2019 U.S.A. Junior Mathematical Olympiad Solutions

USAJMO 1. First we show that if the goal can be reached, then ab must be even. Suppose that it is possible to achieve the desired end-state. Each time we perform a move, the sum of the positions of the apples increases by 1. Since the sum starts out as $1 + 2 + \cdots + a$ and ends up as $(b + 1) + (b + 2) + \cdots + (b + a)$, the total number of moves must be

$$[(b + 1) + (b + 2) + \cdots + (b + a)] - [1 + 2 + \cdots + a] = ab.$$

On the other hand, note that when we move an apple from i to $i + 1$ while moving a pear from j to $j - 1$ with $i - j$ even, the number of fruits in even-indexed bowls either goes up by 2 or down by 2. Since the number of fruits in even-indexed bowls is the same at the end of the process as it is at the beginning, the sum of all of these ± 2's must be equal to zero. But the sum of an odd number of terms each equal to ± 2 is twice an odd number and therefore cannot be zero, so we see that the total number of moves must be even. Therefore ab must be even.

Next we show that as long as ab is even, the goal can be reached. Since ab is even, either a or b is itself even. We may assume that a is even (the case of b even is proven symmetrically).

First, if $a = 2$ and $b = 1$, then it is easy to see that the apple initially in bowl 1 and the pear (in bowl 3) can be made to switch places in two moves. Next, consider the case where a is any even number and $b = 1$. We can apply the previous case $a/2$ times, each time making the pear switch places with the apple that is currently 2 bowls to its left, so that we end up with the pear in bowl 1 and apples in bowls $2, 3, \ldots, a + 1$. Finally, for arbitrary even a and arbitrary b, we can apply the previous case b times, moving one pear past the row of a apples at each step, to reach the desired end-state.

Remark: Here is a different proof that ab must be even: At each step, consider the sum of the *squares* of each fruit's position. When we move an apple from i to $i + 1$ while moving a pear from j to $j - 1$, this sum of squares changes by

$$[(i + 1)^2 + (j - 1)^2] - [(i)^2 + (j)^2] = 2(i - j) + 2.$$

Because $i - j$ is even, this quantity is congruent to 2 mod 4. Since the sum of the squares of the fruits' positions needs to be the same at the end of the process as it is at the beginning, the sum of these changes across all moves must be 0. And since the sum of an odd numbers of integers congruent to 2 (mod 4) is congruent to 2 (mod 4) and hence nonzero, the total number of moves must be even.

This problem was proposed by Jim Propp.

USAJMO 2. The answer is (k, k) and $(-k, k)$ for any integer k. To see that these are possible: for $(a, b) = (k, k)$ we take $f(x) = x$ and $g(x) = x + k$ for all x; for $(a, b) = (-k, k)$ we take $f(x) = -x$ and $g(x) = -x + k$.

Now we show that $|a| = |b|$ in any valid pair, so that there are no other solutions. Note that

$$f(x+b) = f(g(f(x))) = f(x) + a. \tag{1}$$

Consequently, if $b = 0$ then $f(x) = f(x) + a$, implying $a = 0$ as well. A similar argument shows that if $a = 0$ then $b = 0$. So we can focus on the case where neither a nor b is zero.

Since $g(f(x)) = x + b$ it follows f is injective, i.e. we never have $f(x) = f(y)$ unless $x = y$.

If $|b| > |a|$, then by the pigeonhole principle, among the $|b|$ numbers $f(1), f(2), \ldots, f(|b|)$ there must be some two, say $f(r)$ and $f(s)$, that are congruent modulo $|a|$; that is, $f(r) - f(s)$ is a multiple of $|a|$. By swapping r and s if needed, we can assume $f(r) - f(s) = k \cdot a$ with $k \geq 0$. Then, applying (1) repeatedly, we find that $f(s + kb) = f(s) + ka = f(r)$. By injectivity, this implies $s + kb = r$. But since r, s are different elements of $\{1, 2, \ldots, |b|\}$, their difference is not divisible by b, so we have a contradiction.

Similarly, $|a| > |b|$ is impossible. So we must have $|a| = |b|$, as claimed.

This problem was proposed by Ankan Bhattacharya.

