XIV. Classroom Accessories

AMC 10 Student Practice Questions

You will find these and additional problems for the AMC 10 and AMC 12 on AMC's web site: http://www.unl.edu/amc, available from the current and previous AMC 10/12 Teacher Manuals, (http://www.unl.edu/amc/e-exams/e6-amc12/archive12.shtml) or from our Problems page archives (http://www.unl.edu/amc/a-activities/a7-problems/problem81012archive.shtml).

The larger of two consecutive odd integers is three times the smaller. What is their sum?

(A) 4 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** 20

2007 AMC 10 A, Problem #4—
2007 AMC 12 A, Problem #3—
"Set up an equation to represent the relation of the two integers."

Solution

Answer (A): Let the smaller of the integers be x. Then the larger is $x+2$. So $x+2=3x$, from which $x=1$. Thus the two integers are 1 and 3, and their sum is 4.

Difficulty: Easy

NCTM Standard: Algebra Standard: represent and analyze mathematical situations and structures using algebraic symbols.

Mathworld.com Classification: Number Theory > Integers > Odd number

The 2007 AMC 10 will be scored by awarding 6 points for each correct response, 0 points for each incorrect response, and 1.5 points for each problem left unanswered. After looking over the 25 problems, Sarah has decided to attempt the first 22 and leave only the last 3 unanswered. How many of the first 22 problems must she solve correctly in order to score at least 100 points?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

2007 AMC 10 B, Problem #6—
2007 AMC 12 B, Problem #5—
"Sarah must earn at least 95.5 points on the first 22 problems."

Solution

Answer (D): Sarah will receive 4.5 points for the three questions she leaves unanswered, so she must earn at least $100 - 4.5 = 95.5$ points on the first 22 problems. Because

$$15 < \frac{95.5}{6} < 16,$$

she must solve at least 16 of the first 22 problems correctly. This would give her a score of 100.5.

Difficulty: Medium-easy
NCTM Standard: Problem Solving Standard: solve problems that arise in mathematics and in other contexts.
Mathworld.com Classification: Calculus and Analysis > Inequalities
Number Theory > Arithmetic > Addition and Subtraction

The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

2007 AMC 10 A, Problem #10—
"Set up equations with two variables, and use the conditions above to eliminate one. "

Solution

Answer (E): Let N represent the number of children in the family and T represent the sum of the ages of all the family members. The average age of the members of the family is 20, and the average age of the members when the 48-year-old father is not included is 16, so

$$20 = \frac{T}{N+2} \quad \text{and} \quad 16 = \frac{T-48}{N+1}.$$

This implies that

$$20N + 40 = T \quad \text{and} \quad 16N + 16 = T - 48,$$

so

$$20N + 40 = 16N + 64.$$

Hence $4N = 24$ and $N = 6$.

Difficulty: Medium-hard

NCTM Standard: Data Analysis and Probability Standard: select and use appropriate statistical methods to analyze data.

Mathworld.com Classification: Algebra > Algebraic Equations > Linear Equation
Calculus and Analysis > Special Functions > Means > Arithmetic Mean

How many pairs of positive integers (a, b) are there such that a and b have no common factors greater than 1 and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer?

(A) 4 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** infinitely many

2007 AMC 10 B, Problem #25—
2007 AMC 12 B, Problem #24—
"Play with the restriction: $\frac{a}{b} + \frac{14b}{9a}$ is an integer."

Solution

Answer (A): Let $u = a/b$. Then the problem is equivalent to finding all positive rational numbers u such that $u + \frac{14}{9u} = k$, for some integer k. This equation is equivalent to $9u^2 - 9uk + 14 = 0$, whose solutions are

$$u = \frac{9k \pm \sqrt{81k^2 - 504}}{18} = \frac{k}{2} \pm \frac{1}{6}\sqrt{9k^2 - 56}.$$

Hence u is rational if and only if $\sqrt{9k^2 - 56}$ is rational, which is true if and only if $9k^2 - 56$ is a perfect square. Suppose that $9k^2 - 56 = s^2$ for some positive integer s. Then $(3k - s)(3k + s) = 56$. The only factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56, so $(3k - s, 3k + s)$ is one of the ordered pairs $(1, 56)$, $(2, 28)$, $(4, 14)$, or $(7, 8)$. The cases $(1, 56)$ and $(7, 8)$ yield no integer solutions. The cases $(2, 28)$ and $(4, 14)$ yield $k = 5$ and $k = 3$, respectively. If $k = 5$, then $u = 1/3$ or $u = 14/3$. If $k = 3$, then $u = 2/3$ or $u = 7/3$. Therefore there are four pairs (a, b) that satisfy the given conditions, namely $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$.

OR

Rewrite the equation $\dfrac{a}{b} + \dfrac{14b}{9a} = k$, in two different forms. First, multiply both sides by b and subtract a to obtain

$$\frac{14b^2}{9a} = bk - a.$$

Because a, b, and k are integers, $14b^2$ must be a multiple of a, and because a and b have no common factors greater than 1, it follows that 14 is divisible by a. Next, multiply both sides of the original equation by $9a$ and subtract $14b$ to obtain

$$\frac{9a^2}{b} = 9ak - 14b.$$

This shows that $9a^2$ is a multiple of b, so 9 must be divisible by b. Thus if (a, b) is a solution, then $b = 1$, 3, or 9, and $a = 1$, 2, 7, or 14. This gives a total of twelve possible solutions (a, b), each of which can be checked quickly. The only such pairs for which

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer are when (a, b) is $(1, 3)$, $(2, 3)$, $(7, 3)$, or $(14, 3)$.

