$N-1+y$, where $0 \le y < 1$. Thus, $d(n) = N = 1 + \log_{10} F_n - y = \lfloor 1 + \log_{10} F_n \rfloor$, where $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. Using the approximation $F_n \approx \frac{\phi^n}{\sqrt{5}}$ we get $d(n) = \lfloor 1 + \log_{10}(\frac{\phi^n}{\sqrt{5}}) \rfloor \approx \lfloor 1 + n \log_{10} \phi - \log_{10} \sqrt{5} \rfloor \approx \lfloor 0.209n + 0.651 \rfloor$. Notice that $d(n+k) - d(n) \approx \lfloor 0.209n + 0.651 + 0.209k \rfloor - \lfloor 0.209n + 0.651 \rfloor$. Notice that for $k = 1, 2, 3$ or 4, $d(n+k) - d(n) = 0$ or 1. But $d(n+5) - d(n) = 1$ or 2. When $d(n+5) - d(n) = 1$, we get a run of five Fibonacci numbers with the same number of digits. However, when $d(n+5) - d(n) = 2$, we get only four. Using a computer, we found that $d(n+5) - d(n) = 2$ at $n = 16, 35, 59, 83, 102, 126, 150, 169, 193, \ldots$. For example, F_{16} to F_{21} are 987, 1597, 2584, 4181, 6765, 10946, which shows a run of only four Fibonacci numbers with four digits.

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Lagrange Multipliers Can Fail To Determine Extrema

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The method of Lagrange multipliers is the usual approach taught in multivariable calculus courses for locating the extrema of a function of several variables subject to one or more constraints. It must, however, be applied with care because the method can miss the sought extremal values. This capsule discusses some simple examples in which Lagrange multipliers fails to locate extrema.

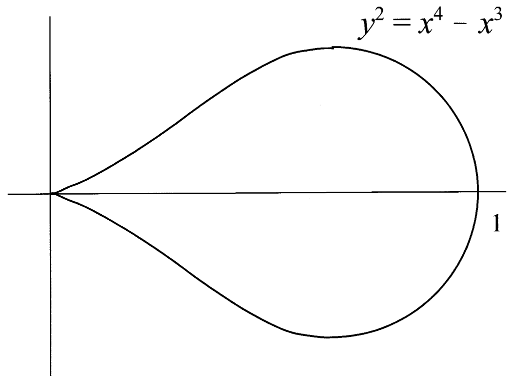
Recall that the method of Lagrange multipliers proceeds as follows in the simplest two dimensional setting. To find the extrema of a function $f(x, y)$ subject to the constraint $g(x, y) = k$ when all functions are C^1 smooth, we compute the gradient vectors $\nabla f(x, y)$ and $\nabla g(x, y)$ and solve the simultaneous system in three variables x, y, and λ

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k. \tag{1}$$

Then if the geometry is right, a constrained extremum must occur at a point (x_0, y_0) among the solutions to (1). Since this set is often finite, the location of the extrema can be determined by surveying all possibilities. But to be assured that the method succeeds, we must know that the geometry is right—that is, the set defined by $g(x, y) = k$ is a smooth curve in the plane. Here the Implicit Function Theorem is useful; it guarantees that a level set $g(x, y) = k$ is a smooth curve with nonvanishing tangent vector in a neighborhood of a point (a, b) if $\nabla g(a, b) \neq \mathbf{0}$. Thus, when seeking constrained extrema, we should also examine all critical points of $g(x, y)$.

Consider the very simple situation where $f(x, y) = x + y$ and $g(x, y) = x^2 + y^2$. If the constrained set is defined by $g(x, y) = 1$, then g's gradient is always nonvanishing and the method works beautifully. But suppose that $g(x, y) = 0$, so the constrained set is the single point defined by $x^2 + y^2 = 0$. Trivially, the function f has 0 as both its maximum and minimum value. But $\nabla f(0, 0) = (1, 1)$ and $\nabla g(0, 0) = (0, 0)$, so there is no value of λ for which $\nabla f(0, 0) = \lambda \nabla g(0, 0)$. Thus, in this example, by neglecting g's critical point we miss f's extrema.

For a prettier and less trivial example, let us minimize $f(x, y) = x$ on the piriform curve defined by $g(x, y) = 0$, where $g(x, y) = y^2 + x^4 - x^3$. As the plot below shows, this curve has a singular point at the origin where g's graph is not smooth. At any such singular point, $\nabla g = (0, 0)$ as predicted by the Implicit Function Theorem. (If the gradient were nonzero, the level set must be locally a smooth curve.) Although f's minimum clearly occurs at (0, 0), this point does not satisfy the Lagrange condition that $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ for any value λ.



