Goals for School Mathematics

The Report of the Cambridge Conference on School Mathematics

PUBLISHED FOR
EDUCATIONAL SERVICES INCORPORATED
BY
Houghton Mifflin Company · Boston
NEW YORK · ATLANTA · GENEVA, ILL. · DALLAS · PALO ALTO

CAVEAT

The reader is urged to recognize the report that follows for what it is and for nothing more. A small number of professional mathematicians have attempted to express their tentative views upon the shape and content of a pre-college mathematics curriculum that might be brought into being over the next few decades. These views are intended to serve as a basis for widespread further discussion and, above all, experimentation by mathematicians, teachers, and all others who share the responsibility for the processes and goals of American education. At this stage of their development they can not pretend to represent guidelines for school administrators or mathematics teachers, and they should not be read as such. If this report, however, fulfills its purposes by provoking general debate and bold experimentation, those guidelines may ultimately emerge.

PARTICIPANTS

*The following participated in the Cambridge Conference on School Mathematics. Participants identified with * were members of the Steering Committee.*

MAURICE AUSLANDER, Brandeis University
*EDWARD G. BEGLE, Stanford University
*JEROME S. BRUNER, Harvard University
R. CREIGHTON BUCK, University of Wisconsin
GEORGE FRANCIS CARRIER, Harvard University
JULIAN D. COLE, California Institute of Technology
ROBERT DAVIS, Syracuse University
ROBERT P. DILWORTH, California Institute of Technology
BERNARD FRIEDMAN, University of California
H. L. FRISCH, Bell Telephone Laboratories; Yeshiva University
*ANDREW M. GLEASON, Harvard University
PETER J. HILTON, Cornell University
J. L. HODGES, JR., University of California
*MARK KAC, Rockefeller Institute
SEYMOUR H. KOENIG, IBM Watson Laboratories; Columbia University
C. C. LIN, Massachusetts Institute of Technology
EARLE L. LOMON, Massachusetts Institute of Technology
*WILLIAM TED MARTIN, Massachusetts Institute of Technology
*EDWIN E. MOISE, Harvard University
FREDERICK MOSTELLER, Harvard University
HENRY O. POLLAK, Bell Telephone Laboratories
*MINA S. REES, City University of New York
MAX M. SCHIFFER, Stanford University
GEORGE SPRINGER, University of Kansas
*PATRICK SUPPES, Stanford University
A. H. TAUB, University of Illinois
*STEPHEN WHITE, Educational Services Incorporated
*SAMUEL S. WILKS, Princeton University
*JERROLD R. ZACHARIAS, Massachusetts Institute of Technology

The Conference was organized and administered by Educational Services Incorporated, 108 Water Street, Watertown, Massachusetts, under a grant from the National Science Foundation.

CONTENTS

FOREWORD

BY FRANCIS KEPPEL

United States Commissioner of Education

If one were to look for the most significant development in education over the past decade, it would be reasonable to single out the wave of curriculum reform which has swept the school system, and appears to be maintaining its vigor undiminished. Beginning with mathematics and the physical sciences, it has spread in scope until almost every discipline represented in the primary and secondary school curriculum has been in some degree affected.

These recent reforms have several characteristics that differentiate them from the steady stream of curriculum reform of earlier years. They have been for the most part national, or at least regional, efforts. They have drawn on university scholarship and skilled teachers not only for leadership but for the immediate demands of day-to-day operation; to some extent they have served to destroy (or at least to lower) the wall that has traditionally separated the scholar from the teacher. Almost without exception they have passed from the determination of policy and program directly into the preparation of materials for use in the schools.

For the most part, they have been eminently successful, and in the light of their successes it has sometimes been difficult to distinguish their shortcomings. Yet the shortcomings are there, and they are by no means insignificant. It can be argued, in fact, that the deficiencies of the present reform movement are grave enough to threaten the expressed goals of the movements themselves.

These deficiencies derive from the inherent inconsistency that characterizes most curriculum reforms. On the one hand, there is the intention to represent in the revised curriculum the discipline in question as the scholar himself regards that discipline, complete with its sense of adventure, its unsolved questions, and its groping toward the future. Inseparably associated with this intention is the belief that the student can be brought into contact with the frontiers of knowledge, and that his capacity to learn is far beyond anything we have been accustomed to attribute to him.

But these ambitions are immediately dampened by the awareness that serious limitations are imposed upon the student's ability to learn by the instructor's ability to teach. If the student is to be brought to the frontiers of knowledge, the teacher must know the whereabouts of those frontiers. If the student is to be encouraged to grope, the teacher must at least be able to suggest which of his roads are likely to be blind alleys.

