# Mathematics

## CRAFTY Curriculum Foundations Project
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## Summary

The most important task of the first two years is to move students from a procedural/computational understanding of mathematics to a broad understanding encompassing logical reasoning, generalization, abstraction, and formal proof. This should be achieved as soon as possible in a student's undergraduate career.

The transition should be accompanied by an increased sophistication of student attitudes towards mathematics. For example, students should develop an understanding that they can learn from failure. They should come to understand that solutions are often multi-staged and require significant creativity, time and patience, and that they can gain considerable insight by analyzing whatever went wrong. Courses should instill confidence that mathematics makes sense, is worthwhile, and has intellectual vitality.

## Narrative

Essential themes—fundamental ideas or principles that apply to varied subject matter—should be threaded through different courses. Departments should think about how to provide coherent integration of these themes across courses. Some themes are about mathematics as a whole:

- The nature of mathematical language: basic logic, the difference between a statement and its converse, semantics of quantifiers, the role of definitions.
- The nature of mathematical knowledge: the nature of proof, why theorems are important, why the universality of a statement is important in mathematics.
- Interrelations between concepts, skills, and topics.

Some themes are more specific:

- Concept of function (as distinct from formula).
- Approximation.
- Algorithm.
- Linearity.
- Dimension.

Students should gain proficiency in the following skills in the first two years:

- Understanding and doing standard computations
- Visualization and geometric skills, especially 3-dimensional visualization and solid geometry.
- Translating mathematics into words and translating words (statement of problems) into mathematics.

- Generalization and checking general statements via specific examples and experimentation.
- Recognizing invalid arguments and incorrect answers; knowing when the result/answer is reasonable.
- Analyzing the problem (is it well defined?), attacking the problem (starting without knowing the answer), using all techniques and results available (not quitting after one attempt).
- Stating problems clearly and setting them up (e.g., defining variables, deciding which variable is to be solved for, setting up coordinates, defining notation).
- Communicating mathematics in writing and orally, using precise reasoning and genuine analysis.

There should be a balance between the development of computational skill, conceptual understanding, theoretical reasoning, and applications. Connections between these should be brought out: computational technique can be used to illustrate and develop theory, and applications can be used to give flesh to conceptual understanding. There has been, on average, an overemphasis on developing computational skill in the introductory courses. Similarly, in advanced courses there is often a stress on theoretical reasoning before the necessary conceptual understanding has been achieved. There should be an attempt to phase in logical language starting in the freshman year, rather than a sudden jump in the sophomore or junior year.

Calculus and linear algebra are likely to remain the primary core of the first two years of undergraduate mathematics. We should strive to make such courses reflect the core of mathematical culture: the value and validity of careful reasoning, of precise definition, and close argument. Doing so in an appropriate and effective way is the challenge of the current decade.

Students should have the opportunity to sample a wider variety of topics. Such topics should include mathematics that is of recent origin or current interest, illustrating our discipline as a continually growing and developing body of knowledge, driven by creativity, a passionate search for fundamental structure and interrelationships, and a methodology that is both powerful and intellectually compelling. Students should encounter mathematics in an interesting historical, aesthetic or useful context. Prospective high school teachers need a deepening of their understanding of high school mathematics. Alternatives to calculus and linear algebra should be available in the first two years. Some possibilities are: discrete mathematics, number theory, geometry, and knot theory.

Realism dictates recognition that students are headed in many directions besides graduate school in mathematics: K–12 teaching, graduate school in various other disciplines, industry, business, and government. Many do not yet have clear plans but have a general interest in mathematics. It is unrealistic to suppose that students can always be sorted according to their different goals, but we should prepare students to start making decisions about different tracks that might open up in the last two years.

Similarly, we should recognize our students' great variety of individual, cultural, and educational backgrounds. Students come with vastly different experience, skills, and learning styles. Introductory courses should provide experiences flexible enough to allow every student the opportunity both to reinforce existing strengths and to fill gaps.

## Technology

Prospective mathematics majors should be familiar with appropriate use of technology. They should, in particular, have experience with a computer algebra system during the first two years of undergraduate training. In addition to its obvious value for visualization and for lengthy computations, technology is useful as a diagnostic tool. Instructors can also use technology as a basis for student experiences requiring active thought. In addition, there are creative web applications to support mathematics learning such as:

- JOMA, the Journal of Online Mathematics and its Applications
- ALEKS, the Aleks Corporation

The existence and ubiquity of high-powered computational technology reduces the importance of student mastery of manual computations of an intricate and specialized nature. Since the interactions between computational fluency and broader learning goals are not understood, caution is needed to avoid loss of technique that would impede later mastery of mathematics. This probably depends to some extent on the future plans of the student. However, there is a concern that introduction of technology is often accompanied by a decrease in computational fluency.

It should be observed that graphing calculators have limited uses outside of the mathematics classroom. Though there are exceptions, graphing calculators are little used, professionally or pedagogically, in other disciplines. A decision to use graphing calculators cannot therefore be based solely on unsubstantiated reports of their ubiquity or general usefulness in other domains.

