$$\begin{aligned}
\omega_{ki} &= \frac{\partial \bar{\bar{x}}^j}{\partial x^k}\frac{d}{dt}\left(\frac{\partial \bar{\bar{x}}^j}{\partial x^i}\right) = \frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^p}\frac{\partial \bar{x}^p}{\partial x^k}\frac{d}{dt}\left(\frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^q}\frac{\partial \bar{x}^q}{\partial x^i}\right) \\
&= \frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^p}\frac{\partial \bar{x}^p}{\partial x^k}\left(\frac{d}{dt}\left(\frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^q}\right)\frac{\partial \bar{x}^q}{\partial x^i} + \frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^q}\frac{d}{dt}\left(\frac{\partial \bar{x}^q}{\partial x^i}\right)\right) \\
&= \frac{\partial \bar{x}^p}{\partial x^k}\frac{\partial \bar{x}^q}{\partial x^i}\bar{\omega}_{pq} + \delta_{pq}\frac{\partial \bar{x}^p}{\partial x^k}\frac{d}{dt}\left(\frac{\partial \bar{x}^q}{\partial x^i}\right) \\
&= \frac{\partial \bar{x}^p}{\partial x^k}\frac{\partial \bar{x}^q}{\partial x^i}\bar{\omega}_{pq} + \frac{\partial \bar{x}^p}{\partial x^k}\frac{d}{dt}\left(\frac{\partial \bar{x}^p}{\partial x^i}\right).
\end{aligned}$$

This completes the proof.

It is simply a matter of algebraic manipulation to show that

$$\bar{\omega}_{rs} = \frac{\partial x^k}{\partial \bar{x}^r}\frac{\partial x^i}{\partial \bar{x}^s}\omega_{ki} + \frac{\partial x^i}{\partial \bar{x}^r}\frac{d}{dt}\left(\frac{\partial x^i}{\partial \bar{x}^s}\right).$$

Of course one should not have expected the components ω_{ik} to have tensor character for they are defined for any system in terms of the coefficients relating that system to the $\bar{\bar{x}}^j$ system (*i.e.*, the coordinates of the preferred Newtonian frame). Hence by definition $\bar{\bar{\omega}}_{ki} = \delta_k^j(d\delta_i^j/dt) = 0$. Since the symbols are zero in one system, tensor character would imply that they were zero in every system. This is contrary to physical experience.

RECOMMENDATIONS OF THE MATHEMATICAL ASSOCIATION OF AMERICA FOR THE TRAINING OF MATHEMATICS TEACHERS

The Committee on the Undergraduate Program in Mathematics (CUPM)* is a committee of the Mathematical Association of America and is supported in part by the National Science Foundation. The general purpose of this committee is to develop a broad program of improvement in the undergraduate mathematics curriculum of the nation's colleges and universities.

As part of its mandate, CUPM established a Panel on Teacher Training.† This panel was instructed to prepare for CUPM a set of recommendations of minimum standards for the training of teachers on all levels. The following report is the result of the work of the Panel on Teacher Training, and has re-

* Members of CUPM: R. C. Buck (*Chairman*), E. G. Begle, L. W. Cohen, W. T. Guy, Jr., R. D. James, J. L. Kelley, J. G. Kemeny, E. E. Moise, J. C. Moore, Frederick Mosteller, H. O. Pollak, G. B. Price, Patrick Suppes, Henry Van Engen, R. J. Walker, A. D. Wallace, R. J. Wisner (*Executive Director*).

† Members of the Panel on Teacher Training: J. G. Kemeny (*Chairman*), E. G. Begle, W. T. Guy, Jr., P. S. Jones, J. L. Kelley, B. E. Meserve, E. E. Moise, Rothwell Stephens, Henry Van Engen, R. C. Buck (*ex officio*), R. J. Wisner (*ex officio*).

ceived the endorsement of the Committee on the Undergraduate Program in Mathematics and of the Board of Governors of the Mathematical Association of America.

The Panel on Teacher Training has been further charged with the implementation of these recommendations and hopes to issue supplementary reports, as well as to hold various regional conferences, to make these minimum standards a reality.

The report consists of the following: General Recommendations, The Five Levels, Recommendations for the Five Levels, Summary of Recommendations, Curriculum-study Courses, Training of Supervisors, Sample Course Descriptions.

Further information and reprints of this report (not available before January 15, 1961) may be obtained by writing to the Executive Director: Professor Robert J. Wisner, Michigan State University Oakland, Rochester, Michigan.

