A1 A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point (2021, 2021)?

A2 For every positive real number \( x \), let

\[
g(x) = \lim_{r \to 0} \left( (x + 1)^{r+1} - x^{r+1} \right)^{\frac{1}{r}}.
\]

Find \( \lim_{x \to \infty} \frac{g(x)}{x} \).

A3 Determine all positive integers \( N \) for which the sphere

\[ x^2 + y^2 + z^2 = N \]

has an inscribed regular tetrahedron whose vertices have integer coordinates.

A4 Let

\[
I(R) = \int \int_{x^2+y^2 \leq R^2} \left( \frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4} \right) \, dx \, dy.
\]

Find

\[
\lim_{R \to \infty} I(R),
\]

or show that this limit does not exist.

A5 Let \( A \) be the set of all integers \( n \) such that \( 1 \leq n \leq 2021 \) and \( \gcd(n, 2021) = 1 \). For every nonnegative integer \( j \), let

\[
S(j) = \sum_{n \in A} n^j.
\]

Determine all values of \( j \) such that \( S(j) \) is a multiple of 2021.

A6 Let \( P(x) \) be a polynomial whose coefficients are all either 0 or 1. Suppose that \( P(x) \) can be written as the product of two nonconstant polynomials with integer coefficients. Does it follow that \( P(2) \) is a composite integer?