# $30^{\text{th}}$ United States of America Mathematical Olympiad

**Part I 9 a.m. – 12 noon**

**May 1, 2001**

1. Each of eight boxes contains six balls. Each ball has been colored with one of $n$ colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Find the smallest integer $n$ for which this is possible.

**Solution:** The smallest such $n$ is 23.

We first show that $n = 22$ cannot be achieved. We present two arguments.

- *First argument* Let $m_{i,j}$ be the number of balls which are the same color as the $j^{\text{th}}$ ball in box $i$ (including that ball). For a fixed box $i$, $1 \le i \le 8$, consider the sums

$$S_i = \sum_{j=1}^{6} m_{i,j} \quad \text{and} \quad s_i = \sum_{j=1}^{6} \frac{1}{m_{i,j}}.$$

For each fixed $i$, since no pair of colors is repeated, each of the reamining seven boxes can contributed at most one ball to $S_i$. Thus $S_i \le 13$. It follows by the convexity of $f(x) = 1/x$ that $s_i$ is minimized when one of the $m_{i,j}$ is equal to 3 and the other five equal to 2. Hence $s_i \ge 17/6$. Note that

$$n = \sum_{i=1}^{8} \sum_{j=1}^{6} \frac{1}{m_{i,j}} \ge 8 \cdot \frac{17}{6} = \frac{68}{3}.$$

Hence there must be 23 colors.

- *Second argument* Assume that some color, say red, occurs four times. Then the first box containing red contains 6 colors, the second contains red and 5 colors not mentioned so far, and likewise for the third and fourth boxes. A fifth box can contain at most one color used in each of these four, so must contain 2 colors not mentioned so far, and a sixth box must contain 1 color not mentioned so far, for a total of 6+5+5+5+2+1=24, a contradiction.

  Next, assume that no color occurs four times; this forces at least four colors to occur three times. In particular, there are two colors that occur at least three times and which both occur in a single box, say red and blue. Now the box containing red and blue contains 6 colors, the other boxes containing red each contain 5 colors not mentioned so far, and the other boxes containing blue each contain 3 colors not mentioned so far (each may contain one color used in each of the boxes containing red but not blue). A sixth box must contain one color not mentioned so far, for a total of 6+5+5+3+3+1=23, again a contradiction.

We now give a construction for $n = 23$, guided by the second argument. We still cannot have a color occur four times, so at least two colors must occur three times. Call these red and green. Put one red in each of three boxes, and fill these with 15 other colors. Put one green in each of three boxes, and fill each of these boxes with one color from each of the three boxes containing red and two new colors. We now have used $1 + 15 + 1 + 6 = 23$ colors, and each box contains two colors that have only been used once so far. Split those colors between the last two boxes. The resulting arrangement is:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 4 | 5 | 6 | 7 |
| 1 | 8 | 9 | 10 | 11 | 12 |
| 1 | 13 | 14 | 15 | 16 | 17 |
| 2 | 3 | 8 | 13 | 18 | 19 |
| 2 | 4 | 9 | 14 | 20 | 21 |
| 2 | 5 | 10 | 15 | 22 | 23 |
| 6 | 11 | 16 | 18 | 20 | 22 |
| 7 | 12 | 17 | 19 | 21 | 23 |