GENERAL RECOMMENDATIONS

The purpose of this report is to present a preliminary outline of the panel's recommendations for the minimal college training program for teachers of mathematics. We have found it a most useful device to arrive at a classification of mathematics teachers which does not, as far as we know, depend on any present scheme of training teachers for their various tasks. The existing classifications seem to have arisen from a series of historical accidents and from fundamental psychological considerations. We hope this report reflects our feeling that we have made a serious attempt to classify teachers according to their position in an over-all sequential schedule of presenting the main ideas of mathematics.

For each classification presented, we give a recommendation as to the type and minimum amount of mathematics which should be taken by the student preparing for a career in teaching. Further, we spell out in some detail the *types* of courses—included as an Appendix—which we recommend to implement the goals described. The courses we describe which are specifically designed for prospective teachers should be taught by persons who are masters of their subject matter and who have, in addition, a knowledge of the problems which teachers face.

These sample courses are given solely for illustrative purposes to explain the type of courses and the levels of advancement desirable for prospective teachers. It should be clearly understood that different institutions will wish to exercise considerable freedom in implementing these recommendations, both as to the way topics are combined into courses and as to the exact choice of topics for individual courses.

There are several very sincere words of warning to be put forth in regard to the reading and interpretation of this report.

First, the classifications are to be taken in the rather loose fashion in which they are described. Their exact delineations will of course depend upon local

conditions of school and curricular organization. It should be noted that the various classifications overlap: this is done deliberately in an attempt to meet just such local conditions.

Second, the recommendations are not motivated by a desire to meet the demands of any special program of mathematics education; nor do the descriptions or outlines of courses to be taken by prospective teachers represent an attempt on the part of this committee to further the goals of any particular school curriculum planning organization. The recommendations are meant to be the minimum which should be required of teachers in any reasonable educational program, and the course descriptions are presented only to illustrate what is meant by the course titles.

Third, it is to be hoped that everyone recognizes good mathematics education to be a sequential experience. Thus, the teacher at any particular level should have an understanding of the mathematics which will confront the student in subsequent courses; and as a consequence, it is desirable that a teacher at a given level be prepared to teach at least some succeeding courses. Ideally, a person preparing for teaching should meet, in addition to the minimal requirements set forth here, as many of the requirements for the next level as his or her college program permits.

Fourth, this report is meant as a guide to the preparation of people who will be teaching any mathematics whatsoever. The suggestions apply, within each level, to all people who teach any mathematics. The recommendations do not in any sense exclude the teacher who is assigned classes scheduled primarily for students of low aptitude.

Fifth, every good teacher knows that mathematics must begin at the concrete level before it can proceed to the more technical or abstract formulation. Motivation for new concepts must be derived and later application of the theory to nature must be included. In each of the outlines to follow, it is assumed that topics will contain a judicious mixture of motivation, theory, and application. A purely abstract course for teachers would be madness, and a course in calculation with no theory would not be mathematics.

Sixth, the phrase "a course" occurs in several places in this report. For purposes of fixing ideas, this phrase is employed in the sense of a three-semester-hour presentation of the subject matter described, and it is not meant to exclude integrated programs or other curricular arrangements.

Finally, the reader should note that the training for Level I teaching is a separate program, while the curricula for the further levels form a cumulative sequence, in which each program is a continuation of the preceding one.

The committee benefited greatly from previous studies on teacher preparation, such as that of the Cooperative Committee on the Teaching of Science and Mathematics, a committee of the American Association for the Advancement of Science. The committee was also guided by discussions with a variety of professional organizations. It is pleased to note the considerable degree of agreement common to all proposals.

It should be emphasized that these recommendations are minimal in nature and that some institutions have already met and exceeded these recommendations. It is expected that as high school curricula are strengthened, these minimum recommendations will be revised.

THE FIVE LEVELS

I. Teachers of elementary school mathematics. This level consists of teachers confronted with the problem of presenting the elements of arithmetic and the associated material now commonly taught in grades K through 6. The committee recognizes that special pedagogical problems may be connected with grades K through 2, and so a special program may be appropriate for teachers of such grades.

II. Teachers of the elements of algebra and geometry. Included here are teachers who are assigned the task of giving introductory year courses in either algebra or geometry, or the less formal preliminary material in these fields. These introductory courses are now commonly taught in grades 7 through 10.

III. Teachers of high school mathematics. These teachers are qualified to teach a modern high school mathematics sequence* in grades 9 through 12.

IV. Teachers of the elements of calculus, linear algebra, probability, etc. This is a mixed level, consisting of teachers of advanced programs in high school, junior college teachers, and staff members employed by universities to teach in the first two years. These teachers should be qualified to present a modern two-year college mathematics program.

V. Teachers of college mathematics. These teachers should be qualified to teach all basic courses offered in a strong undergraduate college curriculum.