USAJMO 3. Note that there can only be one point P on $\overline{AB}$ satisfying the given angle condition, since as P moves from A to B, $\angle APD$ decreases while $\angle BPC$ increases. Consequently, if we can show that there is a single point P on $\overline{AB}$ such that $\angle APD = \angle BPC$ and line PE bisects $\overline{CD}$, then it must coincide with the point in the problem statement, and we will be done. We construct such a point as follows.

Since $AD^2 + BC^2 = AB^2$, there exists a point P on $\overline{AB}$ satisfying

$$AD^2 = AP \cdot AB \quad \text{and} \quad BC^2 = BP \cdot BA.$$

Thus $AP/AD = AD/AB$ and $BP/BC = BC/BA$. We then have similar triangles, $\triangle APD \sim \triangle ADB$ and $\triangle BPC \sim \triangle BCA$, from which $\angle APD = \angle ADB = \angle ACB = \angle BPC$.

Now we show that line PE bisects $\overline{CD}$. Define $K = \overline{AC} \cap \overline{PD}$ and $L = \overline{BD} \cap \overline{PC}$.

The quadrilaterals $APLD$ and $BPKC$ are cyclic, because

$$\measuredangle ADL = \measuredangle ACB = \measuredangle BPC = \measuredangle APL$$

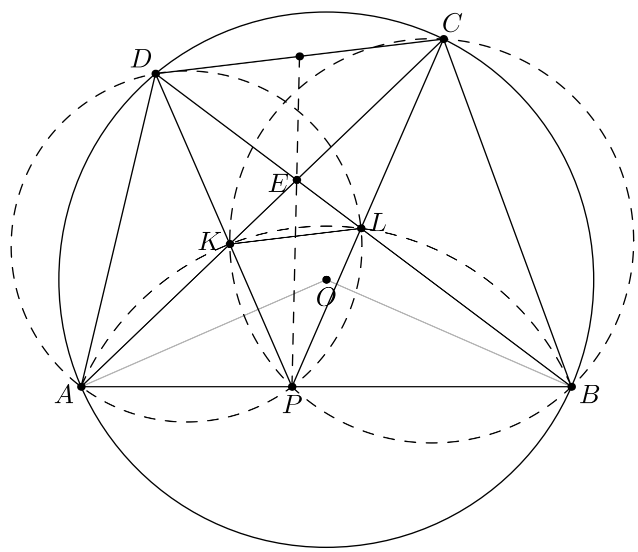
and similarly $\measuredangle KCB = \measuredangle KPB$. (The notation $\measuredangle$ here refers to directed angles taken modulo $180°$.)

Now the quadrilateral $AKLB$ is also cyclic, because

$$\measuredangle AKB = \measuredangle CKB = \measuredangle CPB$$

and similarly $\measuredangle ALB = \measuredangle APD$, and these are equal.

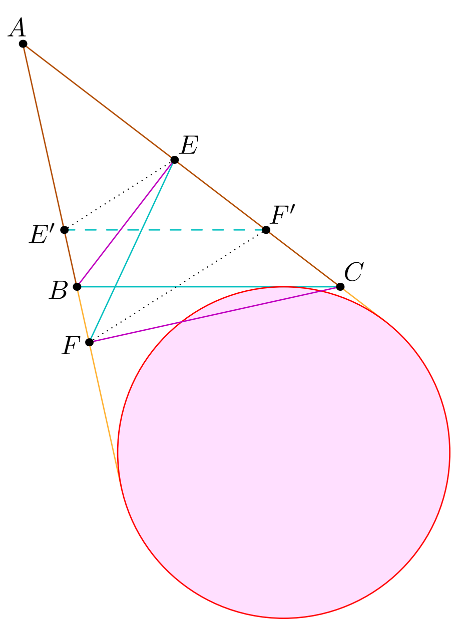
Now the cyclic quadrilaterals imply $\measuredangle KCD = \measuredangle ABD = \measuredangle ABL = \measuredangle AKL = \measuredangle CKL$, from which we conclude $\overline{CD} \parallel \overline{KL}$. Thus $CDKL$ is a trapezoid whose legs intersect at P and whose diagonals



intersect at E. As is well-known (and can be quickly shown using Ceva's theorem), this implies that line PE bisects the bases $\overline{CD}$ and $\overline{KL}$, as desired.

This problem was proposed by Ankan Bhattacharya.

