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53rd Annual American Mathematics Contest 12

AMC 12 - Contest P

Solutions Pamphlet

January 2002

The "P" set of contests were originally developed for a group of schools in Taiwan, to be taken in early January of 2002. Once the contests were taken, the AMC office released the questions here for use as a practice ("P") set of questions for the 2002 AMC 10 and AMC 12.

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606

1. (C) For $4^x5^y6^z = 2^{2x+z}3^z5^y$ to be a perfect square, the exponent on each prime must be even. That is, y and z must be even. Only choice (C) satisfies this condition.

2. (B) Checking the first few values we find

$$\begin{aligned}u_0 &= 4\\ u_1 &= f(4) = 5\\ u_2 &= f(5) = 2\\ u_3 &= f(2) = 1\\ u_4 &= f(1) = 4\\ u_5 &= f(4) = 5.\end{aligned}$$

In general, we see that $u_{4k+j} = u_j$, where k is any integer greater than or equal to zero. Hence, $u_{2002} = u_{4\cdot500+2} = u_2 = 2$.

3. (B) Let a, b, c be the dimensions of the box, $a \le b \le c$. Since $abc = 2002 = 2\cdot7\cdot11\cdot13$, the only possible triples (a, b, c) are (1,1,2002), (1,2,1001), (1,7,286), (1,11,182), (1,13,154), (1,14,143), (1,22,91), (1,26,77), (2,7,143), (2,11,91), (2,13,77), (7,11,26), (7,13,22), and (11,13,14). Among these, the last triple gives the minimum sum, 38.

4. (E) Multiplying both sides of the given equation by $b(b + 10a)$ yields

$$2ab + 10a^2 + 10b^2 = 2b^2 + 20ab$$

which is equivalent to $(a - b)(5a - 4b) = 0$. Since $a \ne b$, we have $5a = 4b$, or $\frac{a}{b} = 0.8$.

5. (C) There are 16 factors of $2002 = 2 \cdot 7 \cdot 11 \cdot 13$. We examine them and find that only three of them, namely 2, 7, and 14, are 2 less than a perfect square.

6. (B) If m and w are the current numbers of men and women, respectively, then we have

$$\frac{m}{1.05} + \frac{w}{1.20} = \frac{m+w}{1.10}$$

or

$$\frac{m}{w} \cdot \left(\frac{1}{1.05} - \frac{1}{1.10}\right) = \frac{1}{1.10} - \frac{1}{1.20}.$$

It follows that $\frac{m}{w} = \frac{7}{4}$ so that $\frac{m+w}{w} = \frac{11}{4}$, and $\frac{w}{m+w} = \frac{4}{11}$.

7. (A) Let U be the set of all three-digit numbers, let S be the set of three-digit numbers that contain no 2s, and let T be the set of three digit numbers that contain no 3s. Then $S \cap T$ is the set of three-digit numbers containing neither a 2 nor a 3 and $U - (S \cup T)$ is the set of three-digit numbers containing at least one 2 and at least one 3. We have $|U| = 900$, $|S| = |T| = 8 \cdot 9^2$, and $|S \cap T| = 7 \cdot 8^2$. Therefore $|S \cup T| = |S| + |T| - |S \cap T| = 848$, and $|U - (S \cup T)| = 52$.

8. (D) From similar right triangles ACE and ECB, $\frac{EC}{AC} = \frac{CB}{EC}$. Therefore $EC^2 = (AC)(CB) = (1)(25)$ and $EC = 5$. Similarly, $FD^2 = (AD)(DB) = (8)(18)$ and $FD = 12$. If we choose G on FD so that EG is parallel to AB then triangle EGF is right with $EG = GF = 7$. Therefore the hypotenuse $EF = 7\sqrt{2}$.

9. (D) Let the fly be x meters from the ceiling. Then the fly and point P determine a major diagonal of the rectangular parallelepiped having dimensions 1, 8, and x. Therefore, $1^2 + 8^2 + x^2 = 9^2$, and it follows that $x = 4$.