The levels having been presented, we are now ready to proceed to a description of our recommendations of the minimal college training requirements for entry into the teaching profession at each level.

RECOMMENDATIONS FOR LEVEL I

(Teachers of elementary school mathematics)

As a prerequisite for the college training of elementary school teachers, we recommend at least two years of college preparatory mathematics, consisting of a year of algebra and a year of geometry, or the same material in integrated courses. It must also be assured that these teachers are competent in the basic techniques of arithmetic. The exact length of the training program will depend on the strength of their preparation. For their college training, we recommend the equivalent of the following courses:

* Such sequences have been recommended by the Commission on Mathematics, the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, and others.

(A) A two-course sequence devoted to the structure of the real number system and its subsystems. (See course-sequence 1.)*
(B) A course devoted to the basic concepts of algebra. (See course 2.)
(C) A course in informal geometry. (See course 3.)

The material in these courses might, in a sense, duplicate material studied in high school by the prospective teacher, but we urge that this material be covered again, this time from a more sophisticated, college-level point of view.

Whether the material suggested in (A) above can be covered in one or two courses will clearly depend upon the previous preparation of the student.

We strongly recommend that at least 20 per cent of the Level I teachers in each school have stronger preparation in mathematics, comparable to Level II preparation but not necessarily including calculus. Such teachers would clearly strengthen the elementary program by their very presence within the school faculty. This additional preparation is certainly required for elementary teachers who are called upon to teach an introduction to algebra or geometry.

RECOMMENDATIONS FOR LEVEL II

(Teachers of the elements of algebra and geometry)

Prospective teachers should enter this program ready for a mathematics course at the level of a beginning course in analytic geometry and calculus (requiring a minimum of three years in college preparatory mathematics). It is recognized that many students will have to correct high school deficiencies in college. (However, such courses as trigonometry and college algebra should not count toward the fulfillment of minimum requirements at the college level.) Their college mathematics training should then include:

(A) Three courses in elementary analysis (including or pre-supposing the fundamentals of analytic geometry). (See course-sequence 4.)

This introduction to analysis should stress basic concepts. However, prospective teachers should be qualified to take more advanced mathematics courses requiring a year of the calculus, and hence calculus courses especially designed for teachers are normally not desirable.

(B) Four other courses: a course in abstract algebra, a course in geometry, a course in probability from a set-theoretic point of view, and one elective. One of these courses should contain an introduction to the language of logic and sets. (See courses 5–7.)

RECOMMENDATIONS FOR LEVEL III

(Teachers of high school mathematics)

Prospective teachers of mathematics beyond the elements of algebra and geometry should complete a major in mathematics and a minor in some field

* Sample courses, by numbers, are to be found in the Appendix.

in which a substantial amount of mathematics is used. This latter should be selected from areas in the physical sciences, biological sciences, and the social studies, but the minor should in each case be pursued to the extent that the student will have encountered substantial applications of mathematics.

The major in mathematics should include, in addition to the work listed under Level II, at least an additional course in each of algebra, geometry, and probability-statistics, and one more elective.

Thus, the minimum requirements for high school mathematics teachers should consist of the following:*

(A) Three courses in analysis. (See course-sequence 4.)
(B) Two courses in abstract algebra. (See course-sequence 5.)
(C) Two courses in geometry beyond analytic geometry. (See course-sequence 6.)
(D) Two courses in probability and statistics. (See course-sequence 7.)
(E) Two upper-class elective courses, *e.g.*, introduction to real variables, number theory, topology, history of mathematics, or numerical analysis (including use of high-speed computing machines).

One of these courses should contain an introduction to the language of logic and sets, which can be used in a variety of courses.

RECOMMENDATIONS FOR LEVEL IV

(Teachers of the elements of calculus, linear algebra, probability, etc.)

On this level we recommend a Master's degree with at least two-thirds of the courses being in mathematics, and for which an undergraduate program at least as strong as Level III training is a prerequisite. A teacher who has completed the recommendations for Level III should use the additional mathematics courses to acquire greater mathematical breadth.

Since these teachers will be called upon to teach calculus, we recommend that the program include the equivalent of at least two courses of theoretical analysis in the spirit of the theory of functions of real and complex variables.

It is important that universities have graduate programs available which can be entered with Level III preparation, recognizing that these students substitute greater breadth for lack of depth in analysis as compared with an ordinary B.A. with a major in mathematics. In other respects, graduate schools should have great freedom in designing the M.A. program for teachers.

RECOMMENDATIONS FOR LEVEL V

(College mathematics teachers)

We recognize the tremendous problems created by the shortage of qualified college mathematics teachers. A recommendation for the alleviation of this problem is now receiving serious attention.