USAJMO 4. The answer is no.



Suppose that line EF is tangent to the A-excircle. Consider the reflection over the bisector of $\angle BAC$. This operation swaps rays AB and AC; suppose E and F are sent to E' and F'. Note that the A-excircle stays fixed under this reflection, so line $E'F'$ must also be tangent to the circle. Since $BEFC$ is cyclic, we obtain $\measuredangle ECB = \measuredangle EFB = \measuredangle EF'E'$, so $\overline{E'F'} \parallel \overline{BC}$. However, as $\overline{EF}$ is a chord in the circle ω with diameter $\overline{BC}$, we have $EF \le BC$.

If $EF < BC$ then $E'F' < BC$ too, so then $\overline{E'F'}$ lies inside $\triangle ABC$ (as in the picture) and line EF cannot be tangent to the A-excircle.

The remaining case is when $EF = BC$. In this case, $\overline{EF}$ is also a diameter of circle ω, so $BECF$ is a rectangle. In particular $\overline{BF} \parallel \overline{CE}$. However, the sides AB and AC of triangle ABC cannot be parallel to each other, so this is a contradiction.

Alternate solution: Define E' and F' as above. Observe that $AE' = AE < AB$, since $\overline{AE}$ is a leg of a right triangle with hypotenuse $\overline{AB}$. Similarly $AF' < AC$. So segment $\overline{E'F'}$ lies inside $\triangle ABC$ and so cannot be tangent to the A-excircle.

This problem was proposed by Ankan Bhattacharya, Zack Chroman, and Anant Mudgal.

USAJMO 5. The answer is $(2n)! \cdot 2^{n^2}$. It may be helpful to view the sets $S_{i,j}$ as being placed in a grid, as shown in Figure 1. We say a choice of sets $S_{i,j}$ is *valid* if it satisfies the two conditions in the problem. In a slight abuse of terminology, we also apply this definition at times when only some of the $(n+1)^2$ total sets are chosen, with the rest left undetermined (in this case, the conditions are ignored when one or more of the sets involved is undetermined).

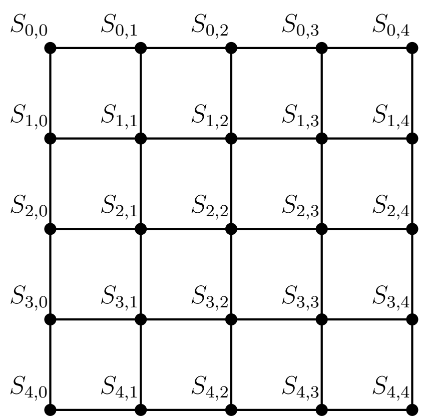


Figure 1: The $S_{i,j}$ arranged in a grid.

Let us define an *initial configuration* to be a valid choice of the sets corresponding to the top row and rightmost column (i.e. sets of the form $S_{0,j}$ and $S_{i,n}$). We first count the number of initial configurations. Since we must have

$$\emptyset = S_{0,0} \subseteq S_{0,1} \subseteq S_{0,2} \subseteq \cdots \subseteq S_{0,n} \subseteq S_{1,n} \subseteq S_{2,n} \subseteq \cdots \subseteq S_{n,n} = \{1, 2, \ldots, 2n\}$$

and recalling that $|S_{i,j}| = i + j$, it follows that the above sequence of sets is obtained by adding different elements of $\{1, 2, \ldots, 2n\}$ one at a time. We may add these $2n$ elements in any order, so the number of initial configurations is $(2n)!$.

Next, for any $0 \le i, j < n$, consider the sets $S_{i,j}$, $S_{i+1,j}$, $S_{i,j+1}$, and $S_{i+1,j+1}$. If they are part of a valid choice, we must have

$$S_{i,j} \subseteq S_{i+1,j+1} \quad \text{and} \quad |S_{i+1,j+1}| = i + j + 2 = |S_{i,j}| + 2,$$

which implies $S_{i+1,j+1} \setminus S_{i,j} = \{x, y\}$ for some distinct $x, y \in \{1, 2, \ldots 2n\}$. Then, $S_{i+1,j}$ and $S_{i,j+1}$ are each either $S_{i,j} \cup \{x\}$ or $S_{i,j} \cup \{y\}$. Let us say the ordered pair (i, j) is *hot* if $S_{i+1,j}$ and $S_{i,j+1}$ are different and *cold* if they are the same. We define a *hot-cold configuration* to consist of a designation of "hot" or "cold" for each of the n^2 ordered pairs (i, j). Clearly, there are 2^{n^2} hot-cold configurations.