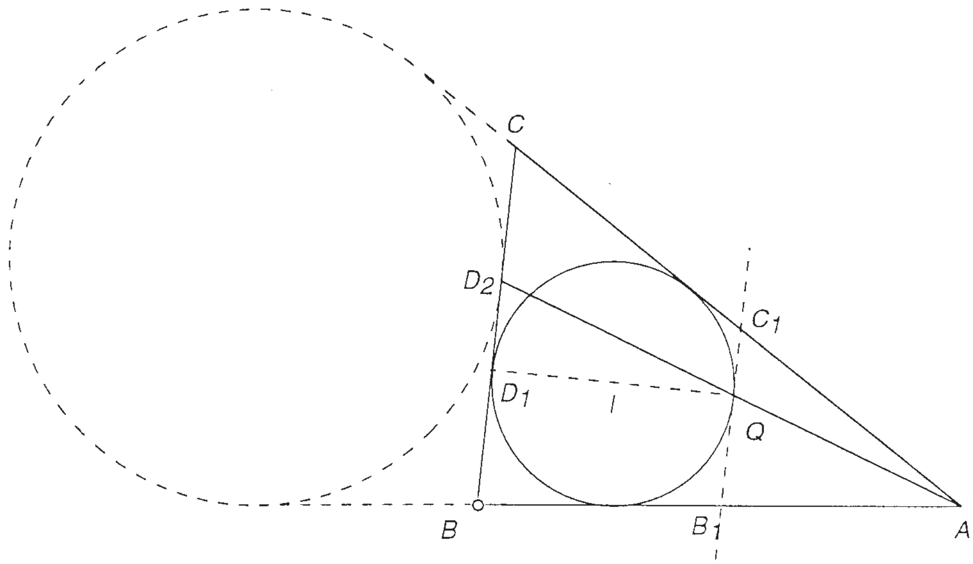
**Note:** Thanks to David Savitt for his help in assembling this solution: he also showed that for 10 boxes of eight balls, the minimum number of colors is 39. The general case of $n + 2$ boxes of $n$ balls, or even more generally of $n + k$ boxes of $n$ balls for other small values of $k$, may be of interest.

2. Let $ABC$ be a triangle and let $\omega$ be its incircle. Denote by $D_1$ and $E_1$ the points where $\omega$ is tangent to sides $BC$ and $AC$, respectively. Denote by $D_2$ and $E_2$ the points on sides $BC$ and $AC$, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by $P$ the point of intersection of segments $AD_2$ and $BE_2$. Circle $\omega$ intersects segment $AD_2$ at two points, the closer of which to the vertex $A$ is denoted by $Q$. Prove that $AQ = D_2P$.

**Solution:**

The key observation is the following lemma.

**Lemma** *Segement $D_1Q$ is a diameter of circle $\omega$.*




*Proof*: Let $I$ be the center of circle $\omega$, i.e., $I$ is the incenter of triangle $ABC$. Extand segement $D_1I$ through $I$ to intersect circle $\omega$ again at $Q'$, and extand segment $AQ'$ through $Q'$ to intersectt segment $BC$ at $D'$. We show that $D_2 = D'$, which in turn implies that $Q = Q'$, that is, $D_1Q$ is a diameter of $\omega$.

Let $\ell$ be the line tangent to circle $\omega$ at $Q'$, and let $\ell$ intersect the segments $AB$ and $AC$ at $B'$ and $C'$, respectively. Then $\omega$ is an **excircle** of triangle $AB'C'$. Let $\mathbf{H}_1$ denote the dialation with its center at $A$ and ratio $AD'/AQ'$. Since $\ell \perp D_1Q'$ and $BC \perp D_1Q$, $\ell \perp BC$. Hence $AB/AB' = AC/AC' = AD'/AQ'$. Thus $\mathbf{H}_1(Q') = D'$, $\mathbf{H}_1(B') = B$, and $\mathbf{H}_1(C') = C$. It also follows that an excircle $\Omega$ of triangle $ABC$ is tangent to the side $BC$ at $D'$.

It is well known that

$$CD_1 = \frac{1}{2}(BC + CA - AB). \tag{1}$$

We compute $BD'$. Let $X$ and $Y$ denote the points of tangency of circle $\Omega$ with rays $AB$ and $AC$, respectively. Then by equal tangents, $AX = AY$, $BD' = BX$, and $D'C = YC$. Hence

$$AX = AY = \frac{1}{2}(AX + AY) = \frac{1}{2}(AB + BX + YC + CA) = \frac{1}{2}(AB + BC + CA).$$

