A1 A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

Answer: 578.

Solution: Each hop can be described by a displacement vector $\langle p, q\rangle$ with $p^2 + q^2 = 25$; the twelve possible vectors are

$$\langle 3,4\rangle; \langle -3,4\rangle; \langle 3,-4\rangle; \langle -3,-4\rangle; \langle 4,3\rangle; \langle -4,3\rangle; \langle 4,-3\rangle; \langle -4,-3\rangle; \langle 5,0\rangle; \langle -5,0\rangle; \langle 0,5\rangle; \langle 0,-5\rangle.$$

One way to write the total displacement as a sum of 578 of these vectors is

$$\langle 2021, 2021\rangle = 288 \cdot \langle 3,4\rangle + 288 \cdot \langle 4,3\rangle + \langle 0,5\rangle + \langle 5,0\rangle.$$

To show that it cannot be done with fewer, note that each hop can increase the sum of the grasshopper's coordinates by at most $3 + 4 = 7$. Because this sum has to reach

$$2021 + 2021 = 4042 = 7 \cdot (577) + 3\,,$$

at least 578 hops are needed.

A2 For every positive real number x, let

$$g(x) = \lim_{r \to 0} \left((x+1)^{r+1} - x^{r+1} \right)^{\frac{1}{r}}.$$

Find $\displaystyle \lim_{x \to \infty} \frac{g(x)}{x}$.

Answer: e.

Solution: Note that for $r > -1$ and any positive x, we have $(x+1)^{r+1} - x^{r+1} > 0$. Thus, by the continuity of the logarithm,

$$\begin{aligned} \log g(x) &= \lim_{r \to 0} \log \left(\left((x+1)^{r+1} - x^{r+1} \right)^{\frac{1}{r}} \right) \\ &= \lim_{r \to 0} \frac{1}{r} \log \left((x+1)^{r+1} - x^{r+1} \right). \end{aligned}$$

Applying L'Hôpital's rule, we get

$$\begin{aligned} \log g(x) &= \lim_{r \to 0} \frac{(x+1)^{r+1} \log(x+1) - x^{r+1} \log x}{(x+1)^{r+1} - x^{r+1}} \\ &= \frac{(x+1)\log(x+1) - x \log x}{(x+1) - x} = \log \left((x+1)^{x+1} x^{-x} \right), \quad \text{so} \\ g(x) &= (x+1)^{x+1} x^{-x} = (x+1) \left(1 + \frac{1}{x} \right)^{x}. \end{aligned}$$

Finally,

$$\lim_{x \to \infty} \frac{g(x)}{x} = \lim_{x \to \infty} \frac{x+1}{x} \left(1 + \frac{1}{x} \right)^{x} = 1 \cdot e = e.$$

A3 Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

has an inscribed regular tetrahedron whose vertices have integer coordinates.

Answer: A necessary and sufficient condition is that N be of the form $N = 3m^2$, where m is a positive integer.

Solution 1: To see that the condition is sufficient, note that the four points

$$(-m, -m, -m),\ (m, m, -m),\ (m, -m, m),\ (-m, m, m)$$

are the vertices of a regular tetrahedron inscribed in the sphere $x^2 + y^2 + z^2 = 3m^2$.

To show that the condition is necessary, we will use two lemmas:

Lemma 1. *If T is a tetrahedron whose vertices have integer coordinates, then its volume is of the form $V(T) = D/6$ for some integer D.*

Lemma 2. *The volume of a regular tetrahedron T inscribed in a sphere of radius R is given by $V(T) = \dfrac{8\sqrt{3}R^3}{27}$.*

Assuming that the sphere $x^2 + y^2 + z^2 = N$ has an inscribed regular tetrahedron T whose vertices have integer coordinates, we can combine the results of these lemmas (for $R = \sqrt{N}$) to get

$$D = 6\,V(T) = \frac{16\sqrt{3}\,N^{3/2}}{9} = \frac{16N}{9}\sqrt{3N}\,.$$

Because D is an integer, it follows that $\sqrt{3N}$ is a rational number. Thus the prime factorization of N must contain an odd number of factors 3 and an even number of factors p for any other prime p; therefore, $N = 3m^2$ for some positive integer m.