* The requirements for Level II preparation have been included in this list.

SUMMARY OF RECOMMENDATIONS

Level	Description	Degree	High School Prerequisites	Minimum Number College Courses
I	Elementary School	B.A.	Two Years of College Preparatory Mathematics	4
II	Elements of Algebra and Geometry	B.A., Mathematics Minor	Preparation for Analytic Geometry and Calculus	7
III	High School	B.A., Mathematics Major	Preparation for Analytic Geometry and Calculus	11
IV	Elements of Calculus, Linear Algebra, Probability, etc.	M.A. in Mathematics	Preparation for Analytic Geometry and Calculus	18 (approx.)

BREAKDOWN BY SUBJECTS

Level	Numbers	Analysis	Algebra	Geometry*	Probability-Statistics	Elective
I	2		1	1		
II		2	1	2	1†	1§
III		2	2	3	2†	2§
IV‡		4	2	3	2	7

* Including analytic geometry.
† An introduction to the language of logic and sets should appear in some one course.
‡ The numbers in this row indicate the approximate number of courses.
§ Preferably from the areas specified.

CURRICULUM-STUDY COURSES

The above recommendations have dealt in detail with the subject-matter training of mathematics teachers. There are many other facets to the education of the scholarly, vigorous, and enthusiastic persons to whom we wish to entrust the education of our youth. One of these merits special mention by us. Effective mathematics teachers must be familiar with such items as:

(A) The objectives and content of the many proposals for change in our curriculum and texts.
(B) The techniques, relative merits, and roles of such teaching procedures as the inductive and deductive approaches to new ideas.
(C) The literature of mathematics and its teaching.

(D) The underlying ideas of elementary mathematics and the manner in which they may provide a rational basis for teaching, unless taken care of by mathematics courses especially designed for teachers.

(E) The chief applications which have given rise to various mathematical subjects. These applications will depend upon the level of mathematics to be taught and are an essential part of the equipment of all mathematics teachers.

Such topics are properly taught in so-called "methods" courses. We would like to stress that adequate teaching of these can be done only by persons who are well informed *both* as to the basic mathematical concepts and as to the nature of American public schools, and as to the concepts, problems, and literature of mathematics education. In particular, we do not feel that this can be done effectively at either the elementary or secondary level in the context of "general" methods courses, or by persons who have not had at least the training of Level IV.

TRAINING OF SUPERVISORS

There is a great need for providing adequately trained supervisors of mathematics, Grades K-12, for our public schools. At present, administrators find no ready supply of such individuals and, hence, are through necessity making appointments which are highly questionable, if not indefensible. For this reason, it is urgent to develop a program for supervisors and to seek adequate support for those individuals who have the desired qualities for supervision and the ability to benefit from advanced training. Such training would prepare the "leaders of teachers" in the local system, (A) to make sound judgments concerning mathematics programs for the schools, (B) to understand thoroughly the recommendations made by national committees, and (C) to enable schools better to articulate school mathematics with college mathematics.

Prerequsite to this program should be a regular Master's degree in mathematics or a Master's degree given as a result of participation in an Academic Year Institute. The program should consist of additional graduate courses in abstract algebra, analysis, and geometry, with courses selected from logic, statistics, theory of numbers, philosophy of education, history of education, history of mathematics, seminar courses on the program of the elementary school and secondary school mathematics, and additional elective courses in algebra, analysis, or geometry to provide some degree of concentration.

The committee feels that action must be taken to fill the need for supervisory personnel, and we recommend such action to the appropriate authorities.

APPENDIX: COURSE DESCRIPTIONS

We list below sample courses that might be used to fulfill the minimum requirements for Levels I through III, and the undergraduate requirements of Levels IV and V. These brief descriptions are included to clarify the meaning of course titles but are not intended as syllabi for actual courses. It must be

recognized that there are other equally good ways of combining various recommended topics, and colleges should be encouraged to work out detailed curricula to suit their own tastes and local conditions. However, the committee hopes that these very brief descriptions will help in indicating the types of courses desirable and the level of advancement.

Level I

1. *Algebraic structure of the number system* (2-course sequence). This is a study of the numbers used in elementary school—whole numbers, common fractions, decimal fractions, irrational numbers.

Emphasis should be on the basic concepts and techniques: properties of addition, multiplication, inverses, systems of numeration, and the number line. The techniques for computation with numbers should be derived from the properties and structure of the number system, and some attention should be paid to approximation. Some elementary number theory, including prime numbers, properties of even and odd numbers, and some arithmetic with congruences should be included.

