30^{th} International Mathematical Olympiad

Braunschweig, Germany

Day I

1. Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets A_i ($i = 1, 2, \ldots, 117$) such that:

   (i) Each A_i contains 17 elements;

   (ii) The sum of all the elements in each A_i is the same.

2. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

   (i) The area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$.

   (ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

3. Let n and k be positive integers and let S be a set of n points in the plane such that

   (i) No three points of S are collinear, and

   (ii) For any point P of S there are at least k points of S equidistant from P.

   Prove that:

$$k < \frac{1}{2} + \sqrt{2n}.$$

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Day II

4. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

5. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

6. A permutation $(x_1, x_2, \ldots, x_m)$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property P than without.