A_2 is less than the area $\frac{1}{2}(1+\frac{1}{2}) = \frac{3}{4}$ of the trapezoid having heights 1 and $\frac{1}{2}$. [The line segment joining points (1, 1) and (2, $\frac{1}{2}$) has equation $y = -\frac{x}{2} + \frac{3}{2}$, and $-\frac{x}{2} + \frac{3}{2} > \frac{1}{x}$ for $x \in (1, 2)$.] Thus, $\frac{1}{2} < A_2 < \frac{3}{4}$. Now (Figure 2) let A_3 be the area under the hyperbola $y = \frac{1}{x}$ between $x = 1$ and $x = 3$. If we divide the interval [1, 3] into eight equal-sized subintervals, then the sum of the area of the "inner" rectangles is $\frac{1}{4}(\frac{4}{5} + \frac{4}{6} + \cdots \frac{4}{12}) \approx 1.0199$. Therefore $A_3 = 1$.

It is intuitively obvious (without the benefit of the Fundamental Theorem of Calculus) that the area under the hyperbola from $x = 1$ on increases continuously as the value of the right-hand endpoint increases. Hence, somewhere strictly between 2 and 3, there lies a unique number $\boldsymbol{e}$ for which the area under the hyperbola from $x = 1$ to $x = \boldsymbol{e}$ is exactly 1. For $x \in [1, \frac{5}{2}]$, students can verify (Figure 3) that the sum of the three trapezoidal areas is

$$\frac{1}{2} \times \frac{1}{2}\left[\left(1+\frac{2}{3}\right)+\left(\frac{2}{3}+\frac{1}{2}\right)+\left(\frac{1}{2}+2\right)\right] = \frac{56}{60} < 1,$$

and so $\boldsymbol{e} > 2.5$.

Reference

1. Dimitric, R. M., Using less calculus in teaching calculus: an historical approach, *Mathematics Magazine* **74** (2001) 201–211.

A Surface Useful for Illustrating the Implicit Function Theorem

Jeffrey Nunemacher (jlnunema@owu.edu), Ohio Wesleyan University, Delaware, OH 43015

While teaching multi-variable calculus last year, I stumbled upon a surface that can be used to make the content of the Implicit Function Theorem concrete and visual. The folium of Descartes, defined by the equation $x^3 + y^3 - 3xy = 0$, is a classic curve often used to illustrate various techniques in single variable calculus. We construct our surface by setting $z = h(x, y) = x^3 + y^3 - 3xy$, so that the level set at $z = 0$ is the folium. Most of the level sets of the defining function for this surface are smooth curves, but there are two points where the hypotheses of this theorem break down, and the level sets at these points display interesting singularities. The surface can also be used to illustrate the complementarity of two and three dimensional graphs for studying a function of two variables.

A curve is locally smooth at a point P if the curve does not intersect itself at P and the direction of the tangent line varies continuously there. The Implicit Function Theorem asserts that a level set of a function $z = f(x, y)$ is locally a smooth curve at a point $P(a, b)$ if $\mathbf{grad}\ f(a, b) \neq \mathbf{0}$. Here $\mathbf{grad}\ f(a, b)$ denotes the vector $(f_x(a, b), f_y(a, b))$ and $\mathbf{0}$ is the vector (0, 0). Many students have trouble appreciating the significance of this theorem. A study of the surface $z = h(x, y)$ sheds some light on this fundamental result.

Since $\mathbf{grad}\, h(x, y) = (3x^2 - 3y, 3y^2 - 3x)$, we see that $\mathbf{grad}\, h = \mathbf{0}$ only at the points $(0, 0)$ and $(1, 1)$. At all other points of h's domain, the Implicit Function Theorem guarantees that the level sets are smooth curves. The point $(0, 0)$ lies on the level set $z = h(x, y) = 0$ and is the place where the folium crosses itself (displays a nodal singularity). Since $h(1, 1) = -1$, the level set of interest is $h(x, y) = x^3 + y^3 - 3xy = -1$. The expression $x^3 + y^3 - 3xy + 1$ factors as $(x + y + 1)(x^2 + y^2 - xy - x - y + 1)$. Thus, the level set $h(x, y) = -1$ is the union of the line $x + y + 1 = 0$ with the solution of $x^2 + y^2 - xy - x - y + 1 = 0$. The discriminant of this quadratic equation in y is $(x + 1)^2 - 4(x^2 - x + 1)$, which simplifies to $-3(x - 1)^2$. Since the discriminant is nonnegative only when $x = 1$, the only solution (and corresponding part of the level set) is the single point $(1, 1)$. Thus, $(1, 1)$ is an isolated point on this level set, another sort of singularity of a real algebraic curve. We display graphs for six of the level sets in Figure 1.