It follows that

$$BD' = BX = AX - AB = \frac{1}{2}(BC = CA - AB). \tag{2}$$

Combining (1) and (2) yields $BD' = CD_1$. Thus

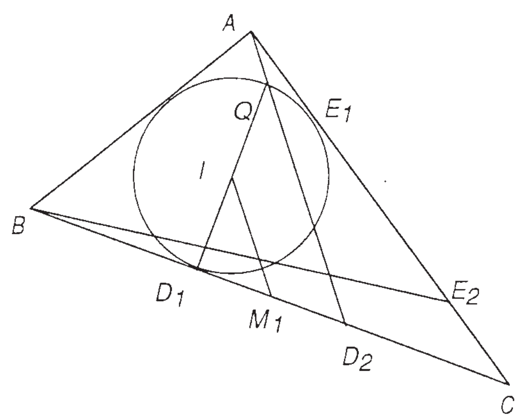
$$BD_2 = BD_1 - D_2D_1 = D_2C - D_2D_1 = CD_1 = BD',$$

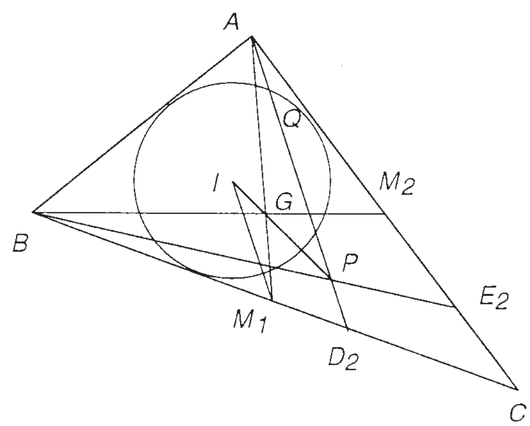
that is, $D' = D_2$, as desired. ■

Now we prove our main result. Let $M_1$ and $M_2$ be the respective midpoints of segments $BC$ and $CA$. Then $M_1$ is also the midpoint of segment $D_1D_2$, from which it follows that $IM_1$ is the midline of triangle $D_1QD_2$. Hence

$$QD_2 = 2IM_1 \tag{3}$$

and $AD_2 \parallel M_1I$. Similarly, we can prove that $BE_2 \parallel M_2I$.




Let $G$ be the centroid of triangle $ABC$. Thus segments $AM_1$ and $BM_2$ intersect at $G$. Define transformation $\mathbf{H}_2$ as the **dialation** with its center at $G$ and ratio $-1/2$. Then $\mathbf{H}_2(A) = M_1$ and $\mathbf{H}_2(B) = M_2$. Under the dilation, parallel lines go to parallel lines and the intersection of two lines goes to the intersection of their images. Since $AD_2 \parallel M_1I$ and $BE_2 \parallel M_2I$, $\mathbf{H}$ maps lines $AD_2$ and $BE_2$ to lines $M_1I$ and $M_2I$, respectively. It also follows that $\mathbf{H}_2(I) = P$ and

$$\frac{IM_1}{AP} = \frac{GM_1}{AG} = \frac{1}{2}$$

or

$$AP = 2IM_1. \tag{4}$$

Combining (3) and (4) yields

$$AQ = AP - QP = 2IM_1 - QP = QD_2 - QP = PD_2,$$

as desired.

**Note:** We used three different diagrams of triangle $ABC$. Each diagram was desgined to assist the reader in understanding a particular part of the proof. We used directed lengths of segements in our calculations to avoid possible complications caused by the different shapes of triangle $ABC$.

