## $21^{\text {st }}$ USA Mathematical Olympiad

April 30, 1992
Time Limit: $3 \frac{1}{2}$ hours

1. Find, as a function of $n$, the sum of the digits of

$$
9 \times 99 \times 9999 \times \cdots \times\left(10^{2^{n}}-1\right)
$$

where each factor has twice as many digits as the previous one.
2. Prove

$$
\frac{1}{\cos 0^{\circ} \cos 1^{\circ}}+\frac{1}{\cos 1^{\circ} \cos 2^{\circ}}+\cdots+\frac{1}{\cos 88^{\circ} \cos 89^{\circ}}=\frac{\cos 1^{\circ}}{\sin ^{2} 1^{\circ}}
$$

3. For a nonempty set $S$ of integers, let $\sigma(S)$ be the sum of the elements of $S$. Suppose that $A=\left\{a_{1}, a_{2}, \ldots, a_{11}\right\}$ is a set of positive integers with $a_{1}<a_{2}<\cdots<a_{11}$ and that, for each positive integer $n \leq 1500$, there is a subset $S$ of $A$ for which $\sigma(S)=n$. What is the smallest possible value of $a_{10}$ ?
4. Chords $\overline{A A^{\prime}}, \overline{B B^{\prime}}, \overline{C C^{\prime}}$ of a sphere meet at an interior point $P$ but are not contained in a plane. The sphere through $A, B, C, P$ is tangent to the sphere through $A^{\prime}, B^{\prime}, C^{\prime}, P$. Prove that $A A^{\prime}=B B^{\prime}=C C^{\prime}$.
5. Let $P(z)$ be a polynomial with complex coefficients which is of degree 1992 and has distinct zeros. Prove that there exist complex numbers $a_{1}, a_{2}, \ldots, a_{1992}$ such that $P(z)$ divides the polynomial

$$
\left(\cdots\left(\left(z-a_{1}\right)^{2}-a_{2}\right)^{2} \cdots-a_{1991}\right)^{2}-a_{1992}
$$

