January 2002

Contest P

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions
Presented by the Akamai Foundation

AMC 10

53rd Annual American Mathematics Contest **10**

This test and the matching AMC 12 P were developed for the use of a group of Taiwan schools, in early January of 2002. When Taiwan had taken the contests, the AMC released the questions here as a set of practice questions for the 2002 AMC 10 and AMC 12 contests.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. The ratio $\frac{(2^4)^8}{(4^8)^2}$ equals

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** 8

2. The sum of eleven consecutive integers is 2002. What is the smallest of these integers?

(A) 175 **(B)** 177 **(C)** 179 **(D)** 180 **(E)** 181

3. Mary typed a six-digit number, but the two 1s she typed didn't show. What appeared was 2002. How many different six-digit numbers could she have typed?

(A) 4 **(B)** 8 **(C)** 10 **(D)** 15 **(E)** 20

4. Which of the following numbers is a perfect square?

(A) $4^4 5^5 6^6$ **(B)** $4^4 5^6 6^5$ **(C)** $4^5 5^4 6^6$ **(D)** $4^6 5^4 6^5$ **(E)** $4^6 5^5 6^4$

5. Let $(a_n)_{n \geq 1}$ be a sequence such that $a_1 = 1$ and $3a_{n+1} - 3a_n = 1$ for all $n \geq 1$. Find a_{2002}.

(A) 666 **(B)** 667 **(C)** 668 **(D)** 669 **(E)** 670

6. The perimeter of a rectangle is 100 and its diagonal has length x. What is the area of this rectangle?

(A) $625 - x^2$ **(B)** $625 - \frac{x^2}{2}$ **(C)** $1250 - x^2$

(D) $1250 - \frac{x^2}{2}$ **(E)** $2500 - \frac{x^2}{2}$

7. The dimensions of a rectangular box in inches are all positive integers and the volume of the box is 2002 in^3. Find the minimum possible sum in inches of the three dimensions.

(A) 36 **(B)** 38 **(C)** 42 **(D)** 44 **(E)** 92

8. How many ordered triples of positive integers (x, y, z) satisfy $(x^y)^z = 64$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

9. The function f is given by the table

x	1	2	3	4	5
$f(x)$	4	1	3	5	2

If $u_0 = 4$ and $u_{n+1} = f(u_n)$ for $n \geq 0$, find u_{2002}.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

10. Let a and b be distinct real numbers for which

$$\frac{a}{b} + \frac{a+10b}{b+10a} = 2.$$

Find $\frac{a}{b}$.

(A) 0.6 **(B)** 0.7 **(C)** 0.8 **(D)** 0.9 **(E)** 1

11. Let $P(x) = kx^3 + 2k^2x^2 + k^3$. Find the sum of all real numbers k for which $x - 2$ is a factor of $P(x)$.

(A) -8 **(B)** -4 **(C)** 0 **(D)** 4 **(E)** 8

12. For $f_n(x) = x^n$ and $a \neq 1$ consider

I. $(f_{11}(a)f_{13}(a))^{14}$
II. $f_{11}(a)f_{13}(a)f_{14}(a)$
III. $(f_{11}(f_{13}(a)))^{14}$
IV. $f_{11}(f_{13}(f_{14}(a)))$

Which of these equal $f_{2002}(a)$?

(A) I and II only **(B)** II and III only **(C)** III and IV only
(D) II, III, and IV only **(E)** all of them

13. Participation in the local soccer league this year is 10% higher than last year. The number of males increased by 5% and the number of females increased by 20%. What fraction of the soccer league is now female?

(A) $\frac{1}{3}$ **(B)** $\frac{4}{11}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{4}{9}$ **(E)** $\frac{1}{2}$

14. The vertex E of square $EFGH$ is at the center of square $ABCD$. The length of a side of $ABCD$ is 1 and the length of a side of $EFGH$ is 2. Side EF intersects CD at I and EH intersects AD at J. If angle $EID = 60^\circ$, the area of quadrilateral $EIDJ$ is

(A) $\frac{1}{4}$ **(B)** $\frac{\sqrt{3}}{6}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{\sqrt{2}}{4}$ **(E)** $\frac{\sqrt{3}}{2}$

15. What is the smallest integer n for which any subset of $\{1, 2, 3, \ldots, 20\}$ of size n must contain two numbers that differ by 8?

(A) 2 **(B)** 8 **(C)** 12 **(D)** 13 **(E)** 15

16. Two walls and the ceiling of a room meet at right angles at point P. A fly is in the air one meter from one wall, eight meters from the other wall, and 9 meters from point P. How many meters is the fly from the ceiling?

(A) $\sqrt{13}$ **(B)** $\sqrt{14}$ **(C)** $\sqrt{15}$ **(D)** 4 **(E)** $\sqrt{17}$

17. There are 1001 red marbles and 1001 black marbles in a box. Let P_s be the probability that two marbles drawn at random from the box are the same color, and let P_d be the probability that they are different colors. Find $|P_s - P_d|$.

(A) 0 **(B)** $\frac{1}{2002}$ **(C)** $\frac{1}{2001}$ **(D)** $\frac{2}{2001}$ **(E)** $\frac{1}{1000}$

18. For how many positive integers n is $n^3 - 8n^2 + 20n - 13$ a prime number?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** more than 4

19. If a, b, c are real numbers such that $a^2 + 2b = 7$, $b^2 + 4c = -7$, and $c^2 + 6a = -14$, find $a^2 + b^2 + c^2$.

(A) 14 **(B)** 21 **(C)** 28 **(D)** 35 **(E)** 49

20. How many three-digit numbers have at least one 2 and at least one 3?

(A) 52 **(B)** 54 **(C)** 56 **(D)** 58 **(E)** 60

21. Let f be a real-valued function such that

$$f(x) + 2f\left(\frac{2002}{x}\right) = 3x$$

for all $x > 0$. Find $f(2)$.

(A) 1000 **(B)** 2000 **(C)** 3000 **(D)** 4000 **(E)** 6000

22. In how many zeros does the number $\frac{2002!}{(1001!)^2}$ end?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 200 **(E)** 400

23. Let

$$a = \frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + \cdots + \frac{1001^2}{2001}$$

and

$$b = \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \cdots + \frac{1001^2}{2003}.$$

Find the integer closest to $a - b$.

(A) 500 **(B)** 501 **(C)** 999 **(D)** 1000 **(E)** 1001

24. What is the maximum value of n for which there is a set of distinct positive integers $k_1, k_2, \ldots, k_n$ for which

$$k_1^2 + k_2^2 + \ldots + k_n^2 = 2002.$$

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

25. Under the new AMC 10, 12 scoring method, 6 points are given for each correct answer, 2.5 points are given for each unanswered question, and no points are given for an incorrect answer. Some of the possible scores between 0 and 150 can be obtained in only one way, for example, the only way to obtain a score of 146.5 is to have 24 correct answers and one unanswered question. Some scores can be obtained in exactly two ways; for example, a score of 104.5 can be obtained with 17 correct answers, 1 unanswered question, and 7 incorrect, and also with 12 correct answers and 13 unanswered questions. There are three scores that can be obtained in exactly three ways. What is their sum?

(A) 175 **(B)** 179.5 **(C)** 182 **(D)** 188.5 **(E)** 201