3. Let $a, b$, and $c$ be nonnegative real numbers such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \le ab + bc + ca - abc \le 2.$$

**First Solution:** From the condition, at least one of $a, b$, and $c$ does not exceed 1, say $a \le 1$. Then

$$ab + bc + ca - abc = a(b + c) + bc(1 - a) \ge 0.$$

Now we prove the upper bound. Let us note that at least two of the three numbers $a, b$, and $c$ are both greater than or equal to 1 or less than or equal to 1. Without loss of generality, we assume that the numbers with this property are $b$ and $c$. Then we have

$$(1 - b)(1 - c) \ge 0. \tag{1}$$

The given equality $a^2 + b^2 + c^2 + abc = 4$ and the inequality $b^2 + c^2 \ge 2bc$ imply that

$$a^2 + 2bc + abc \le 4, \quad \text{or} \quad bc(2 + a) \le 4 - a^2.$$

Dividing both sides of the last inequality by $2 + a$ yields

$$bc \le 2 - a. \tag{2}$$

Combining (1) and (2) gives

$$ab + bc + ac - abc \le ab + 2 - a + ac(1 - b) = 2 - a(1 + bc - b - c) = 2 - a(1 - b)(1 - c) \le 2,$$

as desired.

The last equality holds if and only if $b = c$ and $a(1 - b)(1 - c) = 0$. Hence equality for the upper bound holds if and only if $(a, b, c)$ is one of the triples $(1, 1, 1)$, $(0, \sqrt{2}, \sqrt{2})$, $(\sqrt{2}, 0, \sqrt{2})$, and $(\sqrt{2}, \sqrt{2}, 0)$. Equality for the lower bound holds if and only if $(a, b, c)$ is one of the triples $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

**Second Solution:** The proof for the lower bound is the same as in the first solution. Now we prove the upper bound. It is clear that $a, b, c \le 2$. If $abc = 0$, then the result is trivial. Suppose that $a, b, c > 0$. Solving for $a$ yields

$$a = \frac{-bc + \sqrt{b^2c^2 - 4(b^2 + c^2 - 4)}}{2} = \frac{-bc + \sqrt{(4 - b^2)(4 - c^2)}}{2}.$$

This asks for the trigonometric substitution $b = 2\sin u$ and $c = 2\sin v$, where $0^\circ < u, v < 90^\circ$. Then

$$a = 2(-\sin u \sin v + \cos u \cos v) = 2\cos(u + v),$$

and if we set $u = B/2$ and $v = C/2$, then $a = 2\sin(A/2)$, $b = 2\sin(B/2)$, and $c = 2\sin(C/2)$, where $A, B$, and $C$ are the angles of a triangle. We have

$$\begin{aligned}
ab &= 4\sin\frac{A}{2}\sin\frac{B}{2} = 2\sqrt{\sin A\tan\frac{A}{2}\sin B\tan\frac{B}{2}} = 2\sqrt{\sin A\tan\frac{B}{2}\sin B\tan\frac{A}{2}} \\
&\le \sin A\tan\frac{B}{2} + \sin B\tan\frac{A}{2} \qquad (\text{by the } \textbf{AM-GM inequality}) \\
&= \sin A\cot\frac{A+C}{2} + \sin B\cot\frac{B+C}{2}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
bc &\le \sin B\cot\frac{B+A}{2} + \sin C\cot\frac{C+A}{2}, \\
ca &\le \sin A\cot\frac{A+B}{2} + \sin C\cot\frac{C+B}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
ab + bc + ca &\le (\sin A + \sin B)\cot\frac{A+B}{2} + (\sin B + \sin C)\cot\frac{B+C}{2} + (\sin C + \sin A)\cot\frac{C+A}{2} \\
&= 2\left(\cos\frac{A-B}{2}\cos\frac{A+B}{2} + \cos\frac{B-C}{2}\cos\frac{B+C}{2} + \cos\frac{C-A}{2}\cos\frac{C+A}{2}\right) \\
&= 2(\cos A + \cos B + \cos C) = 6 - 4\left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}\right) \\
&= 6 - (a^2 + b^2 + c^2) = 2 + abc,
\end{aligned}$$

as desired.

# $30^{\text{th}}$ United States of America Mathematical Olympiad

**Part II** 1 p.m. – 4 p.m.

May 1, 2001

4. Let $P$ be a point in the plane of triangle $ABC$ such that the segments $PA$, $PB$, and $PC$ are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to $PA$. Prove that $\angle BAC$ is acute.