Proof of Lemma 1: Let P, Q, R, and S be the vertices of the tetrahedron. As these are all lattice points, the three vectors

$$\vec{PQ} = \langle q_1, q_2, q_3\rangle, \quad \vec{PR} = \langle r_1, r_2, r_3\rangle, \quad \vec{PS} = \langle s_1, s_2, s_3\rangle$$

have integer components. We can use a triple product to express the volume as

$$V = \frac{1}{6}|\vec{PQ} \cdot (\vec{PR} \times \vec{PS})| = \frac{1}{6}\left| \det \begin{pmatrix} q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{pmatrix} \right|,$$

and as the determinant is an integer, we are done.

Proof of Lemma 2: If a regular tetrahedron T is inscribed in a sphere of radius R, we can choose a coordinate system in which the vertices of T are given by

$$P = (-d, -d, -d), \quad Q = (d, d, -d), \quad R = (d, -d, d), \quad S = (-d, d, d)$$

with $3d^2 = R^2$. We then have

$$\vec{PQ} = \langle 0, 2d, 2d\rangle, \quad \vec{PR} = \langle 2d, 0, 2d\rangle, \quad \vec{PS} = \langle 2d, 2d, 0\rangle,$$

and the volume is

$$V = \frac{1}{6}|\vec{PQ} \cdot (\vec{PR} \times \vec{PS})| = \frac{1}{6}\left| \det \begin{pmatrix} 0 & 2d & 2d \\ 2d & 0 & 2d \\ 2d & 2d & 0 \end{pmatrix} \right| = \frac{8}{3}d^3 = \frac{8\sqrt{3}R^3}{27}.$$

Solution 2: As in Solution 1, the condition on N is sufficient. To show that it is necessary, note that if

$$v_i = (x_i, y_i, z_i), \quad i = 1, 2, 3, 4$$

are the vertices of a regular tetrahedron inscribed in the sphere $x^2 + y^2 + z^2 = N$, we can add the four antipodal points

$$w_i = (-x_i, -y_i, -z_i), \quad i = 1, 2, 3, 4$$

to get the eight vertices of a cube. (This is easily seen by choosing an alternate coordinate system as in the proof of Lemma 2 above.) Because this cube is inscribed in the sphere, its space diagonals have length $2\sqrt{N}$; therefore, each edge of the cube has length $2\sqrt{N/3}$ and its volume is $8\left(\sqrt{N/3}\right)^3$. But the volume of the cube is the determinant of three vectors with integer coordinates, so it is an integer, and as in Solution 1 it follows that $N = 3m^2$ for some positive integer m.

A4 Let

$$I(R) = \iint\limits_{x^2+y^2 \le R^2} \left(\frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4} \right) dx\, dy\,.$$

Find

$$\lim_{R\to\infty} I(R)\,,$$

or show that this limit does not exist.

Answer: The limit exists and equals $\dfrac{\pi\sqrt{2}\log 2}{2}$.

Solution: First we symmetrize the integrand. Let

$$f(x,y) = \frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4}\,, \quad \text{so that}$$

$$f(x,y)+f(y,x) = \frac{2+2(x^2+y^2)}{1+x^4+6x^2y^2+y^4} - \frac{2+x^2+y^2}{2+x^4+y^4} \quad \text{and thus}$$

$$2\,I(R) = \iint\limits_{x^2+y^2 \le R^2} \frac{2+2(x^2+y^2)}{1+x^4+6x^2y^2+y^4} - \frac{2+x^2+y^2}{2+x^4+y^4}\, dx\, dy.$$

Now consider the "first part" of this double integral, say

$$J(R) = \iint\limits_{x^2+y^2 \le R^2} \frac{2+2(x^2+y^2)}{1+x^4+6x^2y^2+y^4}\, dx\, dy.$$

