9^{th} United States of America Junior Mathematical Olympiad
Day 1. 12:30 PM – 5:00 PM EDT
April 18, 2018

Note: For any geometry problem whose statement begins with an asterisk ($*$), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 1. For each positive integer n, find the number of n-digit positive integers that satisfy both of the following conditions:

- no two consecutive digits are equal; and
- the last digit is a prime.

USAJMO 2. Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

USAJMO 3. ($*$) Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ = FR$.

9^{th} United States of America Junior Mathematical Olympiad
Day 2. 12:30 PM – 5:00 PM EDT
April 19, 2018

Note: For any geometry problem whose statement begins with an asterisk $(*)$, the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 4. Triangle ABC is inscribed in a circle of radius 2 with $\angle ABC \geq 90^\circ$, and x is a real number satisfying the equation $x^4 + ax^3 + bx^2 + cx + 1 = 0$, where $a = BC, b = CA, c = AB$. Find all possible values of x.

USAJMO 5. Let p be a prime, and let $a_1, a_2, \ldots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \ldots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

USAJMO 6. Karl starts with n cards labeled $1, 2, 3, \ldots, n$ lined up in a random order on his desk. He calls a pair (a, b) of these cards *swapped* if $a > b$ and the card labeled a is to the left of the card labeled b. For instance, in the sequence of cards $3, 1, 4, 2$, there are three swapped pairs of cards, $(3, 1)$, $(3, 2)$, and $(4, 2)$.

He picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on until he has picked up and put back each of the cards $1, 2, \ldots, n$ exactly once in that order. (For example, the process starting at $3, 1, 4, 2$ would be $3, 1, 4, 2 \to 3, 4, 1, 2 \to 2, 3, 4, 1 \to 2, 4, 3, 1 \to 2, 3, 4, 1$.)

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

2018 U.S.A. Junior Mathematical Olympiad Solutions

USAJMO 1.

First solution. Let us call a positive integer *great* if it has no consecutive digits equal and its last digit is prime. Let $p(n)$ denote the number of great n-digit numbers, so the problem is asking us to compute $p(n)$. We claim that $p(n) = 2 \cdot \frac{9^n-(-1)^n}{5}$.

For $n \geq 2$, we say an n-digit number is *good* if it ends in a prime digit and has no two consecutive digits equal among its first $n-1$ digits. Since the first $n-1$ digits and the last digit may be treated independently, the number of good n-digit numbers is $4 \cdot 9^{n-1}$.

Clearly, any great number is good. On the other hand, a good n-digit number fails to be great if its last two digits are equal. By disregarding the last digit, such good-but-not-great numbers are in bijection with great $(n-1)$-digit numbers. Thus, for $n \geq 2$, we have the equation $p(n) = 4 \cdot 9^{n-1} - p(n-1)$. (If $n = 1$, we have $p(1) = 4 \cdot 9^0 = 4$.) Applying this recursively, we find that

$$p(n) = 4 \cdot (9^{n-1} - 9^{n-2} + 9^{n-3} - \cdots + (-1)^{n-2} \cdot 9 + (-1)^{n-1}) = 4 \cdot \frac{9^n - (-1)^n}{10},$$

as claimed.

Second solution. Define great numbers and $p(n)$ as above. For $n \geq 3$, we will count the number of great n-digit numbers by considering two cases:

- If the second digit is 0, then note that the third digit must be non-zero, so the last $n-2$ digits form a great number. Meanwhile, the first digit can be any non-zero digit. Thus, there are $9 \cdot p(n-2)$ great n-digit numbers of this form.
- If the second digit is not 0, then the last $n-1$ digits form a great number, while there are 8 possibilities for the first digit (it can be any non-zero digit not equal to the second digit). This gives $8 \cdot p(n-1)$ great n-digit numbers of this form.

We conclude that $p(n) = 8p(n-1) + 9p(n-2)$ for all $n \geq 3$. This is a second order recurrence, which we may solve by factoring its characteristic polynomial $t^2 - 8t - 9 = (t-9)(t+1)$. The factorization implies that $p(n)$ takes the form $p(n) = A \cdot 9^n + B \cdot (-1)^n$ for some constants A and B. We can solve the system

$$\begin{aligned} 9A - B &= p(1) = 4 \\ 81A + B &= p(2) = 32, \end{aligned}$$

which yields $A = \frac{2}{5}$ and $B = -\frac{2}{5}$, so that

$$p(n) = \frac{2\,(9^n - (-1)^n)}{5}.$$

USAJMO 2.