**First Solution:** Let $A$ be the origin. For a point $Q$, denote by $q$ the vector $\overrightarrow{AQ}$, and denote by $|q|$ the length of $q$. The given condictions may be written as

$$|p-b|^2 + |p-c|^2 < |p|^2,$$

or

$$p \cdot p + b \cdot b + c \cdot c - 2p \cdot b - 2p \cdot c < 0.$$

Adding $2b \cdot c$ on both sides of the last inequality gives

$$|p-b-c|^2 < 2b \cdot c.$$

Since the left-hand side of the last inequality is nonnegative, the right-hand side is positive. Hence

$$\cos \angle BAC = \frac{b \cdot c}{|b||c|} > 0,$$

that is, $\angle BAC$ is acute.

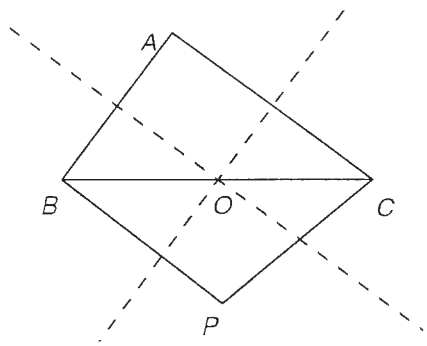
**Second Solution:** For the sake of contradiction, let's assume to the contrary that $\angle BAC$ is not acute. Let $AB = c$, $BC = a$, and $CA = b$. Then $a^2 \geq b^2 + c^2$. We claim that the quadrilateral $ABPC$ is convex. Now applying the generalized Ptolemy's Theorem to the convex quadrilateral $ABPC$ yields

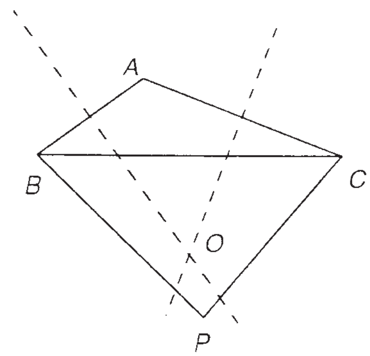
$$a \cdot PA \leq b \cdot PB + c \cdot PC \leq \sqrt{b^2+c^2}\sqrt{PB^2+PC^2} \leq a\sqrt{PB^2+PC^2},$$

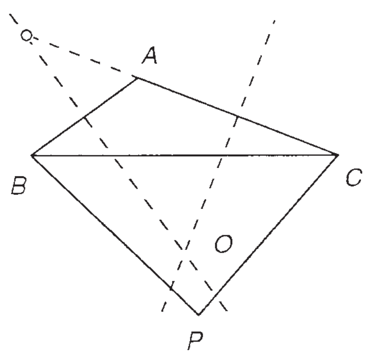
where the second inequality is by Cauchy-Schwarz. This implies $PA^2 \leq PB^2 + PC^2$, in contradiction with the facts that $PA$, $PB$, and $PC$ are the sides of an obtuse triangle and $PA > \max\{PB, PC\}$.

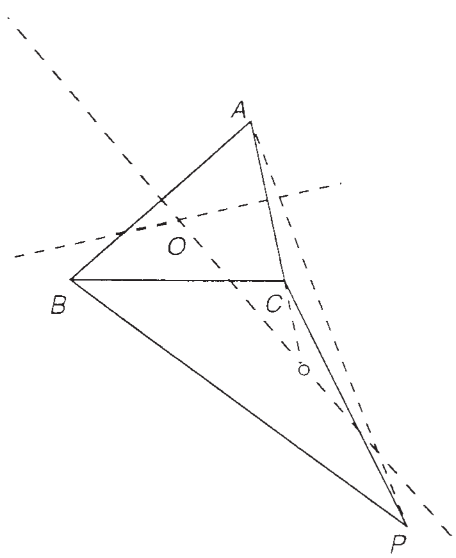
We present two arguments to prove our claim.