Finally, we claim that given any initial configuration and any hot-cold configuration, there is a unique valid choice of sets $S_{i,j}$ for $0 \le i, j \le n$ that agrees with both the initial configuration and the hot-cold configuration. Indeed, we start with the initial configuration of $2n + 1$ sets and choose the remaining sets one by one. We choose them in the following order:

$$\begin{array}{cccc} S_{1,n-1}, & S_{1,n-2}, & \ldots, & S_{1,0}, \\ S_{2,n-1}, & S_{2,n-2}, & \ldots, & S_{2,0}, \\ \vdots & \vdots & \ddots & \vdots \\ S_{n,n-1}, & S_{n,n-2}, & \ldots, & S_{n,0}, \end{array}$$

and we will make sure our choice of sets remains valid at each step. In terms of the grid in Figure 1, this corresponds to going row by row, going right to left in each row.

The above ordering ensures that when we are choosing $S_{i,j}$, the sets $S_{i-1,j}$, $S_{i-1,j+1}$, and $S_{i,j+1}$ have all been chosen already. Based on whether $(i-1, j)$ is required to be hot or cold, we are forced to set $S_{i,j}$ to be $S_{i-1,j} \cup (S_{i,j+1} \setminus S_{i-1,j+1})$ or $S_{i-1,j+1}$, respectively. Moreover, it is straightforward to check that the resulting choice of sets indeed remains valid, because we have ensured that $S_{i-1,j} \subseteq S_{i,j} \subseteq S_{i,j+1}$.

Thus, at the end of the procedure, we arrive at a unique valid choice of all $(n + 1)^2$ of the $S_{i,j}$, establishing the claim. It follows that there are $(2n)! \cdot 2^{n^2}$ valid choices in total, as desired.

This problem was proposed by Ricky Liu.

USAJMO 6. The answer is all (m, n) such that $m + n$ is a power of 2.

First, if $m + n$ is divisible by any prime $p > 2$, we show that any number $\frac{a}{b}$ written on the board will always have $a + b$ divisible by p. Indeed, if $s + t$ and $u + v$ are divisible by p, then the arithmetic mean of $\frac{s}{t}$ and $\frac{u}{v}$ is $\frac{sv+tu}{2tv}$, and we note that

$$sv + tu + 2tv = (s + t)(u + v) + v(s + t) - s(u + v)$$

which is again divisible by p. Since neither t nor v (nor 2) is divisible by p, we see that p still divides the sum of the numerator and denominator after the fraction has been reduced. Similarly, the harmonic mean $\frac{2su}{sv+tu}$ also satisfies the condition.

However, $1 = \frac{1}{1}$, and no prime $p > 2$ divides $1 + 1$, so no such prime can divide $m + n$ if Evan is to ever be able to write 1 on the board. So we need $m + n$ to be a power of 2.

We now show that Evan can fulfill his goal whenever $m + n$ is a power of 2. In fact, he can do this by only using the arithmetic mean. To show this, first notice that since $m + n$ is a power of 2, if he started with the numbers 0 and $m + n$ on the board, by repeatedly taking arithmetic means, he could eventually produce any integer between 0 and $m + n$; in particular, he could obtain the value m. But if $f(x) = cx + d$ is any linear function, the arithmetic mean of $f(x)$ and $f(y)$ is $f\left(\frac{x+y}{2}\right)$, so by replicating the same sequence of steps that gets to m starting from 0 and $m + n$, he can also get to $f(m)$ starting from $f(0)$ and $f(m + n)$. In particular, by taking $c = \frac{n-m}{mn}$ and $d = \frac{m}{n}$, we have $f(0) = \frac{m}{n}$, $f(m + n) = \frac{n}{m}$, and $f(m) = 1$, so by starting from $\frac{m}{n}$ and $\frac{n}{m}$, Evan can eventually reach 1, as needed.

(Note that the harmonic mean operation is never needed.)

This problem was proposed by Yannick Yao.