We recall the simple geometric argument that justifies using Lagrange multipliers to find constrained extrema of $f(x, y)$ at a nonsingular point (a, b) of $g(x, y)$. By the gradient condition, the Implicit Function Theorem asserts that the constrained set $g(x, y) = k$ can be represented locally near (a, b) as a parametrized curve $\mathbf{r}(t)$ with $\mathbf{r}(t_0) = (a, b)$ and $\mathbf{r}'(t_0) \neq \mathbf{0}$. Therefore $g(\mathbf{r}(t)) = k$, and so (by the chain rule) $\nabla g(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$ at $t = t_0$. Since the function of $f(\mathbf{r}(t))$ has an extremum at t_0, we also have (again by the Chain Rule) $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$ at $t_{0.}$. Because both vectors $\nabla g(\mathbf{r}(t_0))$ and $\nabla f(\mathbf{r}(t_0))$ are perpendicular to the nonzero vector $\mathbf{r}'(t_0)$, they must be parallel. So there exists a scalar λ such that $\nabla f(\mathbf{r}(t_0)) = \lambda \nabla g(\mathbf{r}(t_0))$. This completes the argument in the case of two variables. A similar argument applies in higher dimensions.

The same sort of example also works in three dimensions, where the geometry for Lagrange multipliers requires that the constrained set $g(x, y, z) = k$ be a C^1 surface. Suppose that we require the extrema of $f(x, y, z) = x + y$ on the set defined by $g(x, y, z) = x^2 + y^2 = 0$ in $\mathbf{R}^3$. The constrained set is now the z-axis, and $f(0, 0, z) = 0$ at each point of this line. Thus, both the maximum and minimum of f are 0 on this set. However, the equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ yields $(1, 1, 0) = \lambda(2x, 2y, 0)$, which is satisfied at no point of the constrained set. To locate f's extrema, we must also consider g's critical points $(0, 0, z)$, namely, the z-axis.

The moral here is that the geometry matters, and Lagrange multipliers can fail to identify the proper candidate points if $\nabla g = \mathbf{0}$. Therefore, the correct procedure is to consider all points satisfying the equations (1) and also all the critical points of g (i.e., those for which $\nabla g = \mathbf{0}$). This additional consideration is not sufficiently emphasized in current calculus books. Some texts (e.g., [**1**, p. 213] and [**2**, p. 802]) state the Lagrange multipliers theorem correctly but then give an algorithm that works only for the nonsingular case. Thus, they give the false impression that considering the critical points of g in addition to the simultaneous solutions of (1) is just a technicality.

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Off on a Tangent

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It was noted in [**1**] that if a is any positive number other than 1, the tangent line to the curve $y = a^x$ at the point $(1/\ln a, e)$ goes through the origin. The interesting feature here is that the y-coordinate is independent of a. We will show that this result is not as unique as it initially appears; there are many families of curves whose tangent lines at a fixed y-coordinate go through the origin. Shifting attention to the x-coordinate, it turns out that families of curves whose tangent lines at a fixed x-coordinate go through the origin have some interesting properties. In particular, these functions form the kernel of a linear transformation.

The tangent line to a differentiable function f at the point $(c, f(c))$ with $c \neq 0$ goes through the origin if and only if $f'(c) = f(c)/c$. By shifting, scaling, and tilting a given graph, it is possible to make the modified graph have a tangent line go through the origin at any given value of x or y. These geometric operations on a graph can be performed by modifying the original function by linear factors. This is the main idea used in the next paragraph.

Let (a, b) be an open interval that contains the number 1. Suppose that f is differentiable and strictly increasing on (a, b) and that $f'(1) \leq e$. (There is nothing special about the choice of the numbers 1 and e; they could be replaced with any positive real numbers.) For each $c \neq 0$, define a function g_c on the open interval with endpoints ca and cb by

$$g_c(x) = f(x/c) + \frac{e - f'(1)}{c}x + \big(f'(1) - f(1)\big).$$

Since each function g_c is strictly monotone, it is clear that $g_c(x) = e$ only when $x = c$. Since $g'(c) = e/c$, the tangent line to the graph of $y = g_c(x)$ at the point (c, e) goes through the origin; the y-coordinate is independent of the parameter c. Therefore, many functions generate families of curves whose tangent lines when $y = e$ go through the origin. The uniqueness of the function e^x lies in the fact that the linear "correcting factor" is 0. The reader may find it interesting to choose several functions for f (such as x^2 or $\sin x$) and look at the families generated by this method.

Let f be a strictly monotone differentiable function and suppose that the tangent line to the graph of $y = f(x)$ at the point (c, d) goes through the origin. By the properties of inverse functions, the tangent line to the graph of $y = f^{-1}(x)$ at the point