Let $u = x-y$ and $v = x+y$. Then

$$u^2+v^2 = (x+y)^2+(x-y)^2 = 2(x^2+y^2), \qquad u^4+v^4 = (x+y)^4+(x-y)^4 = 2x^4+2y^4+12x^2y^2$$

and $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$, so

$$\begin{aligned} J(R) &= \iint\limits_{x^2+y^2 \le R^2} \frac{2+2(x^2+y^2)}{2+2x^4+12x^2y^2+2y^4}\, 2\, dx\, dy \\ &= \iint\limits_{u^2+v^2 \le 2R^2} \frac{2+u^2+v^2}{2+u^4+v^4}\, du\, dv. \end{aligned}$$

Note that if we rename the variables in this last integral x, y instead of u, v, the integrand will be the same as the integrand of the "second part" of the double integral for $2\,I(R)$ above. Thus we can recombine the parts to get

$$2\,I(R) = \iint\limits_{R^2 < x^2+y^2 \le 2R^2} \frac{2+x^2+y^2}{2+x^4+y^4}\, dx\, dy.$$

Converting to polar coordinates, we get

$$2\,I(R) = \int_{t=0}^{2\pi} \int_{r=R}^{R\sqrt{2}} \frac{2+r^2}{2+r^4(\cos^4 t + \sin^4 t)}\, r\, dr\, dt.$$

As $R \to \infty$, throughout the range of integration r also goes to infinity and

$$\frac{2r + r^3}{2 + r^4(\cos^4 t + \sin^4 t)} = \frac{1}{r(\cos^4 t + \sin^4 t)} + \mathcal{O}(1/r^3)\,,$$

where the error term makes a vanishing contribution to the integral. So

$$2\,I(R) \sim \left[\int_{r=R}^{R\sqrt{2}} \frac{dr}{r}\right]\left[\int_{t=0}^{2\pi} \frac{dt}{\cos^4 t + \sin^4 t}\right].$$

Now

$$\int_{r=R}^{R\sqrt{2}} \frac{dr}{r} = \log(R\sqrt{2}) - \log(R) = \log(\sqrt{2}) = \tfrac{1}{2}\log 2$$

and

$$\begin{aligned}\int_{t=0}^{2\pi} \frac{dt}{\cos^4 t + \sin^4 t} &= \int_{t=0}^{2\pi} \frac{dt}{1 - 2\sin^2 t\cos^2 t} = \int_{t=0}^{2\pi} \frac{2\,dt}{2 - \sin^2(2t)} \\ &= \int_{t=0}^{2\pi} \frac{2\,dt}{2\cos^2(2t) + \sin^2(2t)}\,.\end{aligned}$$

The integrand is periodic with period $\dfrac{\pi}{2}$ and is also even, so we can proceed as follows:

$$\begin{aligned}\int_{t=0}^{2\pi} \frac{dt}{\cos^4 t + \sin^4 t} &= 4\int_{t=-\pi/4}^{\pi/4} \frac{2\,dt}{2\cos^2(2t) + \sin^2(2t)} = 8\int_{t=0}^{\pi/4} \frac{2\,dt}{2\cos^2(2t) + \sin^2(2t)} \\ &= \int_{t=0}^{\pi/4} \frac{16\sec^2(2t)\,dt}{2 + \tan^2(2t)} \\ &= \int_{w=0}^{\infty} \frac{8\,dw}{2 + w^2} = \int_{w=0}^{\infty} \frac{4\,dw}{1 + w^2/2} \\ &= 4\sqrt{2}\tan^{-1}(w/\sqrt{2})\Big|_{w=0}^{\infty} = 2\pi\sqrt{2}\,.\end{aligned}$$

So

$$\lim_{R\to\infty} 2\,I(R) = (\frac{1}{2}\log 2)(2\pi\sqrt{2}) = \pi\sqrt{2}\log 2\,, \quad \text{and} \quad \lim_{R\to\infty} I(R) = \frac{\pi\sqrt{2}\log 2}{2}\,.$$

A5 Let A be the set of all integers n such that $1 \le n \le 2021$ and $\gcd(n, 2021) = 1$. For every nonnegative integer j, let

$$S(j) = \sum_{n \in A} n^j\,.$$

Determine all values of j such that $S(j)$ is a multiple of 2021.