Most instructors are too busy to investigate new technology. However, it is in a department's best interest to acquaint itself with promising new developments. A mathematics department should take some responsibility for helping its instructors understand the different platforms for the teaching of mathematics.

## Instructional Formats and Techniques

For the purpose of this report, we distinguish between instructional *formats* (such as large lecture, small group work, laboratories) and instructional *techniques* (such as quizzes, student presentations, written homework).

Instruction should be viewed as interaction rather than a one-way process. Instructional techniques should be employed that foster discussion, improve the value added by class time, encourage student interaction (either inside or outside the classroom), engage the students in the subject and ignite their curiosity. All instructional techniques must project a positive affirmation of belief in the abilities of our students. We must show that we care and respect our students and that we want them to succeed. Instructional technique is not separate from content. For example, in teaching proofs one should consider the way in which mathematicians work; perhaps sometimes, instead of presenting proofs, one could have students do proofs themselves. It is important to encourage students to take an active role in their own learning (e.g., through reading assignments, discussion of why material is important, standards for homework). Another important habit to cultivate in students is generalization. One should expect students to solve unfamiliar problems; this may require giving them practice and support.

Students can learn from each other; having them work together can also develop a spirit of camaraderie that helps a course succeed. It can also expose them to multiple methods for solving a problem, thus providing a more robust sense of possible approaches. Besides using the technique of in-class group work, one can get students to work together by explicitly organizing them into homework groups, distributing contact information, requiring them to send questions to a list-serve, or making group work a graded component of the course.

Choice of the means of assessment for grading purposes has an effect on learning. Although timed tests have a place in measuring skill development, many aspects of mathematical learning cannot be effectively measured by timed tests. Individual and group projects, take-home tests, on-line quizzes and assignments, in-class and other oral assessments can be effective means to obtain a rounded view of student achievement and to promote learning of mathematics beyond skill acquisition.

In addition to assessments for grading, it is important to have ongoing mechanisms for gathering information about the state of students' knowledge and understanding in order to teach effectively. There are a multitude of instructional techniques that achieve this, such as calling on students to supply steps in a proof or calculation, quick quizzes on each class's material at the end of each class, requiring students to explain their reasoning in writing and asking students to supply questions by email before each class. Individual instructors will have their own preferences and will be more comfortable and more effective with some methods as opposed to others.

Education research, both that specific to mathematics and outside, provides some basic guide to instructional technique (for example, the powerful effect of teacher expectations). Chapter 9 of the NRC report "Adding It Up" provides a guide. (This report is available from the National Academy Press. See `www.nap.edu`. In the draft available on 15 February, the relevant section begins on page 9–18.)

Instructional choices should be made on the basis of effectiveness, not on complacency and comfort. Instructors should be encouraged to examine their preferences and experiment with new techniques. At the same time, instructors should not be forced to use instructional methods that they believe are unsuited to their skills and preferences. Institutions should provide information on different techniques for mathematics instruction, as well as support for experimentation.

## Recruitment

If you want to recruit, teach good courses. We lose a lot of students who come in thinking that they want to be math majors by failing to make real to them the intellectual vitality of mathematics. It is important to assign excellent teachers to the pipeline courses.

Mathematics departments must actively encourage promising students to consider further mathematics courses and ultimately a mathematics major. Instructional techniques that create barriers between students and instructors will hurt recruitment efforts. Instructional techniques must be effective with a broad range of students, not simply with students planning graduate study in mathematics. The profession will pay a dear price if it only encourages the elite to pursue majors in mathematics.

Here are some suggestions for effective student recruitment:

- Put topics and activities into service courses that will attract students to mathematics.
- Present recent developments in mathematics. Students will be excited by material that is interesting, engaging, and intellectually challenging but accessible.
- Use simulations, visualizations and approximations in the presentation of mathematics. Mathematical modeling, which can dramatically illustrate the applicability of mathematics, can draw students into the major.
- Enrich service courses with applications that will both serve students in other departments well and show students the value of mathematics. This possibility is enhanced in calculus courses directed at specific scientific disciplines.
- Promote the idea of mathematics as a profession, with evidence that a major in mathematics is a marketable degree, assistance in obtaining internships, and information about career opportunities in mathematics.
- Seek out students in calculus courses, invite them to be mathematics majors and help them with any necessary paperwork (e.g., by filling out the declaration of major form for them).
- Integrate students into the social as well as the academic life of the department (e.g., by having a departmental lounge accessible to students).
- Employ students in various ways. Jobs, from instructional and tutorial functions to simple clerical duties, can be a powerful draw into the mathematics major. It can greatly add to the students' feelings of involvement in a mathematics community.
- Promote the formation of math clubs, MAA student chapters, Pi Mu Epsilon chapters, Math Circles or other organized mathematics activities.
- Encourage double majors.

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