- *First argument* Without loss of generality, we may assume that $A$, $B$, and $C$ are in counterclockwise order. Let line $\ell_1$ and $\ell_2$ be the perpendicular bisectors of segments $AB$ and $AC$, respectively. Then $\ell_1$ and $\ell_2$ meet at $O$, the circumcenter of triangle $ABC$. Lines $\ell_1$ and $\ell_2$ cut the plane into four regions and $A$ is in the interior of one of these regions. Since $PA > PB$ and $PA > PC$, $P$ must be in the interior of the region that opposes $A$. Since $\angle BAC$ is not acute, ray $AC$ does not meet $\ell_1$ and ray $AB$ does not meet $\ell_2$. Hence $B$ and $C$ must lie in the interiors of the regions adjacent to $A$. Let $\mathcal{R}_X$ denote the region containing $X$. Then $\mathcal{R}_A$, $\mathcal{R}_B$, $\mathcal{R}_P$, and $\mathcal{R}_C$ are the four regions in counterclockwise order. Since $\angle BAC \geq 90°$, either $O$ is on side $BC$ or $O$ and $A$ are on opposite sides of line $BC$. In either case $P$ and $A$ are on opposite sides of line $BC$. Also, since ray $AB$ does not meet $\ell_2$ and ray $AC$ does not meet $\ell_1$, it follows that $\mathcal{R}_P$ is entirely in the interior of $\angle BAC$. Hence $B$ and $C$ are on opposite sides of $AP$. Therefore $ABPC$ is convex.




- *Second argument* Since $PA > PB$ and $PA > PC$, $A$ cannot be inside or on the sides of triangle $PBC$. Since $PA > PB$, we have $\angle ABP > \angle BAP$ and hence $\angle BAC \geq 90° > \angle BAP$. Hence $C$ cannot be inside or on the sides of triangle $BAP$. Symmetrically, $B$ cannot be inside or on the sides of triangle $CAP$. Finally since $\angle ABP > \angle BAP$ and $\angle ACP > \angle CAP$, we have

$$\angle ABP + \angle ACP > \angle BAC \geq 90° \geq \angle ABC + \angle ACB.$$

Therefore $P$ cannot be in inside or on the sides of triangle $ABC$. Since this covers all four cases, $ABPC$ is convex.

5. Let $S$ be a set of integers (not necessarily positive) such that

(A) there exist $a, b \in S$ with $\gcd(a, b) = \gcd(a-2, b-2) = 1$;

(B) if $x$ and $y$ are elements of $S$ (possibly equal), then $x^2 - y$ also belongs to $S$.

Prove that $S$ is the set of all integers.

**Solution:** In the solution below we use the expression $S$ *is stable under* $x \mapsto f(x)$ to mean that if $x$ belongs to $S$ then $f(x)$ also belongs to $S$. If $c, d \in S$, then by (B), $S$ is stable under $x \mapsto c^2 - x$ and $x \mapsto d^2 - x$, hence stable under $x \mapsto c^2 - (d^2 - x) = x + (c^2 - d^2)$. Similarly $S$ is stable under $x \mapsto x + (d^2 - c^2)$. Hence $S$ is stable under $x \mapsto x + n$ and $x \mapsto x - n$ whenever $n$ is an integer linear combination of numbers of the form $c^2 - d^2$ with $c, d \in S$. In particular, this holds for $n = m$, where $m = \gcd\{c^2 - d^2 : c, d \in S\}$.

Since $S \neq \emptyset$ by (A), it suffices to prove that $m = 1$. For the sake of contradiction, assume that $m \neq 1$. Let $p$ be a prime dividing $m$. Then $c^2 - d^2 \equiv 0 \pmod{p}$ for all $c, d \in S$. In other words, for each $c, d \in S$, either $d \equiv c \pmod{p}$ or $d \equiv -c \pmod{p}$. Given $c \in S$, $c^2 - c \in S$ by (B), so $c^2 - c \equiv c \pmod{p}$ or $c^2 - c \equiv -c \pmod{p}$. Hence

$$\text{For each } c \in S, \text{ either } c \equiv 0 \pmod{p} \text{ or } c \equiv 2 \pmod{p}. \qquad (*)$$