10. (E) The expression is an identity. Note first that

$$\begin{aligned} f_6(x) &= (\sin^2 x)^3 + (\cos^2 x)^3 \\ &= (\sin^2 x + \cos^2 x)(f_4(x) - \sin^2 x \cos^2 x) \\ &= f_4(x) - \sin^2 x \cos^2 x. \end{aligned}$$

Therefore,

$$6f_4(x) - 4f_6(x) = 2f_4(x) + 4\sin^2 x \cos^2 x = 2f_2(x).$$

11. (C)

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \ldots + \frac{1}{t_{2002}} =$$

$$\frac{2}{1 \cdot 2} + \frac{2}{2 \cdot 3)} + \frac{2}{3 \cdot 4} + \ldots + \frac{2}{2002 \cdot 2003} =$$

$$(\frac{2}{1} - \frac{2}{2}) + (\frac{2}{2} - \frac{2}{3}) + (\frac{2}{3} - \frac{2}{4}) + \ldots + (\frac{2}{2002} - \frac{2}{2003}) =$$

$$\frac{2}{1} - \frac{2}{2003} = \frac{4004}{2003}.$$

12. (C) Because $n^3 - 8n^2 + 20n - 13 = (n-1)(n^2 - 7n + 13)$, for the value to be prime one factor must equal 1 and the other factor must be prime. For $n - 1 = 1$ we must have $n = 2$, and in this case the other factor is the prime 3. So $n = 2$ is a solution. For $n^2 - 7n + 13 = 1$, we have $n^2 - 7n + 12 = 0 = (n-4)(n-3)$, so we must have $n = 3$ or 4, and in each case the other factor is prime (2 and 3, respectively). Therefore $n^3 - 8n^2 + 20n - 13$ is a prime for three positive integer values of n.

13. (D) Since $1^2 + 2^2 + 3^2 + \ldots + 18^2 > 2002$, we know that $n \leq 17$. Then note that $1^2 + 2^2 + 3^2 + \ldots + 19^2 - 18^2 - 12^2 = 2002$, hence $n = 17$.

14. (D) The real part of the sum is

$$\begin{aligned}-2+4-6+\ldots+2000-2002 &= -2+(4-6)+\ldots+(2000-2002)\\ &= -2\cdot 501 = -1002,\end{aligned}$$

and the imaginary part of the sum is

$$\begin{aligned}1-3+5-\ldots-1999+2001 &= 1+(-3+5)+\ldots+(-1999+2001)\\ &= 1+2\cdot 500 = 1001.\end{aligned}$$

Hence the answer is (D), $-1002+1001i$.

15. (C) There are $\binom{2002}{2} = \frac{2002\cdot 2001}{2\cdot 1} = 1001\cdot 2001$ possible pairs that can be drawn. There are 1001^2 pairs of different colored marbles, so $P_d = \frac{1001^2}{1001\cdot 2001}$. Therefore, $P_s = 1 - P_d = \frac{1000}{2001}$, and $|P_s - P_d| = \frac{1}{2001}$.

16. (C) If the sides of a triangle corresponding to the altitudes 12, 15, and 20 are a, b, and c respectively, then we have

$$12a = 15b = 20c$$

or, dividing by 60,

$$\frac{a}{5} = \frac{b}{4} = \frac{c}{3}.$$

It follows that $a = 5r$, $b = 4r$, and $c = 3r$ and so the triangle is a 3-4-5 right triangle whose largest angle is $90°$.

17. (E) Since $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$, we have

$$\begin{aligned}\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x} &= \sqrt{\sin^4 x - 4\sin^2 x + 4}\\ &\quad - \sqrt{\cos^4 x - 4\cos^2 x + 4}\\ &= \sqrt{(\sin^2 x - 2)^2} - \sqrt{(\cos^2 x - 2)^2}\\ &= \left|\sin^2 x - 2\right| - \left|\cos^2 x - 2\right|\\ &= 2 - \sin^2 x - (2 - \cos^2 x)\\ &= \cos^2 x - \sin^2 x = \cos 2x.\end{aligned}$$

18. (A) Adding the three equations we obtain

$$(a^2 + 6a) + (b^2 + 2b) + (c^2 + 4c) = -14,$$

which is equivalent to

$$(a+3)^2 + (b+1)^2 + (c+2)^2 = 0.$$

Therefore $a = -3$, $b = -1$, $c = -2$, and $a^2 + b^2 + c^2 = 14$.

19. (D) Let the extensions of AB beyond B and DC beyond C meet at E. Then, because angles CBE and BCE both equal 60°, BEC is an equilateral triangle of side 4. The area of triangle BCE is $4\sqrt{3}$ and the area of triangle AED is $\frac{(AE)(DE)\sin 60^\circ}{2} = \frac{63\sqrt{3}}{4}$. Therefore the area of $ABCD$ is $\frac{63\sqrt{3}}{4} - 4\sqrt{3} = \frac{47\sqrt{3}}{4}$.

