# XIV. Classroom Accessories

## AMC 10 Student Practice Questions

You will find these and additional problems for the AMC 10 and AMC 12 on AMC's web site: http://www.unl.edu/amc, available from the 2007 AMC 10/12 Teacher Manual directory, (http://www.unl.edu/amc/d-publication/d1-pubarchive/2003-4pub/04tm12/04amc1012tm.html) or from our Problems page archives (http://www.unl.edu/amc/a-activities/a7-problems/problem81012archive.html).

The ratio of Mary's age to Alice's age is $3:5$. Alice is 30 years old. How many years old is Mary?

**(A)** 15 **(B)** 18 **(C)** 20 **(D)** 24 **(E)** 50

**2006 AMC 10 A, Problem #3—**
**2006 AMC 12 A, Problem #3— "Put the ratio of their ages into a fraction."**

**Solution (B)** Mary is $(3/5)(30) = 18$ years old.

**Difficulty:** Easy

**NCTM Standard:** Number and Operations Standard: understand meanings of operations and how they relate to one another

**Mathworld.com Classification:** Number Theory > Arithmetic > Fractions > Ratio

A player pays \$5 to play a game. A die is rolled. If the number on the die is odd, the game is lost. If the number on the die is even, the die is rolled again. In this case the player wins if the second number matches the first and loses otherwise. How much should the player win if the game is fair? (In a fair game the probability of winning times the amount won is what the player should pay.)

**(A)** \$12 **(B)** \$30 **(C)** \$50 **(D)** \$60 **(E)** \$100

## 2006 AMC 10 A, Problem #13— "What is the probability that the player will win?"

**Solution (D)** Let $x$ represent the amount the player wins if the game is fair. The chance of an even number is $1/2$, and the chance of matching this number on the second roll is $1/6$. So the probability of winning is $(1/2)(1/6) = 1/12$. Therefore $(1/12)x = \$5$ and $x = \$60$.

**Difficulty:** Medium
**NCTM Standard:** Data Analysis and Probability Standard: understand and apply basic concepts of probability
**Mathworld.com Classification:** Probability and Statistics > Probability > Probability

Leap Day, February 29, 2004, occurred on a Sunday. On what day of the week will Leap Day, February 29, 2020, occur?

**(A)** Tuesday **(B)** Wednesday **(C)** Thursday **(D)** Friday **(E)** Saturday

**2006 AMC 10 B, Problem #16—**
**"In the years from 2004 through 2020, Each Leap Day occurs $3 \cdot 365 + 366 = 1461$ days after the preceding Leap Day."**

**Solution (E)** In the years from 2004 through 2020, Each Leap Day occurs $3 \cdot 365 + 366 = 1461$ days after the preceding Leap Day. When 1461 is divided by 7 the remainder is 5. So the day of the week advances 5 days for each 4-year cycle. In the four cycles from 2004 to 2020, the Leap Day will advance 20 days. So Leap Day in 2020 will occur one day of the week earlier than in 2004, that is, on a Saturday.

**Difficulty:** Hard

**NCTM Standard:** Measurement Standard: understand measurable attributes of objects and the units, systems, and processes of measurement

**Mathworld.com Classification:** Number Theory > Congruences > Modulus

Let $a$ and $b$ be the roots of the equation $x^2-mx+2=0$. Suppose that $a+(1/b)$ and $b+(1/a)$ are the roots of the equation $x^2-px+q=0$. What is $q$?

**(A)** $\frac{5}{2}$ **(B)** $\frac{7}{2}$ **(C)** 4 **(D)** $\frac{9}{2}$ **(E)** 8

## 2006 AMC 10 B, Problem #14— "Write $x^2-mx+2=0$ as $(x-a)(x-b)$."

**Solution (D)** Since $a$ and $b$ are roots of $x^2-mx+2=0$, we have

$$x^2-mx+2=(x-a)(x-b) \quad \text{and} \quad ab=2.$$

In a similar manner, the constant term of $x^2-px+q$ is the product of $a+(1/b)$ and $b+(1/a)$, so

$$q=\left(a+\frac{1}{b}\right)\left(b+\frac{1}{a}\right)=ab+1+1+\frac{1}{ab}=\frac{9}{2}.$$

**Difficulty:** Hard
**NCTM Standard:** Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols
**Mathworld.com Classification:** Calculus and Analysis > Roots > Root

Oscar buys 13 pencils and 3 erasers for \$1.00. A pencil costs more than an eraser, and both items cost a whole number of cents. What is the total cost, in cents, of one pencil and one eraser?

**(A)** 10 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 20

**2006 AMC 12 A, Problem #9—**
**"Let $p$ be the cost (in cents) of a pencil, and let $s$ be the cost (in cents) of a set of one pencil and one eraser."**

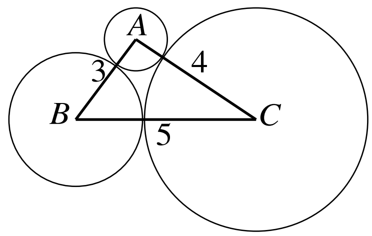
**Solution (A)** Let $p$ be the cost (in cents) of a pencil, and let $s$ be the cost (in cents) of a set of one pencil and one eraser. Because Oscar buys 3 sets and 10 extra pencils for \$1.00, we have

$$3s + 10p = 100.$$

Thus $3s$ is a multiple of 10 that is less than 100, so $s$ is 10, 20, or 30. The corresponding values of $p$ are 7, 4, and 1. Since the cost of a pencil is more than half the cost of the set, the only possibility is $s = 10$.

