47^{th} United States of America Mathematical Olympiad
Day 1. 12:30 PM – 5:00 PM EDT
April 18, 2018

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 1. Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

USAMO 2. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

USAMO 3. For a given integer $n \geq 2$, let $\{a_1, a_2, \ldots, a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by m for every positive integer k.

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