By (A), there exist some $a$ and $b$ in $S$ such that $\gcd(a, b) = 1$, that is, at least one of $a$ or $b$ cannot be divisible by $p$. Denote such an element of $S$ by $\alpha$; thus, $\alpha \not\equiv 0 \pmod{p}$. Similarly, by (A), $\gcd(a-2, b-2) = 1$, so $p$ cannot divide both $a - 2$ and $b - 2$. Thus, there is an element of $S$, call it $\beta$, such that $\beta \not\equiv 2 \pmod{p}$. By $(*)$, $\alpha \equiv 2 \pmod{p}$ and $\beta \equiv 0 \pmod{p}$. By (B), $\beta^2 - \alpha \in S$. Taking $c = \beta^2 - \alpha$ in $(*)$ yields either $-2 \equiv 0 \pmod{p}$ or $-2 \equiv 2 \pmod{p}$, so $p = 2$. Now $(*)$ says that all elements of $S$ are even, contradicting (A). Hence our assumption is false and $S$ is the set of all integers.

6. Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

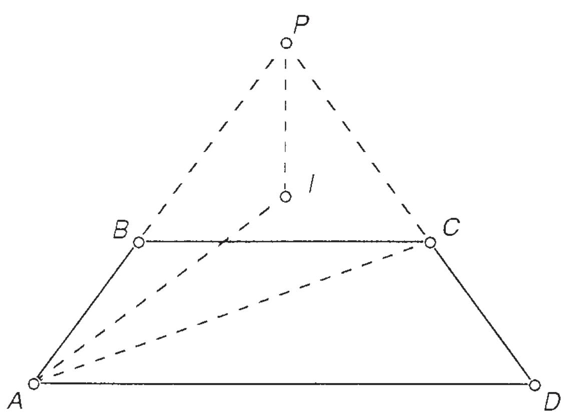
**Solution:** We label each upper case point with the corresponding lower case letter as its assigned number. The key step is the following lemma.

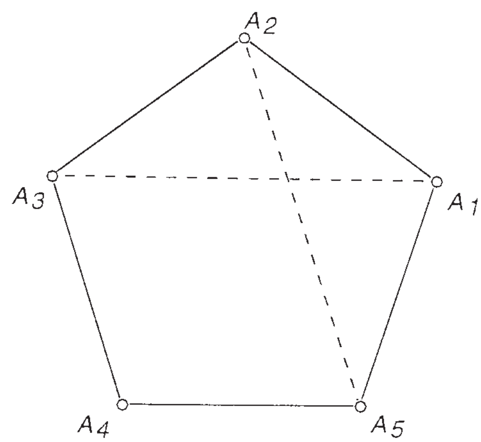
**Lemma** *If $ABCD$ is an isosceles trapezoid, then $a+c=b+d$.*

*Proof:* Assume without loss of generality that $BC \parallel AD$, and that rays $AB$ and $DC$ meet at $P$. Let $I$ be the incenter of triangle $PAC$, and let line $\ell$ bisect $\angle APD$. Then $I$ is on $\ell$, so reflecting everything across line $\ell$ shows that $I$ is also the incenter of ttriangle $PDB$. Therefore,

$$\frac{p+a+c}{3} = i = \frac{p+b+d}{3}.$$

Hence $a+c=b+d$, as desired. ■




For any two distinct points $A_1$ and $A_2$ in the plane, we construct a regular pentagon $A_1A_2A_3A_4A_5$. Applying the lemma to isosceles trapezoids $A_1A_3A_4A_5$ and $A_2A_3A_4A_5$ yields

$$a_1 + a_4 = a_3 + a_5 \quad \text{and} \quad a_2 + a_4 = a_3 + a_5.$$

Hence $a_1 = a_2$. Since $A_1$ and $A_2$ were arbitrary, all points in the plane are assigned the same number.