $n-k-1$ for n in (2) gives

$$S(n-k-1,k) = 2S(n-k-2,k) + F(n-k-1,k).$$

From (3) and some algebra we find that $F(n,k) = 2F(n-1,k) - F(n-k-1,k)$. Using the induction hypothesis for one $F(n-1,k)$, here, we deduce that $F(n,k) = F(n-1,k) + F(n-2,k) + \cdots + F(n-k,k)$, which completes the induction argument and therefore establishes the proposition. ∎

In closing, we note that other occurrences of generalized Fibonacci sequences can be found in [**1**] and [**2**].Thus, the Fibonacci sequence and its generalizations continue to arise in unexpected situations, and interesting induction arguments are often used to establish properties of these sequences.

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The Non-Attacking Queens Game

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The well-known n-queens problem poses the following: place n queens on an $n \times n$ chessboard so that no two queens attack each other. Here a square is said to be "attacked" if a queen can legally move to it. The problem, first proposed by Max Bezzel in 1848 [**2**] for the standard 8×8 board, was solved only two years later by Carl Friedrich Gauss ([**3, 4**]), who proved that there are 92 solutions (twelve of them distinct, the others generated by symmetries), one of which is shown in Figure 1. That the problem is not trivial may be seen by placing queens on a board so that they do not attack each other; on an 8×8 board every square can be attacked with as few as five queens.

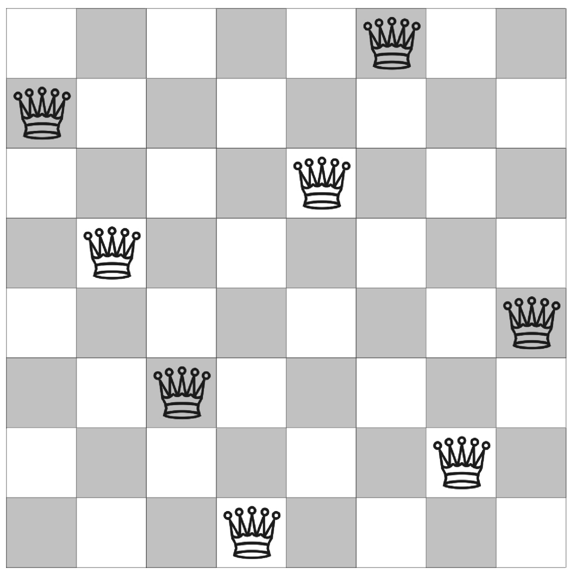


Figure 1. One solution to the 8-queens problem

We define the n-queens *game* as follows: two players successively place queens on the board so that no two attack each other. The winner is the player who places a queen so that all remaining squares are attacked, leaving her opponent no legal moves. Thus, on an 8×8 board the first player wins if all squares are attacked after five or seven moves; the second player wins if this occurs at six or eight moves. Curiously, this game has apparently not been considered before except for a restricted version designed by Georg Schrage in 1989 [**6**], in which he required the queens to be placed in successive columns.

This is an example of the sort of game studied in *combinatorial game theory*, a field more or less established in 1982 by *Winning Ways for Your Mathematical Plays*, recently reissued in four volumes [**1**]. Combinatorial games are defined as follows:

- two players move alternately according to clearly defined rules;
- both players have complete information about the game state and no chance element is involved;
- play is guaranteed to end eventually when one player is unable to move (that player loses the game).

Chess and checkers are not combinatorial games since there is no guarantee that they will end, but games such as Nim and Dots-and-Boxes are. Our n-queens game is also *impartial*, in that the moves available to a player at any stage are the same as those that would be available to her opponent if he were to move instead. One startling result of combinatorial game theory is that any impartial game is isomorphic to a Nim game with reversible moves ([**5, 7**]).