2. *Algebra.* Basic ideas and structure of algebra, including equations, inequalities, positive and negative numbers, absolute value, graphing of truth sets of equations and inequalities, examples of other algebraic systems—definitely including finite ones—to emphasize the structure of algebra as well as simple concepts and language of sets.

3. *Intuitive foundations of geometry.* A study of space, plane, and line as sets of points, considering separation properties and simple closed curves; the triangle, rectangle, circle, sphere, and the other figures in the plane and space considered as sets of points with their properties developed intuitively; the concept of deduction and the beginning of deductive theory based on the properties that have been identified in the intuitive development; concepts of measurement in the plane and space, angle measurement, measurement of the circle, volumes of familiar solids; treatment of coordinate geometry through graphs of simple equations.

Levels II–V

4. *Analytic geometry and calculus* (3-course sequence). Approximately one-third of the sequence should be devoted to analytic geometry, taught either in coordination with calculus or after the calculus sequence. This should include the coordinate plane, functions, polar coordinates, the algebraic description of subsets of the plane—related to solutions of equations—and parametrically as the range of a function, change of coordinates, and brief treatment of conic sections.

The sequence should also give a thorough treatment of the calculus for functions of one variable, with stress on the basic ideas, but with adequate attention to manipulative skills. The course should introduce differentiation, integration, the rational, trigonometric, and exponential functions, as well as a brief treatment of series and some very elementary differential equations.

5. *Abstract Algebra* (2-course sequence). One course in this sequence constitutes an introduction to algebraic structures, such as groups, rings, fields, etc. The basic approach is to proceed from the concrete to the abstract. Use should be made of algebraic systems familiar to the student in order to motivate the abstract axioms. On the one hand, stress should be placed on rigorous algebraic proofs to convince the student that geometry is not the only area for axiomatic treatment. On the other hand, to keep the abstract procedure tied to the student's experience, various "concrete" applications should be given for theorems. Examples should be drawn from number systems, geometry, and other areas.

The other course should be devoted to linear algebra, restricted to real, finite-dimensional cases. This can be introduced by concrete manipulation of vectors and matrices, after which the student should be motivated to free himself from the accident of the choice of a basis. The student

should be taught the handling of vector equations and inequalities along with an intuitive introduction to linear programming and games. A good treatment of linear functions and transformations is needed, including a thorough understanding of the solution of m equations in n unknowns.

6. *Geometry* (2-course sequence). These recommendations have in general been based on the idea that advanced courses for teachers should be designed in such a way as to deepen understanding of the material which they will be teaching. In geometry, such a program involves special problems, because here some of the appropriate background material is not ordinarily thought of as being geometry at all, and much of it is not ordinarily taught on the undergraduate level.

The foundations of geometry, in the sense of Hilbert, is only one among many topics. Some further examples are as follows:

(1) Generalization of the idea of congruence to include rigid motions, that is, one-to-one correspondences preserving distances.
(2) A corresponding generalization of the idea of similarity.
(3) Enough measure theory to turn the familiar area and volume formulas into theorems, and to justify Cavalieri's Principle.
(4) "Pure analytic geometry," in which points, lines, and so on are defined and treated in terms of a coordinate system, without the use of any synthetic postulates at all. This is quite different from conventional analytic geometry in which the synthetic postulates are used in the very construction of coordinate systems. The "purely analytic" treatment can be used to give a consistency proof for the synthetic postulates.

These topics are given merely as illustrations of the sort of material that is needed. The choice of topics and the order of priority may require considerable study. The course might well take the form of a series of fairly long digressions from an outline of a high school course, with each advanced topic taken up at the point where it seems most relevant.

7. *Probability and statistics* (2-course sequence). The purpose of this sequence is to introduce the student to probability theory from a set-theoretic point of view, and to apply basic probability theory to problems of statistical inference.

The first course should be an introduction to random variables on a finite space. It must include motivation, axiomatic treatment of a measure on a finite space, and the proof of a few key theorems. There should be numerous applications from elementary statistics, stochastic processes, and everyday life.

In the second course more stress should be placed on stochastic processes, and probabilities on a continuous sample space should be treated. A substantial amount of time could be devoted to the development of principles of statistical inference.

Note: One of the course sequences, 4 through 7, should include an introduction to the language of logic and sets, so that these concepts may be used wherever appropriate. This introduction could be restricted to a brief treatment of the propositional calculus and of Boolean algebra, stressing the isomorphism between the two structures.

Electives (For Levels II–V)

8. *Introduction to real variables.*

9. *Number theory.*

10. *Elementary topology.*

11. *History of mathematics.*

12. *Numerical analysis, with the use of machines.*