First solution. Assume without loss of generality that $c = \min(a, b, c)$. By the AM-GM inequality and the given condition, we have

$$\begin{aligned} 4c(a+b+c) + 4ab &\geq 2\sqrt{16 \cdot abc(a+b+c)} \\ &= 2\sqrt{16\left(\frac{a+b+c}{4}\right)^3 (a+b+c)} \\ &= (a+b+c)^2. \end{aligned}$$

Subtracting $2(ab + bc + ca)$ from both sides, this gives

$$2(ab + bc + ca) + 4c^2 \geq a^2 + b^2 + c^2,$$

as desired.

Remark. The equality in the AM-GM step occurs if and only if $c(a + b + c) = ab$. Solving for $a+b+c$ and substituting into the condition $a+b+c = 4\sqrt[3]{abc}$, this implies $8c^2 = ab$. Substituting this back into the equation $c(a + b + c) = ab$, we conclude that

$$c(a + b + c) = 8c^2 \implies a + b = 7c.$$

We then have

$$a - b = \pm\sqrt{(a+b)^2 - 4ab} = \pm\sqrt{49c^2 - 32c^2} = \pm\sqrt{17}c.$$

It follows that $\{2a, 2b\} = \{(7 - \sqrt{17})c, (7 + \sqrt{17})c\}$. Hence, equality holds if and only if (a, b, c) is a permutation of

$$\left((7 - \sqrt{17})r, (7 + \sqrt{17})r, 2r\right)$$

for some positive real number r.

Second solution. Suppose, as above, that $c = \min(a, b, c)$, and write $A = a/c$, $B = b/c$, and $D = A+B$. The given condition becomes $A+B+1 = 4\sqrt[3]{AB}$, or equivalently, $AB = (D+1)^3/64$. In terms of A and B, the problem asks us to prove that

$$2(AB + A + B) + 4 \geq A^2 + B^2 + 1,$$

which can be rearranged as

$$2(A + B) + 3 - (A + B)^2 + 4AB \geq 0.$$

After substituting in D, this inequality becomes

$$2D + 3 - D^2 + (D + 1)^3/16 \geq 0.$$

Since the left-hand side factors as $(D + 1)(D - 7)^2/16$, the inequality always holds.

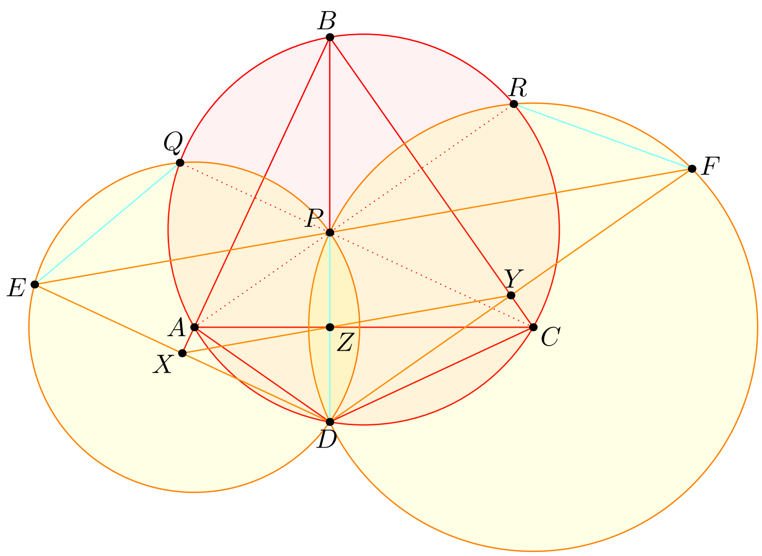
Third solution: Assuming that $c = \min(a, b, c)$ and by adding $2(ab + bc + ca)$ to both sides, our inequality becomes

$$4c(a + b + c) + 4ab \geq (a + b + c)^2.$$

Since both the given condition and the desired claim are homogeneous, we may assume without loss of generality that $a+b+c=8$, so our task is to prove that if $ab=8/c$, then $32c+4ab\geq 64$. This clearly holds, since for any positive real number c we have $32\left(c+\frac{1}{c}\right)\geq 64$.

USAJMO 3.

Let X and Y be the feet of the perpendiculars from D to lines BA and BC, respectively, and let Z be the intersection of lines BD and AC. By Simson's theorem, the points X, Y, Z are collinear. A homothety with ratio 2 about D maps X, Y, Z to E, F, P', respectively, where P' is the orthocenter of $\triangle ABC$. Hence, P' lies on line EF as well as line BD, so $P'=P$.