Answer: All j that are not multiples of 42 or 46.

Solution: Note that modulo 2021, the set A consists precisely of the elements of the multiplicative group. Multiplying by an element of that group permutes the elements, so if x is relatively prime to 2021, then

$$x^j \cdot S(j) = \sum_{n \in A} (xn)^j \equiv \sum_{m \in A} m^j \equiv S(j) \pmod{2021}.$$

Therefore,

$$(x^j - 1)\,S(j) \equiv 0 \pmod{2021}.$$

Also note that $2021 = 2025 - 4 = 45^2 - 2^2 = 43 \cdot 47$ gives the prime factorization of 2021. Let x be a primitive root modulo 43 (that is, an integer between 1 and 42 that is a generator of the cyclic group $(\mathbb{Z}/43\,\mathbb{Z})^*$, which is the multiplicative group of the field with 43 elements). Then $x^j - 1 \equiv 0 \pmod{43}$ if and only if j is a multiple of 42; also, x is relatively prime to 2021. In particular, if j is not a multiple of 42 we have

$$(x^j - 1)\,S(j) \equiv 0 \pmod{2021} \Rightarrow (x^j - 1)\,S(j) \equiv 0 \pmod{43} \Rightarrow S(j) \equiv 0 \pmod{43}\,.$$

Similarly, if y is a primitive root modulo 47 and $y \neq 43$, we have

$$(y^j - 1)\,S(j) \equiv 0 \pmod{2021} \Rightarrow (y^j - 1)\,S(j) \equiv 0 \pmod{47} \Rightarrow S(j) \equiv 0 \pmod{47}$$

whenever j is not a multiple of 46. So if j is not a multiple of 42 or 46, then $S(j)$ is a multiple of both 43 and 47, hence of 2021.

Conversely, suppose that j is a multiple of 42. Then $n^j \equiv 1 \pmod{43}$ for all n in the sum, and $S(j)$ is therefore not a multiple of 43 (or of 2021), as

$$S(j) \equiv \sum_{n \in A} 1 = 42 \cdot 46 \cdot 1 \equiv -3 \equiv 40 \pmod{43}.$$

Similarly, if j is a multiple of 46, then $S(j) \equiv 5 \pmod{47}$.

A6 Let $P(x)$ be a polynomial whose coefficients are all either 0 or 1. Suppose that $P(x)$ can be written as the product of two nonconstant polynomials with integer coefficients. Does it follow that $P(2)$ is a composite integer?

Solution: Yes, we will show that $P(2)$ must be composite. Let $P(x) = F(x)G(x)$ have degree N, where $F(x)$ and $G(x)$ are nonconstant polynomials with integer coefficients, and suppose that $P(2) = p$ were prime. Then either $F(2)$ or $G(2)$ would be a unit, so without loss of generality we may assume that $G(2) = 1$. We have $N \geq 2$ because $F(x)$ and $G(x)$ are nonconstant, and we can write

$$P(x) = \sum_{n=0}^{N} \sigma_n x^n \,,$$

where each σ_n is either 0 or 1 and $\sigma_N = 1$.