Most curriculum reforms, practically enough, have chosen to limit their ambitions in the light of these realities. They have tended to create such new courses as existing teachers, after enjoying the benefits of brief retraining, can competently handle. They have done so fully aware that they are thus setting an upper limit, and an upper limit that is uncomfortably close.

If the matter were to end there, the result might well be disastrous. New curricula would be frozen into the educational system that would come to possess, in time, all the deficiencies of curricula that are now being swept away. And in all likelihood, the present enthusiasm for curriculum reform will have long since been spent; the "new" curricula might remain in the system until, like the old, they become not only inadequate but in fact intolerable. Given the relative conservatism of the educational system, and the tendency of the scholar to retreat to his own direct concerns, the lag may well be at least as long as it has been during the first half of this century.

The present report is a bold step toward meeting this problem. It is characterized by a complete impatience with the present capacities of the educational system. It is not only that most teachers will be completely incapable of teaching much of the mathematics set forth in the curricula proposed here; most teachers would be hard put to comprehend it. No brief period of retraining will suffice. Even the first grade curriculum embodies notions with which the average teacher is totally unfamiliar.

None the less, these are the curricula toward which the schools should be aiming. If teachers cannot achieve them today, they must set their courses so that they may begin to achieve them in ten years, or twenty years, or thirty. If this is what the teacher of the future must know, the schools of education of the present must begin at once to think how to prepare those teachers. There must still be short-term curriculum reforms, they must look upon themselves as constituting a stage toward the larger goals, and they must at all costs be consistent with those larger goals.

Can the goals set forth in this report be trusted? Can we be confident that the curricula set forth here will indeed be the optimum curricula for 1990? It should not disturb us to realize that any such confidence would be completely unwarranted. None of us knows what sort of society we will have in thirty years, nor precisely what role mathematics will play in such a society. This report simply states the views of twenty-nine outstanding mathematicians and natural scientists as to the direction in which we should now be going. They have set forth goals for the future simply so that we may have some informed notion of the steps we should be taking, right now, if we are ever to make real progress. As the years pass, these goals may well change, but at least we will be in motion in the general direction of the new goals, and in a fair way to get there sooner or later.

It would be a mistake to read this report solely in terms of the mathematics curriculum. The step which has been taken here by mathematicians is one that scholars in all the disciplines must sooner or later attempt to take. If curriculum reform is to continue to play any vital role in American education, it must be bold enough to look far beyond immediate needs and immediate resources. To accomplish this will be far more difficult in disciplines which do not share the inherent simplicity of mathematics, but it is certainly to be hoped that the precedent that has been set in this report will be followed, no matter how great the difficulty, by those who pursue all the other disciplines represented in general education.

Section 1

INTRODUCTION

Purpose of the Conference

There is no disagreement today — nor will there be in the foreseeable future — on the vital importance of mathematics, both to the scientist, engineer, or other specialist called upon to use mathematics in his work, and to the intelligent layman in his everyday life. Mathematical education, to fulfill the needs of an advanced and advancing community, must be under continual scrutiny and undergo constant change, and it is the responsibility of all mathematicians, working in university, school, or industry, to concern themselves with the problem of keeping mathematical education vital and up-to-date.

During the summer of 1963, a group of twenty-five professional mathematicians and mathematics-users took time off from their normal work to review school mathematics and to establish goals for mathematical education. The present report sets down our tentative and highly provisional thinking on the nature of a good curriculum. It includes considerable detail, not so much for the sake of the detail itself but because the direction of the group's thinking is often better clarified by the provision of detail than by the bare enunciation of general principles.

At the outset, it is important to recognize that this report is a discussion document, and not a prescription. It would certainly not be appropriate, here and now, as a guide for general teaching, if only because one would be able to find so few teachers capable of teaching it. At this stage of the effort at least, the aims are more modest: it is hoped that our intentions and initiatives will commend this report to all who think about mathematical education, whether or not they are actually engaged in experimental work in this field.

Organization of the Conference

A brief account of the manner in which the Cambridge Conference was organized will be useful in considering the report which it has produced.

In the summer of 1962, Professors J. R. Zacharias and William Ted Martin, of the Massachusetts Institute of Technology, invited several Cambridge mathematicians and representatives of the National Science Foundation to an informal discussion of the state of mathematics instruction in the schools, both primary and secondary. At that meeting, it was agreed that although various groups were already doing important and excellent work in improving the curriculum, it was advisable to begin at once to move toward more radical revisions than any at that time under way.