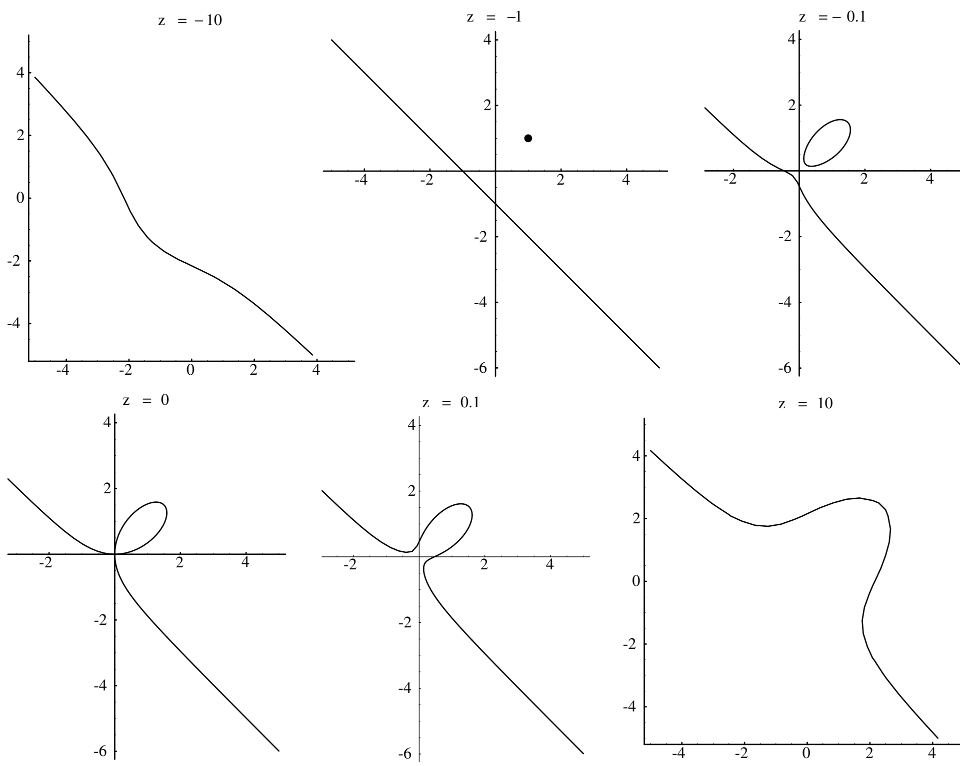

Figure 1.

The nature of the surface can be understood by visualizing the two-dimensional slices as z increases from, say, -10 to 10. The surface has a local low point at $(1, 1, -1)$. As z increases from -1, the corresponding level set has the two branches shown in the third graph in Figure 1. With further increase in z, the oval branch first touches the other branch at the singularity at the origin, and then the two branches merge into a single branch with a bulge. The condition $\mathbf{grad}\, h = \mathbf{0}$ has thus detected the two most important points on this surface. Figure 2 displays a partial three dimensional plot of the surface.

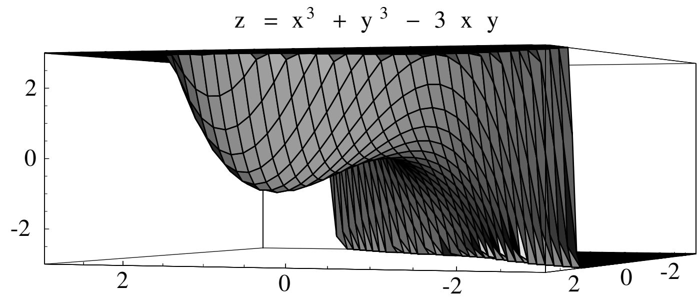

Figure 2.

It is worth pointing out that level sets of a function need not always be (geometrically) singular when the gradient vanishes. Although the function $g(x, y) = x^3 - y^3$ has **grad** $g = \mathbf{0}$ at the origin, the level set through (0, 0) is simply the line $y = x$. But the vanishing gradient condition detects all candidate singularities and thus identifies all possible points at which the level sets may not be smooth curves.

Are Mathematicians Weird?

Norton Starr (Amherst College, nstarr@amherst.edu) doesn't think so, and submits as evidence a portion of a 1979 interview with Richard Feynman (which can be found in *The Pleasure of Finding Things Out*, chapter 9) in which Feynman says

> You know, it's not true that what is called "abstruse" math is so difficult.

and goes on

> I don't believe in the idea that there are a few peculiar people capable of understanding math, and the rest of the world is normal. Math is a human discovery, and it's no more complicated than humans can understand. I had a calculus book once that said, "What one fool can do, another can." What we've been able to work out about nature may look abstract and threatening to someone who hasn't studied it, but it was fools who did it, and in the next generation, all the fools will understand it.