20. (B) Setting $x = 2$, and then $x = 1001$, we have $f(2)+2f(1001) = 6$ and $f(1001)+2f(2) = 3003$. Subtracting the first equation from twice the second equation we obtain $3f(2) = 6000$, so $f(2) = 2000$. Note that $f(x) = \frac{4004}{x} - x$ is such a function.

21. (C) Let $x = \log_c a$ and $y = \log_c b$. Then we are given

$$2\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{9}{x+y}.$$

This is equivalent to $2(x+y)^2 = 9xy$ or

$$2x^2 - 5xy + 2y^2 = (2x - y)(x - 2y) = 0.$$

Therefore, $2\log_c a = \log_c b$ or $\log_c a = 2\log_c b$, and so $\log_a b = \frac{1}{2}$ or 2. The larger value is 2.

22. (D) If there are c ($c \ge 0$) correct answers and u ($u \ge 0$) unanswered questions and $c + u \le 25$, then the score is $6c + 2.5u$. If c is sufficiently large and u is sufficiently small, the same score will be obtained with $c - 5$ correct answers and $u + 12$ unanswered questions (this requires $c + u \le 18$), and also with $c - 10$ correct answers and $u + 24$ unanswered questions. Note that in the latter case we must have $c \ge 10$ and $c + u \le 11$. Therefore, for there to be three ways to obtain the score $6c + 2.5u$ we can only have $c = 10$ and $u = 0$, or $c = 10$ and $u = 1$, or $c = 11$ and $u = 0$. The three such scores are 60, 62.5, and 66, and their sum is 188.5

23. (A) Let $z = w - 14i$. Then the equation becomes

$$(w - 14i)(w - 13i)(w - 11i) = 2002i.$$

This simplifies to

$$w^3 - 38iw^2 - 479w = 0.$$

The zeros of this cubic are 0 and $19i \pm \sqrt{118}$. Therefore the zeros of the original equation are $-14i$ and $5i \pm \sqrt{118}$, and the zero satisfying the given conditions is $\sqrt{118} + 5i$, so $a = \sqrt{118}$.

24. (B) Let a be the length of $ABCD$'s edges. Denote x, y, z the distances from E to the faces DAB, DBC, DCA respectively, and by u, v, w the distances from E to edges AB, BC, CA, respectively. Then

$$\begin{aligned} V_{ABCD} &= V_{EDAB} + V_{EDBC} + V_{EDCA} \\ &= \frac{1}{3}(K_{DAB} \cdot x + K_{DBC} \cdot y + K_{DCA} \cdot z) \end{aligned}$$

We have $V_{ABCD} = \frac{1}{3}K_{ABC} \cdot h$, where h is the altitude of tetrahedron $ABCD$. Then h is a leg in a right triangle whose other leg is $\frac{1}{3}a\frac{\sqrt{3}}{2} = \frac{a\sqrt{3}}{6}$ and whose hypotenuse is $\frac{a\sqrt{3}}{2}$. Then $h = \sqrt{\frac{3a^2}{4} - \frac{3a^2}{36}} = \frac{a\sqrt{6}}{3}$. Since ABC, DAB, DBC, and DCA are all equilateral triangles, $K_{ABC} = K_{DAB} = K_{DBC} = K_{DCA}$, so $s = x + y + z = h = \frac{a\sqrt{6}}{3}$. Similarly, in triangle ABC, $K_{ABC} = K_{EAB} + K_{EBC} + K_{ECA} = \frac{1}{2}(AB \cdot u + BC \cdot v + CA \cdot w)$. So $\frac{a^2\sqrt{3}}{4} = \frac{a}{2}(u + v + w)$. Hence, $S = u + v + w = \frac{a\sqrt{3}}{2}$ and $\frac{s}{S} = \frac{\frac{a\sqrt{6}}{3}}{\frac{a\sqrt{3}}{2}} = \frac{2\sqrt{2}}{3}$.

25. (C) Squaring each equation and adding gives

$$(\sin^2 a + \cos^2 a) + (\sin^2 b + \cos^2 b) + 2(\cos a \cos b + \sin a \sin b) = 2$$

or

$$\cos(a - b) = 0.$$

Multiplying the two equations together gives

$$(\sin a \cos b + \sin b \cos a) + (\sin a \cos a + \sin b \cos b) = \frac{\sqrt{3}}{2}$$

or

$$\sin(a + b) + \sin(a + b)\cos(a - b) = \frac{\sqrt{3}}{2}.$$

Substituting $\cos(a - b) = 0$, we have $\sin(a + b) = \frac{\sqrt{3}}{2}$.