The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

3rd Annual American Mathematics Contest 10

AMC 10 - Contest P

Solutions Pamphlet

January 2002

The “P” set of contests were originally developed for a group of schools in Taiwan, to be taken in early January of 2002. Once the contests were taken, the AMC office released the questions here for use as a practice (“P”) set of questions for the 2002 AMC 10 and AMC 12.

This Pamphlet gives at least one solution for each problem on this year’s contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

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1. (C) Since $(2^4)^8 = 2^{32}$ and $(4^8)^2 = 4^{16} = 2^{32}$, we have

$$\frac{(2^4)^8}{(4^8)^2} = \frac{2^{32}}{2^{32}} = 1.$$

2. (B) Let n be the smallest of the eleven integers. Then

$$\begin{aligned} 2002 &= n + (n+1) + (n+2) + \ldots + (n+10) \\ &= 11n + (1+2+3+\ldots+10) \\ &= 11n + (10)(11)/2 \\ &= 11n + 55. \end{aligned}$$

Solving $11n + 55 = 2002$ yields $n = 177$.

3. (D) Imagine six spaces to hold the six digits. We can place the two 1s in two of the spaces in $\binom{6}{2} = \frac{6!}{2!4!} = 15$ ways. (The digits 2, 0, 0, and 2 can then occupy the remaining four spaces.)

4. (C) For $4^x 5^y 6^z = 2^{2x+z} 3^z 5^y$ both to be a perfect square, the exponent on each prime must be even. That is, y and z must be even. Only choice (C) satisfies this condition.

5. (C) Since

$$\begin{aligned} a_{n+1} &= a_n + \frac{1}{3} \\ &= a_{n-1} + \frac{2}{3} \\ &= a_{n-2} + \frac{3}{3} \\ &\vdots \\ &= a_1 + \frac{n}{3} \end{aligned}$$

We see that $a_{2002} = 1 + \frac{2001}{3} = 668$.

6. (D) Let the sides of the rectangle be l and w. We are given that $l + w = 50$ and that $l^2 + w^2 = x^2$. Now observe that

$$\begin{aligned} 2lw &= (l+w)^2 - (l^2 + w^2) \\ &= 2500 - x^2 \end{aligned}$$

and hence the area of the rectangle is $lw = \frac{2500 - x^2}{2} = 1250 - \frac{x^2}{2}$.

7. (B) Let a, b, c be the dimensions of the box, $a \leq b \leq c$. Since $abc = 2002 = 2\cdot 7\cdot 11\cdot 13$, the only possible triples (a, b, c) are (1,1,2002), (1,2,1001), (1,7,286), (1,11,182), (1,13,154), (1,14,143), (1,22,91), (1,26,77), (2,7,143), (2,11,91), (2,13,77), (7,11,26), (7,13,22), and (11,13,14). Among these, the last triple gives the minimum sum, 38.

8. (E) Because $64 = 2^6$, we see that 64 is a sixth power, a cube, a square, as well as a first power. Therefore, $z = 6, 3, 2$, or 1. For $z = 6$ we have only $(x, y) = (2, 1)$; for $z = 3$ we have $(x, y) = (4, 1)$, and $(2, 2)$; for $z = 2$ we have $(x, y) = (8, 1)$ and $(2, 3)$; and for $z = 1$ we have $(x, y) = (64, 1)$, $(8, 2)$, and $(2, 6)$. There are nine solutions in all.

9. (B) Checking the first few values, we find

$$\begin{aligned} u_0 &= 4 \\ u_1 &= f(4) = 5 \\ u_2 &= f(5) = 2 \\ u_3 &= f(2) = 1 \\ u_4 &= f(1) = 4 \\ u_5 &= f(4) = 5 \\ u_6 &= f(5) = 2. \end{aligned}$$

In general, we see that $u_{4k+j} = u_j$, where k is any integer greater than or equal to zero. Hence, $u_{2002} = u_{4\cdot 500+2} = u_2 = 2$.