**Difficulty:** Easy
**NCTM Standard:** Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols
**Mathworld.com Classification:** Algebra > Algebraic Equations > Linear Equation

The vertices of a $3-4-5$ right triangle are the centers of three mutually externally tangent circles, as shown. What is the sum of the areas of these circles?




**(A)** $12\pi$ **(B)** $\frac{25\pi}{2}$ **(C)** $13\pi$ **(D)** $\frac{27\pi}{2}$ **(E)** $14\pi$

## 2006 AMC 12 A, Problem #13— "Label the radii. What do we know about the right triangle?"

**Solution (E)** Let $r$, $s$, and $t$ be the radii of the circles centered at $A$, $B$, and $C$, respectively. Then $r+s=3$, $r+t=4$, and $s+t=5$, from which $r=1$, $s=2$, and $t=3$. Thus the sum of the areas of the circles is

$$\pi(1^2+2^2+3^2)=14\pi.$$

**Difficulty:** Medium
**NCTM Standard:** Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
**Mathworld.com Classification:** Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Right Triangle

An object in the plane moves from one lattice point to another. At each step, the object may move one unit to the right, one unit to the left, one unit up, or one unit down. If the object starts at the origin and takes a ten-step path, how many different points could be the final point?

**(A)** 120 **(B)** 121 **(C)** 221 **(D)** 230 **(E)** 231

**2006 AMC 12 B, Problem #18**— "Each step changes either the $x$-coordinate or the $y$-coordinate of the object by 1."

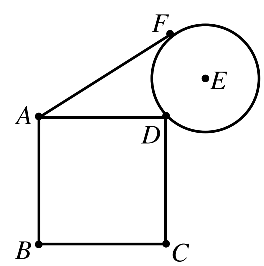
**Solution (B)** Each step changes either the $x$-coordinate or the $y$-coordinate of the object by 1. Thus if the object's final point is $(a, b)$, then $a + b$ is even and $|a| + |b| \leq 10$. Conversely, suppose that $(a, b)$ is a lattice point with $|a| + |b| = 2k \leq 10$. One ten-step path that ends at $(a, b)$ begins with $|a|$ horizontal steps, to the right if $a \geq 0$ and to the left if $a < 0$. It continues with $|b|$ vertical steps, up if $b \geq 0$ and down if $b < 0$. It has then reached $(a, b)$ in $2k$ steps, so it can finish with $5 - k$ steps up and $5 - k$ steps down. Thus the possible final points are the lattice points that have even coordinate sums and lie on or inside the square with vertices $(\pm 10, 0)$ and $(0, \pm 10)$. There are 11 such points on each of the 11 lines $x + y = 2k$, $-5 \leq k \leq 5$, for a total of 121 different points.

**Difficulty:** Hard

**NCTM Standard:** Measurement Standard: apply appropriate techniques, tools, and formulas to determine measurements

**Mathworld.com Classification:** Discrete Mathematics > Point Lattices > Point Lattice

Square $ABCD$ has side length $s$, a circle centered at $E$ has radius $r$, and $r$ and $s$ are both rational. The circle passes through $D$, and $D$ lies on $\overline{BE}$. Point $F$ lies on the circle, on the same side of $\overline{BE}$ as $A$. Segment $AF$ is tangent to the circle, and $AF = \sqrt{9+5\sqrt{2}}$. What is $r/s$?




**(A)** $\frac{1}{2}$ **(B)** $\frac{5}{9}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{5}{3}$ **(E)** $\frac{9}{5}$

## 2006 AMC 12 A, Problem #17— "Look at right triangle $AFE$."

**Solution (B)** Let $B = (0,0)$, $C = (s,0)$, $A = (0,s)$, $D = (s,s)$, and $E = \left(s + \frac{r}{\sqrt{2}}, s + \frac{r}{\sqrt{2}}\right)$. Apply the Pythagorean Theorem to $\triangle AFE$ to obtain

$$r^2 + \left(9 + 5\sqrt{2}\right) = \left(s + \frac{r}{\sqrt{2}}\right)^2 + \left(\frac{r}{\sqrt{2}}\right)^2,$$

from which $9 + 5\sqrt{2} = s^2 + rs\sqrt{2}$. Because $r$ and $s$ are rational, it follows that $s^2 = 9$ and $rs = 5$, so $r/s = 5/9$.

OR

Extend $\overline{AD}$ past $D$ to meet the circle at $G \neq D$. Because $E$ is collinear with $B$ and $D$, $\triangle EDG$ is an isosceles right triangle. Thus $DG = r\sqrt{2}$. By the Power of a Point Theorem,

$$9+5\sqrt{2} = AF^2 = AD \cdot AG = AD \cdot (AD + DG) = s\left(s + r\sqrt{2}\right) = s^2 + rs\sqrt{2}.$$

As in the first solution, conclude that $r/s = 5/9$.

**Difficulty:** Hard

**NCTM Standard:** Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

**Mathworld.com Classification:** Geometry > Plane Geometry > Circles > Circle