Because the coefficients of $P(x)$ are nonnegative integers, $P(x)$ cannot have a positive real root, so $G(x)$ cannot have a positive real root either. Thus, as $G(2)$ is positive, $G(x)$ must be positive for all $x > 0$. In particular, because $\sigma_N = 1$ is the product of the leading coefficients of $F(x)$ and $G(x)$, the polynomial $G(x)$ must be monic. Let $r_1, \ldots, r_k$ be the (complex, not necessarily distinct) roots of $G(x)$, so that

$$G(x) = \prod_{j=1}^{k} (x - r_j)\,.$$

Consider the integer $G(1)$. Because $G(x) > 0$ for $x > 0$, we have $G(1) \geq 1 = G(2)$. In particular, $|G(1)| \geq |G(2)|$, and using the factorization of $G(x)$ we get

$$\prod_{j=1}^{k} |1 - r_j| \geq \prod_{j=1}^{k} |2 - r_j| \,.$$

It follows that $G(x)$ must have at least one root ρ with $|\rho - 1| \geq |\rho - 2|$, which is equivalent to $\mathrm{Re}(\rho) \geq \frac{3}{2}$. This implies that $|\rho| \geq \frac{3}{2}$; note that ρ is also a root of $P(x)$.

First consider the case $N = 2$. Dividing $P(\rho) = 0$ by ρ yields

$$\rho + \sigma_1 + \frac{\sigma_0}{\rho} = 0.$$

We have $\mathrm{Re}(\rho) \geq \frac{3}{2} > 0$ and hence

$$\mathrm{Re}\left(\frac{1}{\rho}\right) = \mathrm{Re}\left(\frac{\bar{\rho}}{|\rho|^2}\right) = \frac{1}{|\rho|^2}\mathrm{Re}\,(\rho) > 0.$$

But then

$$\mathrm{Re}(\rho) \leq \mathrm{Re}\left(\rho + \sigma_1 + \frac{\sigma_0}{\rho}\right) = 0\,,$$

which is a contradiction. (Alternatively, one can check the four possible polynomials $P(x)$ of degree 2 with coefficients from $\{0, 1\}$.)

For $N > 2$, we again divide $P(\rho) = 0$ by ρ^{N-1}, which now yields

$$\rho + \sigma_{N-1} + \frac{\sigma_{N-2}}{\rho} = -\frac{\sigma_{N-3}}{\rho^2} - \cdots - \frac{\sigma_0}{\rho^{N-1}}.$$

Once again, the terms on the left have nonnegative real parts. The triangle inequality gives

$$\mathrm{Re}(\rho) \leq \mathrm{Re}\left(\rho + \sigma_{N-1} + \frac{\sigma_{N-2}}{\rho}\right) \leq \left|\rho + \sigma_{N-1} + \frac{\sigma_{N-2}}{\rho}\right| \leq \frac{\sigma_{N-3}}{|\rho|^2} + \cdots + \frac{\sigma_0}{|\rho|^{N-1}}\,,$$

and we can estimate the sum on the right using an infinite geometric series:

$$\begin{aligned}\frac{\sigma_{N-3}}{|\rho|^2} + \cdots + \frac{\sigma_0}{|\rho|^{N-1}} &\leq \frac{1}{|\rho|^2} + \cdots + \frac{1}{|\rho|^{N-1}} \\ &\leq \frac{1}{|\rho|^2}\left(1 + \frac{1}{|\rho|} + \frac{1}{|\rho|^2} + \cdots\right) \\ &= \frac{1}{|\rho|^2} \cdot \frac{1}{1 - 1/|\rho|} = \frac{1}{|\rho|(|\rho| - 1)}\,.\end{aligned}$$

But $\dfrac{1}{x(x-1)}$ is a decreasing function of x for $x > 1$, and $|\rho| \geq \operatorname{Re}(\rho) \geq \dfrac{3}{2}$, so we get

$$\frac{3}{2} \leq \operatorname{Re}(\rho) \leq \frac{1}{|\rho|(|\rho| - 1)} \leq \frac{1}{\frac{3}{2}(\frac{3}{2} - 1)} = \frac{4}{3}\,,$$

a contradiction.

Remark. There are polynomials like $x^7 + x^2 + x + 1 = (x+1)(x^2+1)(x^4 - x^3 + 1)$ or $x^7 + x^3 + x^2 + x + 1$ (which is irreducible) which have roots with real part greater than 1. The polynomial $x^{11} + x^3 + x^2 + x + 1$ has a root r with $|r - 2| < 1$. Hence one needs to take some care with this argument.