10. (C) Multiplying both sides of the given equation by $b(b + 10a)$ yields

$$2ab + 10a^2 + 10b^2 = 2b^2 + 20ab$$

which is equivalent to $(a - b)(5a - 4b) = 0$. Since $a \neq b$, we have $5a = 4b$, or $\frac{a}{b} = 0.8$.

11. (A) The expression $x-2$ is a factor of $P(x)$ if and only if $P(2) = 0 = 8k+8k^2+k^3 = k(k^2 + 8k + 8)$. The values of k are therefore 0 and $\frac{-8\pm\sqrt{32}}{2}$ and the sum of these real numbers is -8.

12. (C) Evaluating the four values we find that

$$\begin{aligned} (f_{11}(a)f_{13}(a))^{14} &= (a^{24})^{14} = a^{336} \\ f_{11}(a)f_{13}(a)f_{14}(a) &= a^{38} \\ (f_{11}(f_{13}(a)))^{14} &= (a^{13\cdot 11})^{14} = a^{2002} \\ f_{11}(f_{13}(f_{14}(a))) &= (a^{14\cdot 13})^{11} = a^{2002} \end{aligned}$$

Hence the answer is (C).

13. (B) If m and w are the current numbers of men and women, respectively, then we have

$$\frac{m}{1.05} + \frac{w}{1.20} = \frac{m+w}{1.10}$$

or

$$\frac{m}{w} \cdot \left(\frac{1}{1.05} - \frac{1}{1.10} \right) = \frac{1}{1.10} - \frac{1}{1.20}.$$

It follows that $\frac{m}{w} = \frac{7}{4}$ so that $\frac{m+w}{w} = \frac{11}{4}$, and $\frac{w}{m+w} = \frac{4}{11}$.

14. (A) In fact, regradless of the size of angle EID, the area of quadrilateral $EIDJ$ will always be $\frac{1}{4}$. Let K be the foot of the perpendicular from E to CD and let L be the foot of the perpendicular from E to AD. Then, because right triangle EKI is congruent to right triangle ELJ, the area of quadrilateral $EIDJ$ equals the area of square $EKDL$ which equals $\frac{1}{4}$.

15. (D) Consider the set of eight disjoint pairs, {(1,9), (2,10), (3,11), (4,12), (5,13), (6,14), (7,15), (8,16)}. Because only four members of $\{1, 2, 3, \ldots, 20\}$ are not members of these pairs, any subset of size 13 must contain both members of at least one of these pairs and thus must contain two numbers that differ by 8. The answer is (D) because the twelve element subset {1, 2, 3, 4, 5, 6, 7, 8, 17, 18, 19, 20} contains no two numbers differing by 8.

16. (D) Let the fly be x meters from the ceiling. Then the fly and point P determine a major diagonal of the rectangular parallelepiped having dimensions 1, 8, and x. Therefore, $1^2 + 8^2 + x^2 = 9^2$, and it follows that $x = 4$.

17. (C) There are $\binom{2002}{2} = \frac{2002 \cdot 2001}{2 \cdot 1} = 1001 \cdot 2001$ possible pairs that can be drawn. There are 1001^2 pairs of different colored marbles, so $P_d = \frac{1001^2}{1001 \cdot 2001}$. Therefore, $P_s = 1 - P_d = \frac{1000}{2001}$, and $|P_s - P_d| = \frac{1}{2001}$.

18. (C) Because $n^3 - 8n^2 + 20n - 13 = (n-1)(n^2 - 7n + 13)$, for the value to be prime one factor must equal 1 and the other factor must be prime. For $n - 1 = 1$ we must have $n = 2$, and in this case the other factor is the prime 3. So $n = 2$ is a solution. For $n^2 - 7n + 13 = 1$, we have $n^2 - 7n + 12 = 0 = (n-4)(n-3)$, so we must have $n = 3$ or 4, and in each case the other factor is prime (2 and 3, respectively). Therefore $n^3 - 8n^2 + 20n - 13$ is a prime for three positive integer values of n.

