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Contest P

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by the Akamai Foundation

AMC 12

53rd Annual American Mathematics Contest 12

This test and the matching AMC 10 P were developed for the use of a group of Taiwan schools, in early January of 2002. When Taiwan had taken the contests, the AMC released the questions here as a set of practice questions for the 2002 AMC 10 and AMC 12 contests.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. Which of the following numbers is a perfect square?

(A) $4^4 5^5 6^6$ **(B)** $4^4 5^6 6^5$ **(C)** $4^5 5^4 6^6$ **(D)** $4^6 5^4 6^5$ **(E)** $4^6 5^5 6^4$

2. The function f is given by the table

x	1	2	3	4	5
$f(x)$	4	1	3	5	2

If $u_0 = 4$ and $u_{n+1} = f(u_n)$ for $n \geq 0$, find u_{2002}.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

3. The dimensions of a rectangular box in inches are all positive integers and the volume of the box is 2002 in^3. Find the minimum possible sum in inches of the three dimensions.

(A) 36 **(B)** 38 **(C)** 42 **(D)** 44 **(E)** 92

4. Let a and b be distinct real numbers for which

$$\frac{a}{b} + \frac{a+10b}{b+10a} = 2.$$

Find $\frac{a}{b}$.

(A) 0.4 **(B)** 0.5 **(C)** 0.6 **(D)** 0.7 **(E)** 0.8

5. For how many positive integers m is

$$\frac{2002}{m^2 - 2}$$

a positive integer?

(A) one **(B)** two **(C)** three **(D)** four **(E)** more than four

6. Participation in the local soccer league this year is 10% higher than last year. The number of males increased by 5% and the number of females increased by 20%. What fraction of the soccer league is now female?

(A) $\frac{1}{3}$ **(B)** $\frac{4}{11}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{4}{9}$ **(E)** $\frac{1}{2}$

7. How many three-digit numbers have at least one 2 and at least one 3?

(A) 52 **(B)** 54 **(C)** 56 **(D)** 58 **(E)** 60

8. Let AB be a segment of length 26, and let points C and D be located on AB such that $AC = 1$ and $AD = 8$. Let E and F be points on one of the semicircles with diameter AB for which EC and FD are perpendicular to AB. Find EF.

(A) 5 **(B)** $5\sqrt{2}$ **(C)** 7 **(D)** $7\sqrt{2}$ **(E)** 12

9. Two walls and the ceiling of a room meet at right angles at point P. A fly is in the air one meter from one wall, eight meters from the other wall, and nine meters from point P. How many meters is the fly from the ceiling?

(A) $\sqrt{13}$ **(B)** $\sqrt{14}$ **(C)** $\sqrt{15}$ **(D)** 4 **(E)** $\sqrt{17}$

10. Let $f_n(x) = \sin^n x + \cos^n x$. For how many x in $[0, \pi]$ is it true that

$$6f_4(x) - 4f_6(x) = 2f_2(x)?$$

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** more than 8

11. Let $t_n = \frac{n(n+1)}{2}$ be the nth triangular number. Find

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \ldots + \frac{1}{t_{2002}}.$$

(A) $\dfrac{4003}{2003}$ **(B)** $\dfrac{2001}{1001}$ **(C)** $\dfrac{4004}{2003}$ **(D)** $\dfrac{4001}{2001}$ **(E)** 2

12. For how many positive integers n is $n^3 - 8n^2 + 20n - 13$ a prime number?

(A) one **(B)** two **(C)** three **(D)** four **(E)** more than four

13. What is the maximum value of n for which there is a set of distinct positive integers $k_1, k_2, \ldots, k_n$ for which

$$k_1^2 + k_2^2 + \ldots + k_n^2 = 2002.$$

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

14. Find $i + 2i^2 + 3i^3 + \ldots + 2002i^{2002}$.

(A) $-999 + 1002i$ **(B)** $-1002 + 999i$ **(C)** $-1001 + 1000i$
(D) $-1002 + 1001i$ **(E)** i

15. There are 1001 red marbles and 1001 black marbles in a box. Let P_s be the probability that two marbles drawn at random from the box are the same color, and let P_d be the probability that they are different colors. Find $|P_s - P_d|$.