Suppose now we extend ray $\overrightarrow{CP}$ to meet ω again at Q'. Then line BA is the perpendicular bisector of both $\overline{PQ'}$ and $\overline{DE}$; consequently, $PQ'ED$ is an isosceles trapezoid. In particular, it is cyclic, and so $Q'=Q$. In the same way, R is the second intersection of ray $\overrightarrow{AP}$ with ω.

Now, because of the two isosceles trapezoids we have found, we conclude

$$EQ=PD=FR,$$

as desired.

USAJMO 4.

The given equation can be rewritten as

$$\left(x^2+\frac{ax}{2}\right)^2+\left(b-\frac{a^2+c^2}{4}\right)x^2+\left(\frac{cx}{2}+1\right)^2=0.$$

Noting that we must have $x\neq 0$, the equation holds if and only if

$$b=\frac{a^2+c^2}{4} \quad \text{and} \quad x=-\frac{a}{2}=-\frac{2}{c}.$$

The assumption $\angle ABC \geq 90^\circ$ and the fact that the circle's diameter is 4 imply $a^2 + c^2 \leq b^2 \leq 4b$; but since we saw that $b = (a^2 + c^2)/4$, both of these inequalities are equalities. We conclude that $\angle ABC = 90^\circ$, $b = 4$, $a^2 + c^2 = 16$, and $ac = 4$. These last two equations imply $(a + c)^2 = 16 + 2 \cdot 4 = 24$ and $(a - c)^2 = 16 - 2 \cdot 4 = 8$. Since $a, c > 0$, we have $a + c = 2\sqrt{6}$ and $a - c = \pm 2\sqrt{2}$. Hence the only possible values of $x = -a/2$ are $-\frac{1}{2}(\sqrt{6} + \sqrt{2})$ or $-\frac{1}{2}(\sqrt{6} - \sqrt{2})$. Conversely, these are indeed possible, by having a right triangle with sides $a = \sqrt{6} + \sqrt{2}, b = 4, c = \sqrt{6} - \sqrt{2}$ or $a = \sqrt{6} - \sqrt{2}, b = 4, c = \sqrt{6} + \sqrt{2}$, respectively.

Remark. One can also show that the acute angles of the triangle are 15 degrees and 75 degrees.

USAJMO 5.

The statement is trivial for $p = 2$, so assume $p = 2q + 1$ is odd. Create a $p \times p$ table of numbers, as follows:

$$\begin{array}{cccc}
a_1 + 1 \cdot 0 & a_2 + 2 \cdot 0 & \cdots & a_p + p \cdot 0 \\
a_1 + 1 \cdot 1 & a_2 + 2 \cdot 1 & \cdots & a_p + p \cdot 1 \\
\vdots & \vdots & \ddots & \vdots \\
a_1 + 1 \cdot (p-1) & a_2 + 2 \cdot (p-1) & \cdots & a_p + p \cdot (p-1)
\end{array}$$

Interpret all the numbers above modulo p. Examine two different columns, say columns i and j. We claim they agree (modulo p) in exactly one row. Indeed, $a_i + ik \equiv a_j + jk \pmod{p}$ holds if and only if $(i - j)k \equiv a_j - a_i \pmod{p}$. Since p is prime and $i \not\equiv j \pmod{p}$, this condition holds for a unique value of k (namely, $k \equiv (a_j - a_i)(i - j)^{-1} \pmod{p}$).

Thus, there are $\binom{p}{2} = \frac{p(p-1)}{2} = pq$ pairs of integers that are congruent modulo p and lie in the same row of the table. Since there are only p rows, some row, say $\{a_n + nk\}_n$, must contain at most q such pairs.

We claim that this k satisfies our requirement. Indeed, if we read the p entries in this row one by one, each entry either is distinct from all the previous ones, or is congruent to at least one previous entry and thereby completes a pair. Since the latter case happens at most q times, there must be at least $p - q = (p + 1)/2$ distinct entries (modulo p), completing the proof.

USAJMO 6.

Consider the following alternative procedure: When Karl removes the card labeled 1, before he inserts it, he adds n to its label to make it a card labeled $n + 1$. Then he reinserts the card as in the original procedure. Now, the new arrangement of cards has the same number of swapped pairs as before, since the 1 used to be part of i swapped pairs using the cards to its left, and now the $n + 1$ is part of i swapped pairs using the cards to its right.

By the same argument, if he next removes the card labeled 2 and adds n to its label before reinserting it in its new position, and so on, he ends up with a permutation of $n + 1, n + 2, \ldots, 2n$ that has the same number of swapped pairs as the one he started with. But this permutation clearly corresponds to the ending permutation from Karl's original procedure upon subtracting n from all the labels, and this subtraction doesn't change the number of swapped pairs. This completes the proof.