Using Theorems 1, 2, and 3, the reader can obtain many additional area relations on n by n skewed chessboards. Finally, we return to the original question. The cross-segments in bold print of Figure 1 divide the skewed chessboard $A_{00}A_{08}A_{88}A_{80}$ into sixteen 2 by 2 blocks. By Theorem 3, the sum of the areas of the black blocks is equal to the sum of the areas of the white blocks for each of the sixteen 2 by 2 blocks. Therefore, the total area of the white blocks equals the total area of the black blocks. Do comparable relationships hold for cubes in a skewed 3-dimensional chessboard?

References

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On the Monotonicity of $\left(1+\frac{1}{n}\right)^n$ and $\left(1+\frac{1}{n}\right)^{n+1}$

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Since the function $f(t) = 1/t$ is decreasing on $(0, +\infty)$, for $0 < a < b$ we have

$$f(b)[b-a] \le \int_a^b f(t)\,dt \le f(a)[b-a].$$

For $a = n$ and $b = n+1$, this reduces to

$$\frac{1}{n+1} \le \log\left(1+\frac{1}{n}\right) \le \frac{1}{n}. \tag{1}$$

The inequalities (1) imply (upon multiplication by n) that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$. Although $\{(1+\frac{1}{n})^n\}$ is an increasing sequence and $\{(1+\frac{1}{n})^{n+1}\}$ is a decreasing sequence, this cannot be proved by (1) alone; one must use, for example, the Mean Value Theorem, or the Binomial Theorem, or the Arithmetic-Geometric Mean Inequality [**4, 3, 2**]. Below we refine inequalities (1) to prove these two results, and we get a little bit more.

For any convex function $F(t)$,

$$F\left(\frac{a+b}{2}\right)[b-a] \le \int_a^b F(t)\,dt \le \frac{F(a)+F(b)}{2}[b-a]. \tag{2}$$

(The right-hand side is the area of the trapezoid circumscribed at the endpoints, and the left-hand side is the area of the trapezoid inscribed at the midpoint. This is known as Hadamard's Inequality [**1**].) For $F(t) = 1/t$, with $a = n$ and $b = n+1$, the inequalities (2) become

$$\frac{2}{2n+1} < \log\left(1+\frac{1}{n}\right) < \frac{2n+1}{2n(n+1)}. \tag{3}$$

If we multiply (3) by n, we get

$$\frac{2n}{2n+1} < \log\left(1+\frac{1}{n}\right)^n < \frac{2n+1}{2n+2}.$$

Now the left-hand side here, with $n+1$ instead of n, is greater than the right-hand side. Therefore, $\{(1+\frac{1}{n})^n\}$ is an increasing sequence.

Similarly, multiplying (3) by $n+1$ we get

$$\frac{2n+2}{2n+1} < \log\left(1+\frac{1}{n}\right)^{n+1} < \frac{2n+1}{2n}.$$

Since the right-hand side, with $n+1$ instead of n, is less than the left-hand side, the sequence $\{(1+\frac{1}{n})^{n+1}\}$ is decreasing.

Inequalities (3) can be written

$$1 < \log\left(1+\frac{1}{n}\right)^{\frac{2n+1}{2}} \quad \text{and} \quad \log\left(1+\frac{1}{n}\right)^{\frac{2n(n+1)}{2n+1}} < 1,$$

which imply

$$\left(1+\frac{1}{n}\right)^{n(1+\frac{1}{2n+1})} < e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}.$$

These inequalities refine the inequalities $(1+\frac{1}{n})^n < e < (1+\frac{1}{n})^{n+1}$ furnished by (1).

The left-hand side of (3) is the midpoint approximation M to $\int_n^{n+1}(1/t)\,dt$, and the right-hand side is the trapezoid approximation T. Using Simpson's rule,

$$S = \frac{2}{3}M + \frac{1}{3}T \approx \int_n^{n+1}\frac{1}{t}\,dt = \log\left(1+\frac{1}{n}\right),$$

we have

$$\log\left(1+\frac{1}{n}\right)^{\frac{1}{S}} \approx 1 \quad \text{and} \quad \left(1+\frac{1}{n}\right)^{\frac{1}{S}} \approx e.$$

Since

$$S = \frac{2}{3}\left(\frac{2}{2n+1}\right) + \frac{1}{3}\left(\frac{2n+1}{2n(n+1)}\right) = \frac{12n^2+12n+1}{6n(n+1)(2n+1)},$$

we obtain

$$\frac{1}{S} = n\left(1+\frac{6n+5}{12n^2+12n+1}\right).$$

Thus,

$$e \approx \left(1+\frac{1}{n}\right)^{\frac{1}{S}} = \left(1+\frac{1}{n}\right)^{n(1+\frac{6n+5}{12n^2+12n+1})}.$$

For example, $n = 100$ gives e correct to nine decimal places.

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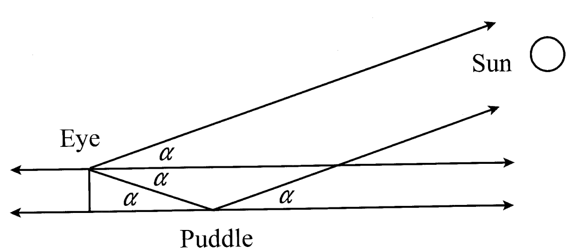
Lost Horizon

Richard Kubelka (San Jose State University, kubelka@math.sjsu.edu) sends the following, which may turn out to be useful sometime.

In the Dark Ages before the 21st century and the Global Positioning System, the second most important navigational instrument (after the compass) was the sextant. Essentially an optical protractor, the sextant is a device for measuring the angular elevation of a celestial body—the sun, for example—above the horizon. But what does one do when there is no horizon? Explorers in the Arctic, for instance, could not determine where the white sea left off and white sky began. It turns out that simple geometry and a basic principle of optics allow one to "shoot the sun"—determine its elevation with a sextant—without finding the horizon at all.

The trick these explorers and others used was to measure the angle between the sun in the sky and its reflection in a pan of mercury placed on the ground. (Mercury was used because it provided a stable and highly reflective horizontal surface; in a pinch a puddle will do.) the angle obtained in this way will be exactly twice the angle of the sun above the horizon, as the figure demonstrates.



The two acute angles in the puddle are equal because "the angle of incidence equals the angle of reflection." This is the principle of optics. In addition, because celestial bodies are so far away that all lines of sight to them may be considered parallel, the angle of elevation of the sun at eye level is the same as its elevation at the level of the puddle. The two *horizon*tal lines are of course parallel, so the alternate interior angles created by the segment joining the eye of the observer to the reflection in the puddle must be equal.