(A) 0 **(B)** $\frac{1}{2002}$ **(C)** $\frac{1}{2001}$ **(D)** $\frac{2}{2001}$ **(E)** $\frac{1}{1000}$

16. The altitudes of a triangle are 12, 15, and 20. The largest angle in this triangle is

(A) 72° **(B)** 75° **(C)** 90° **(D)** 108° **(E)** 120°

17. Let $f(x) = \sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$. An equivalent form of $f(x)$ is

(A) $1 - \sqrt{2}\sin x$ **(B)** $-1 + \sqrt{2}\cos x$ **(C)** $\cos\frac{x}{2} - \sin\frac{x}{2}$

(D) $\cos x - \sin x$ **(E)** $\cos 2x$

18. If a, b, c are real numbers such that $a^2 + 2b = 7$, $b^2 + 4c = -7$, and $c^2 + 6a = -14$, find $a^2 + b^2 + c^2$.

(A) 14 **(B)** 21 **(C)** 28 **(D)** 35 **(E)** 49

19. In quadrilateral $ABCD$, $m\angle B = m\angle C = 120^\circ$, $AB = 3$, $BC = 4$, and $CD = 5$. Find the area of $ABCD$.

(A) 15 **(B)** $9\sqrt{3}$ **(C)** $\frac{45\sqrt{3}}{4}$ **(D)** $\frac{47\sqrt{3}}{4}$ **(E)** $15\sqrt{3}$

20. Let f be a real-valued function such that

$$f(x) + 2f\left(\frac{2002}{x}\right) = 3x$$

for all $x > 0$. Find $f(2)$.

(A) 1000 **(B)** 2000 **(C)** 3000 **(D)** 4000 **(E)** 6000

21. Let a and b be real numbers greater than 1 for which there exists a positive real number c, different from 1, such that

$$2(\log_a c + \log_b c) = 9\log_{ab} c.$$

Find the largest possible value of $\log_a b$.

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $\sqrt{6}$ **(E)** 3

22. Under the new AMC 10, 12 scoring method, 6 points are given for each correct answer, 2.5 points are given for each unanswered question, and no points are given for an incorrect answer. Some of the possible scores between 0 and 150 can be obtained in only one way, for example, the only way to obtain a score of 146.5 is to have 24 correct answers and one unanswered question. Some scores can be obtained in exactly two ways; for example, a score of 104.5 can be obtained with 17 correct answers, 1 unanswered question, and 7 incorrect, and also with 12 correct answers and 13 unanswered questions. There are (three) scores that can be obtained in exactly three ways. What is their sum?

(A) 175 **(B)** 179.5 **(C)** 182 **(D)** 188.5 **(E)** 201

23. The equation $z(z+i)(z+3i) = 2002i$ has a zero of the form $a+bi$, where a and b are positive real numbers. Find a.

(A) $\sqrt{118}$ **(B)** $\sqrt{210}$ **(C)** $2\sqrt{210}$ **(D)** $\sqrt{2002}$ **(E)** $100\sqrt{2}$

24. Let $ABCD$ be a regular tetrahedron and let E be a point inside the face ABC. Denote by s the sum of the distances from E to the faces DAB, DBC, DCA, and by S the sum of the distances from E to the edges AB, BC, CA. Then $\frac{s}{S}$ equals

(A) $\sqrt{2}$ **(B)** $\dfrac{2\sqrt{2}}{3}$ **(C)** $\dfrac{\sqrt{6}}{2}$ **(D)** 2 **(E)** 3

25. Let a and b be real numbers such that $\sin a + \sin b = \frac{\sqrt{2}}{2}$ and $\cos a + \cos b = \frac{\sqrt{6}}{2}$. Find $\sin(a+b)$

(A) $\dfrac{1}{2}$ **(B)** $\dfrac{\sqrt{2}}{2}$ **(C)** $\dfrac{\sqrt{3}}{2}$ **(D)** $\dfrac{\sqrt{6}}{2}$ **(E)** 1