19. (A) Adding the three equations we obtain

$$(a^2 + 6a) + (b^2 + 2b) + (c^2 + 4c) = -14,$$

which is equivalent to

$$(a+3)^2 + (b+1)^2 + (c+2)^2 = 0.$$

Therefore $a = -3$, $b = -1$, $c = -2$, and $a^2 + b^2 + c^2 = 14$.

20. (A) Let U be the set of all three-digit numbers, let S be the set of three-digit numbers that contain no 2s, and let T be the set of three digit numbers that contain no 3s. Then $S \cap T$ is the set of three-digit numbers containing neither a 2 nor a 3 and $U - (S \cup T)$ is the set of three-digit numbers containing at least one 2 and at least one 3. We have $|U| = 900$, $|S| = |T| = 8 \cdot 9^2$, and $|S \cap T| = 7 \cdot 8^2$. Therefore $|S \cup T| = |S| + |T| - |S \cap T| = 848$, and $|U - (S \cup T)| = 52$.

21. (B) Setting $x = 2$, and then $x = 1001$, we have $f(2) + 2f(1001) = 6$ and $f(1001) + 2f(2) = 3003$. Subtracting the first equation from twice the second equation we obtain $3f(2) = 6000$, so $f(2) = 2000$. Note that $f(x) = \frac{4004}{x} - x$ is such a function.

22. (B) The number of zeros at the end of $n!$ is the largest power of 10 that is a factor of $n!$. It is also the largest power of 5 that divides $n!$. In general, the largest power of the prime p in the prime factorization of $n!$ is

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots .$$

Therefore the largest power of 5 that is a factor of $\frac{2002!}{(1001!)^2}$ is

$$\begin{aligned}&\left\lfloor \frac{2002}{5} \right\rfloor + \left\lfloor \frac{2002}{5^2} \right\rfloor + \left\lfloor \frac{2002}{5^3} \right\rfloor + \left\lfloor \frac{2002}{5^4} \right\rfloor \\ &\quad - 2\left(\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{5^2} \right\rfloor + \left\lfloor \frac{1000}{5^3} \right\rfloor + \left\lfloor \frac{1000}{5^4} \right\rfloor \right) = (400 + 80 + 16 + 3) \\ &\qquad\qquad - 2(200 + 40 + 8 + 1) \\ &\qquad = 1.\end{aligned}$$

23. (B) We have

$$a - b = \frac{1^2}{1} + \frac{2^2 - 1^2}{3} + \frac{3^2 - 2^2}{5} + \frac{4^2 - 3^2}{7} + \ldots + \frac{1001^2 - 1000^2}{2001} - \frac{1001^2}{2003}.$$

Since $\frac{(k+1)^2 - k^2}{2k+1} = \frac{2k+1}{2k+1} = 1$ for all $k \geq 0$, it follows that

$$\begin{aligned}a - b &= \underbrace{1 + 1 + \cdots + 1}_{1001 \text{ times}} - \frac{1001^2}{2003} \\ &= 1001 - \frac{1001^2}{2003} \\ &\approx 1001 - 500.25 \approx 500.75.\end{aligned}$$

So the closest integer is 501.

24. (D) Since $1^2 + 2^2 + 3^2 + \ldots + 18^2 > 2002$, it follows that $n \leq 17$. Then note that $1^2 + 2^2 + 3^2 + \ldots + 19^2 - 18^2 - 12^2 = 2002$, hence $n = 17$.

25. (D) If there are c ($c \geq 0$) correct answers and u ($u \geq 0$) unanswered questions and $c + u \leq 25$, then the score is $6c + 2.5u$. If c is sufficiently large and u is sufficiently small, the same score will be obtained with $c - 5$ correct answers and $u + 12$ unanswered questions (this requires $c + u \leq 18$), and also with $c - 10$ correct answers and $u + 24$ unanswered questions. Note that in the latter case we must have $c \geq 10$ and $c + u \leq 11$. Therefore, for there to be three ways to obtain the score $6c + 2.5u$ we can only have $c = 10$ and $u = 0$, or $c = 10$ and $u = 1$, or $c = 11$ and $u = 0$. The three such scores are 60, 62.5, and 66, and their sum is 188.5