Conclusion. These two bisection problems present material involving modeling, arc length, and area calculations suited for a first-year calculus course. Instructors looking to avoid parameterizing the ellipse could discuss only the area problem and the semi-circular case of the arc length problem. It is also interesting to note that employing the usual approximation for the perimeter of a semi-ellipse, $\pi\sqrt{(a^2+b^2)/2}$, does not result in a simpler problem. Students and instructors looking to extend this problem could model the bread slices with curves other than ellipses.

References

1. G. B. Thomas, Jr. and R. L. Finney, *Calculus and Analytic Geometry*, 9th ed., Addison-Wesley, 1996.
2. S. Wolfram, *The Mathematica Book*, 3rd ed., Wolfram Media, 1996.

Limits of Functions of Two Variables

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A common way to show that a function of two variables is not continuous at a point is to show that the 1-dimensional limit of the function evaluated over a curve varies according to the curve that is used. For example one can show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is discontinuous at (0, 0) by showing that

$$\lim_{(x,mx)\to(0,0)} f(x, y) = \frac{m}{1+m^2},$$

which varies with m. The caveat is that the natural converse to this technique cannot be used to demonstrate that a function is continuous. One reminds students that

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

exists only when the limit of f exists as (x, y) approaches (a, b) over *all* curves that run through (a, b).

There is often some vagueness as to what is meant by *all* curves (e.g., all continuous curves, all differentiable curves) and we will see that such vagueness can lead to trouble.

A classic example (e.g., [**1**, exercise 8, p. 165]) is to demonstrate that for the function

$$f(x, y) = \begin{cases} 0 & \text{if } y \le 0 \quad \text{or} \quad y \ge x^n \\ 1 & \text{if } 0 < y < x^n \end{cases}$$

(n a fixed positive integer), the limit of f as (x, y) approaches (0, 0) along any path $\{(x, y) : y = mx^k\}$, $k < n$ through the origin is 0, yet f is discontinuous at the origin. Note that this example fails to be continuous (away from (0, 0)) along two entire curves.

This example can be easily improved. One can modify the definition of f to obtain an example $h(x, y)$ so as to have h be continuous at all points except for (0, 0) and to have the limit of h equal zero as (x, y) approaches (0, 0) along any curve $y = mx^n$ (n a positive integer). In addition, h has another interesting property: the limit of h as $(x, z(x))$ approaches (0, 0) is 0 where z is any function that is real analytic at 0 and $z(0) = 0$.

The modified example is:

$$h(x, y) = \begin{cases} 0 & \text{if } y > 2g(x) \\ 2 - r & \text{if } y = rg(x),\ 1 \le r \le 2,\ (x, y) \ne (0, 0) \\ 2r - 1 & \text{if } y = rg(x),\ \frac{1}{2} < r < 1,\ (x, y) \ne (0, 0) \\ 0 & \text{if } y < \frac{1}{2}g(x) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

where

$$g(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Figure 1 and Figure 2 show cut-away views of the graph of h and Figure 3 shows some level curves of h.

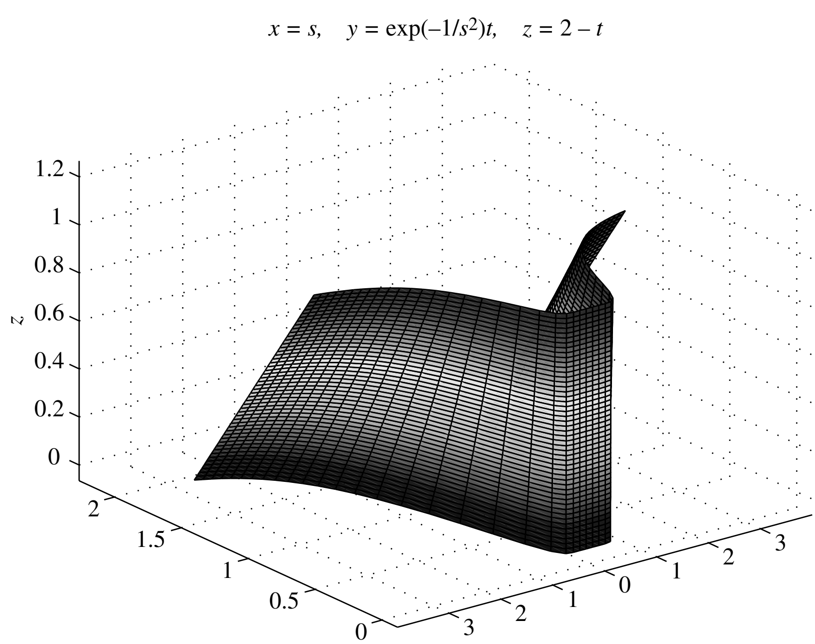

Figure 1.