Difficulty: Hard

NCTM Standard: Algebra for Grades 9-12: Analyze change in various contexts .

Mathworld.com Classification: Number Theory > Prime Numbers > Prime Factorization > Factor

A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

2007 AMC 10 A, Problem #21—
2007 AMC 12 A, Problem #11—
"The sphere inscribed within the cube has diameter 2 meters, which is also the length of the diagonal of the cube inscribed in the sphere."

Solution

Answer (C): Since the surface area of the original cube is 24 square meters, each face of the cube has a surface area of $24/6 = 4$ square meters, and the side length of this cube is 2 meters. The sphere inscribed within the cube has diameter 2 meters, which is also the length of the diagonal of the cube inscribed in the sphere. Let l represent the side length of the inscribed cube. Applying the Pythagorean Theorem twice gives

$$l^2 + l^2 + l^2 = 2^2 = 4.$$

Hence each face has surface area

$$l^2 = \frac{4}{3} \text{ square meters.}$$

So the surface area of the inscribed cube is $6 \cdot (4/3) = 8$ square meters.

Difficulty: Hard
NCTM Standard: Geometry Standard: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Squares

Let a, b, c, d, and e be distinct integers such that

$$(6-a)(6-b)(6-c)(6-d)(6-e) = 45.$$

What is $a+b+c+d+e$?

(A) 5 **(B)** 17 **(C)** 25 **(D)** 27 **(E)** 30

2007 AMC 12 A, Problem #14— "Consider $(6-a), (6-b), (6-c), (6-d), (6-e)$ as five distinct integer factors of 45."

Solution

Answer (C): If 45 is expressed as a product of five distinct integer factors, the absolute value of the product of any four is at least $|(-3)(-1)(1)(3)| = 9$, so no factor can have an absolute value greater than 5. Thus the factors of the given expression are five of the integers $\pm1, \pm3$, and ±5. The product of all six of these is $-225 = (-5)(45)$, so the factors are -3, -1, 1, 3, and 5. The corresponding values of a, b, c, d, and e are 9, 7, 5, 3, and 1, and their sum is 25.

Difficulty: Medium-hard

NCTM Standard: Algebra Standard: use mathematical models to represent and understand quantitative relationships.

Mathworld.com Classification: Number Theory > Prime Numbers > Prime Factorization > Factors

Point P is inside equilateral $\triangle ABC$. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB?

(A) 4 **(B)** $3\sqrt{3}$ **(C)** 6 **(D)** $4\sqrt{3}$ **(E)** 9

2007 AMC 10 B, Problem #17— 2007 AMC 12 B, Problem #14— "Area of $\triangle ABC$ equals the sum of areas of $\triangle APB$, $\triangle BPC$, and $\triangle CPA$."

Solution

Answer (D): Let the side length of $\triangle ABC$ be s. Then the areas of $\triangle APB$, $\triangle BPC$, and $\triangle CPA$ are, respectively, $s/2$, s, and $3s/2$. The area of $\triangle ABC$ is the sum of these, which is $3s$. The area of $\triangle ABC$ may also be expressed as $(\sqrt{3}/4)s^2$, so $3s = (\sqrt{3}/4)s^2$. The unique positive solution for s is $4\sqrt{3}$.

Difficulty: Hard

NCTM Standard: Geometry Standard for Grades 9-12: explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Equilateral Triangles

The first 2007 positive integers are each written in base 3. How many of these base-3 representations are palindromes? (A palindrome is a number that reads the same forward and backward.)

(A) 100 **(B)** 101 **(C)** 102 **(D)** 103 **(E)** 104

2007 AMC 12 B, Problem #21—

"$3^6 = 729 < 2007 < 2187 = 3^7$."

Solution

Answer (A): Because $3^6 = 729 < 2007 < 2187 = 3^7$, it is convenient to begin by counting the number of base-3 palindromes with at most 7 digits. There are two palindromes of length 1, namely 1 and 2. There are also two palindromes of length 2, namely 11 and 22. For $n \geq 1$, each palindrome of length $2n+1$ is obtained by inserting one of the digits 0, 1, or 2 immediately after the nth digit in a palindrome of length $2n$. Each palindrome of length $2n+2$ is obtained by similarly inserting one of the strings 00, 11, or 22. Therefore there are 6 palindromes of each of the lengths 3 and 4, 18 of each of the lengths 5 and 6, and 54 of length 7. Because the base-3 representation of 2007 is 2202100, that integer is less than each of the palindromes 2210122, 2211122, 2212122, 2220222, 2221222, and 2222222. Thus the required total is $2+2+6+6+18+18+54-6 = 100$.

Difficulty: Hard

NCTM Standard: Algebra Standard for Grade 9-12: Understand numbers, ways of representing numbers, relationships among numbers, and number systems.

Mathworld.com Classification: Number Theory > Special Numbers > Palindromic Numbers > Palindrome
Number Theory > Arithmetic > Number Bases > Base