Nim games may be analyzed using an arithmetic theory of *nimbers*, described in detail in *Winning Ways*. The value of a certain game in progress is 0 if the player about to move must lose under optimal strategies. A game with value $*n$ is equivalent to a Nim game with a single stack of n blocks and is thus a winning position for the next player, since she can simply remove the whole pile and change the game's value for her opponent to 0.

For the n-queens game, strategies for small chessboards are easily worked out. On a 3×3 board, placing a queen in the center square wins the game whereas any other move loses, since the other player will cover all remaining squares with his next move. Thus for Player 1 the only square with a nimber value of 0 is the center. On a 4×4 board a move to one of the four center squares wins the game, since the opponent must leave precisely one square unattacked on his move (see Figure 2). Thus the nimbers of all four center squares are 0. We leave it to the reader to describe a winning strategy that begins by placing the first queen in a corner square of a 4×4 board, or (for instance) by placing the queen in a square adjacent to the center of a 5×5 board. This simple exercise should illustrate how much the complexity of the game increases with board size.

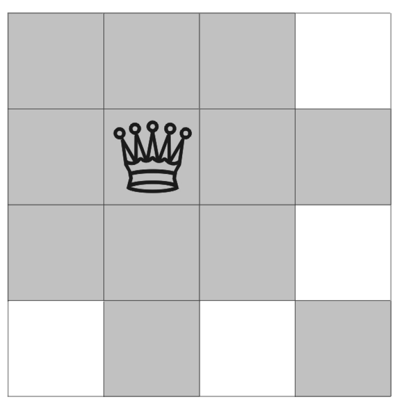
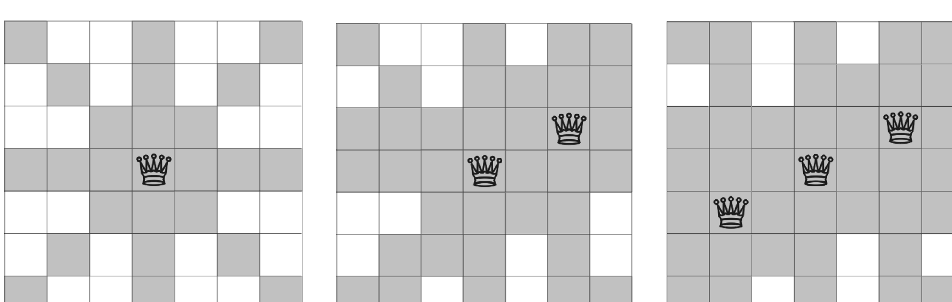


Figure 2. A 4×4 board after player 1 has placed one queen. The gray squares are attacked; the white squares are available to player 2.



(a) (b) (c)

Figure 3. A tit-for-tat strategy for odd-sized boards

Nevertheless, it is possible to describe a fairly simple strategy that wins the game for player 1 for all odd board sizes. She begins by moving to the middle, leaving the board in the symmetric position of Figure 3(a). Wherever player 2 moves next, the symmetry of the board assures that the square directly opposite that move will still be free (Figure 3(b)), and a move by player 1 to that position will preserve the game state's 180° rotational symmetry (Figure 3(c)). Thus she is always guaranteed a move, and will win the game. This is an example of a *tit-for-tat* strategy, where a player consistently responds to an opponent's move with a move that restores some property of the board. On even-sized boards there is no central square; thus symmetry cannot be preserved, and the tit-for-tat strategy fails.

Our computer analysis of the nimbers associated with opening moves on board sizes up to 9×9 is given in Figure 4. Some interesting patterns emerge; note, for instance, that on even-sized boards up to 8×8 a move to one of the four central squares guarantees a win. However, more computer work reveals, to our surprise, that none of the nimbers associated with possible moves for a 10×10 board are zero. Thus a winning strategy exists for the *second* player on a 10×10 board! We suspect, though, that to write it out explicitly might fill the pages of more than a few issues of *CMJ*.