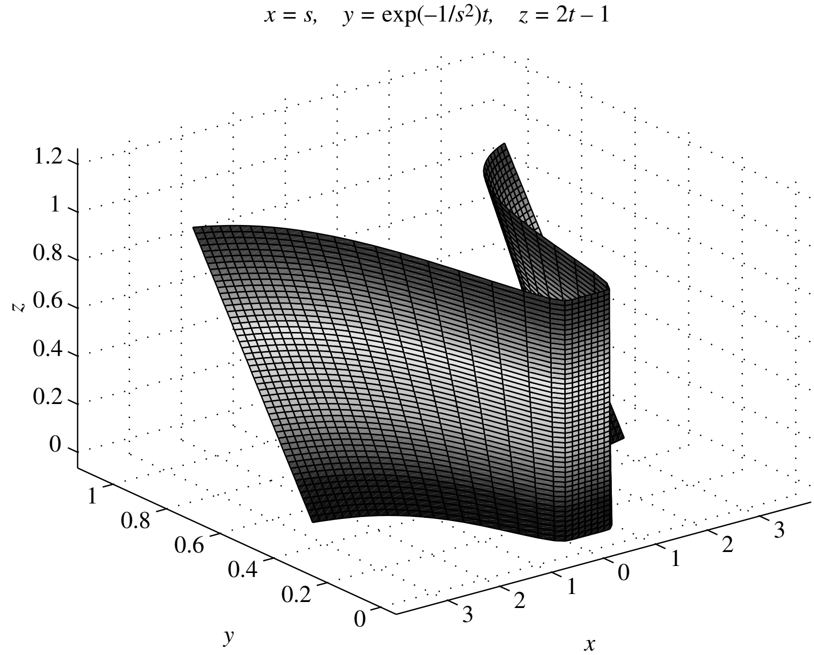

Figure 2.

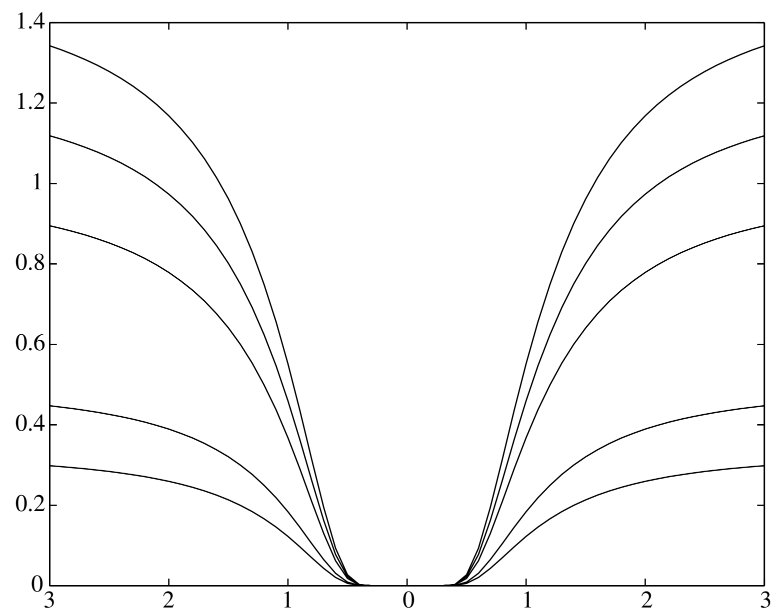

Figure 3.

Showing that h has the desired properties starts with a pleasant exercise involving L'Hôpital's rule: Let n be any positive integer. Then

$$\lim_{x\to 0} \frac{g(x)}{x^n} = 0$$

(see, e.g., [**2**, exercise 55, p. 740]).

To finish showing that the limit is 0 along paths $\{(x, mx^n)\}$ for each $n > 0$, note that because g and x^n are both continuous at 0, there is some $\delta > 0$ so that $g(x) < |x^n|$ for $0 < |x| < \delta$. It follows that

$$\lim_{(x,mx^n)\to(0,0)} h(x, y) = 0$$

for all m and all positive integers n.

Now suppose z is a function such that $z(0) = 0$ and z is real analytic at $x = 0$. (i.e., z has an absolutely convergent power series expansion in some open neighborhood of 0). Then $z = x^m z_0 +$ higher order terms, where $z_0(0) \neq 0$ and $m > 0$. Then

$$\lim_{x\to 0} \frac{g(x)}{z(x)} = \lim_{x\to 0} \frac{g(x)/x^m}{z_0(x)} = \frac{0}{z_0(0)} = 0.$$

Thus there exists some $\delta > 0$ so that $g(x) < |z(x)|$ for $0 < |x| < \delta$; it follows that

$$\lim_{(x,z(x))\to(0,0)} h(x, y) = 0.$$

Thus the limit of h exists as (x, y) approaches (0, 0) along the graph of *any* analytic function that runs through (0, 0).

It should be pointed out that what makes all of this work is that the g used in the definition of h is continuous at 0, and in fact has derivatives of all orders at 0, but does not have a valid power series expansion on any open neighborhood about zero (see, e.g., [**2**, exercise 55, p. 740]).

References

1. T. M. Apostol, *Calculus*, *Vol. II*, Blaisdell Publishing Co., 1962.
2. G. J. Etgen, *Salas and Hille's Calculus*, 7th Ed., John Wiley & Sons, 1995.

Mathematicians on Beauty in Mathematics

As for everything else, so for a mathematical theory: beauty can be perceived but not explained. —Arthur Cayley (Presidential address to the British Association for the Advancement of Science, 1883)

The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics. —G. H. Hardy (*A Mathematician's Apology*, 1941)

Mathematics possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture. —Bertrand Russell (*Mysticism and Logic*, 1918)