1	1	1
1	0	1
1	1	1

0	3	3	0
3	0	0	3
3	0	0	3
0	3	3	0

2	4	1	4	2
4	4	0	4	4
1	0	0	0	1
4	4	0	4	4
2	4	1	4	2

0	2	0	0	2	0
2	0	0	0	0	2
0	0	0	0	0	0
0	0	0	0	0	0
2	0	0	0	0	2
0	2	0	0	2	0

3	3	3	3	3	3	3
3	1	3	0	3	1	3
3	3	3	1	3	3	3
3	0	1	0	1	0	3
3	3	3	1	3	3	3
3	1	3	0	3	1	3
3	3	3	3	3	3	3

1	2	2	2	2	2	2	1
2	2	2	2	2	2	2	2
2	2	1	1	1	1	2	2
2	2	1	0	0	1	2	2
2	2	1	0	0	1	2	2
2	2	1	1	1	1	2	2
2	2	2	2	2	2	2	2
1	2	2	2	2	2	2	1

N	1	2	3	4	5	6	7	8	9	10
Nimber of $n \times n$ board	*1	*1	*2	*1	*3	*1	*2	*3	*1	0

Figure 4. Nimbers associated with moves on boards of size 3×3 to 8×8

Many questions remain. Which player is the winner for even larger boards? Are there other easily stated strategies for specific board sizes or situations? Are variations of the game (say, rectangular boards or different chess pieces) amenable to analysis?

These problems could make for interesting research topics for a course in game theory or for undergraduate research. We await results with anticipation.

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The Existence of Multiplicative Inverses

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It is a common in undergraduate abstract algebra courses to show that if c is square-free, the ring $\mathbb{Q}\left[\sqrt[n]{c}\right]$ is a field. The proof usually involves the notions of minimal irreducible polynomials, quotient fields, isomorphisms, and algebraic extensions (see [**1**, Ch.13] or [**2**, Ch.8], for example).

In this note, we show that every nonzero element in this ring has a multiplicative inverse, using only *basic ideas* from linear algebra and number theory.

Let c be a square-free integer. The set $\mathbb{Q}\left[\sqrt[n]{c}\right]$ is defined as

$$\left\{a_0 + a_1\sqrt[n]{c} + a_2(\sqrt[n]{c})^2 + \cdots + a_{n-1}(\sqrt[n]{c})^{n-1} | a_i \in \mathbb{Q} \text{ for all } i\right\}.$$

Let a be a nonzero element in $\mathbb{Q}\left[\sqrt[n]{c}\right]$. Our problem is to find another element $x = x_0 + x_1\sqrt[n]{c} + x_2(\sqrt[n]{c})^2 + \cdots + x_{n-1}(\sqrt[n]{c})^{n-1}$ such that $a \cdot x = 1$.

By multiplying by common denominators, we see that the problem can be restated as follows:

Let $a = a_0 + a_1\sqrt[n]{c} + a_2(\sqrt[n]{c})^2 + \cdots + a_{n-1}(\sqrt[n]{c})^{n-1}$ with $a_i \in \mathbb{Z}$. If $a \neq 0$, show that there exists an element $x = x_0 + x_1\sqrt[n]{c} + x_2(\sqrt[n]{c})^2 + \cdots + x_{n-1}(\sqrt[n]{c})^{n-1}$ with $x_i \in \mathbb{Q}$ such that $a \cdot x = m \in \mathbb{Q}$.

Furthermore, since $a \neq 0$, we can assume the coefficients of a are relatively prime, that is $\gcd(a_0, a_1, \ldots, a_{n-1}) = 1$. So, assume $a_i \in \mathbb{Z}$, and let α denote $\sqrt[n]{c}$. Multiply the two *polynomials* in the variable α, using the fact that c is square-free and $\alpha^n = c \in \mathbb{Z}$.

The existence of an inverse is then equivalent to the following condition: the coefficient of α^j is 0 for $j = 1, 2, \ldots, n-1$, and the coefficient of α^0 is some $m \in \mathbb{Q}$. We obtain a linear system of equations in the variables x_i, which written in matrix form