Consequently, it was decided to organize a conference to deal with curricular reform in school mathematics. The facilities of Educational Services Incorporated were enlisted to help organize the program, a steering committee was recruited, and a formal proposal was submitted to the National Science Foundation, which quickly made available the necessary support. Professor Martin and Professor Andrew M. Gleason of Harvard University were named as directors of the project.*

Basis of Procedure

In its discussions, the steering committee found itself in agreement upon two major aspects of the general procedure that might be followed. These areas of agreement, which to a large extent dictated the terms of reference and the membership of the conference, may be summarized as follows:

1. The work of the University of Illinois Curriculum Study in Mathematics (UICSM) and the School Mathematics Study Group (SMSG) and others had shown that much can be accomplished in a relatively short time. Their efforts, however, had been affected by many factors, of which the most important have been the scarcity of adequately trained teachers and the necessity to work within the basic framework of the classical curriculum. For several years into the future the main thrust of curriculum reform will have to be in the direction of introducing into the schools the fresh outlook developed by the new groups, and in implementing recommendations for improved training of teachers such as those put forth by the Committee on Undergraduate Program in Mathematics (CUPM). What was now required were the outlines of a new reform program which may begin to become effective when current tasks have been accomplished and a new generation of mathematics teachers has begun to appear. Consequently, it was decided that the conference should deal primarily with the goals of school mathematics, leaving aside the relationship of these goals to existing educational resources.

2. The question of what is or what is not worth teaching must be approached, initially at least, in terms of all the possibilities that are inherent in the subject matter; the question of what is teachable and what is not

* The steering committee is identified on page iii.

depends largely upon the organization of that subject matter. Only the very top-level of expertise is likely to be sufficient to make the necessary determinations, and to set the stage for broader discussions in which all who are concerned with the goals of mathematics, and not merely the mathematicians themselves, may take part. Accordingly, it was decided to invite to the Cambridge Conference only those persons holding university positions or the equivalent. This did not imply a restriction to pure mathematicians; the importance of the applications of mathematics made it mandatory that other fields be included. The list drawn up by the steering committee included pure and applied mathematicians, statisticians, physicists, chemists, and economists. Because of the speed with which the conference was organized, many who would have liked to attend were unable to reschedule their summers upon the short notice that was given; nonetheless, a glance at the list of participants will show that persons of widely divergent interests in mathematics did indeed take part.

The Task of the Conference

The conference, as it in fact proceeded, bore a satisfactory resemblance to the general plans made by the steering committee, except that Professor Martin, who with Professor Gleason was to have been co-chairman, was unable to join the deliberations until the group reassembled in Boulder, Colorado, late in August.

The task of the conference, as we conceived it, was exploratory thinking with a view to a long-range future. We were therefore not concerned at all with the sort of practical considerations which govern the work of the next few years. Thus we ignored the whole problem of teacher training, and acted on the assumption that if a teachable program were developed, teachers would be trained to handle it. We also ignored many fine points of pedagogic technique, partly because our ideas were not developed in sufficient detail for such questions to come up, and partly because very few of us were professionally qualified to teach young children. In this spirit, we have not attempted to discuss the use of Cuisenaire® rods, and we have not considered the question of how much (if any) of the mathematics curriculum might lend itself to the methods of programmed learning. Some of us have opinions, pro and con, on the latter question, but these are not professional opinions, and they are not discussed in this report.

We made no attempt to take account of recent researches in cognitive psychology. It has been argued by Piaget and others that certain ideas and degrees of abstraction cannot be learned until certain ages. We regard this question as open, partly because there are cognitive psychologists on both sides of it, and partly because the investigations of Piaget, taken at face value, do not justify any conclusion relevant to our task. The point is that Piaget is not a teacher but an observer — he has tried to find out what it is that children understand, at a given age, when they have been taught in conventional ways. The essence of our enterprise is to alter the data which have formed, so far, the basis of his research. If teaching furnishes experiences which few children now have, then in the future such observers as Piaget may observe quite different things. We therefore believe that no predictions, either positive or negative, are justified, and that the only way to find out when and how various things can be taught is to try various ways of teaching them.

We have also neglected a practical problem which figures in present curricular work, namely, the development of political accommodations. The members of the conference made no attempt to form such accommodations, even with each other, let alone with an unpredictable future public. This point is essential for an understanding of the spirit in which the report is written, for two reasons.

In the first place, the report includes, in many places, expositions of mutually incompatible ideas. This should not be regarded as a sign of vacillation or confusion. Our idea, rather, was that the disagreements which remained at the end of the conference should be reported straightforwardly, along with the agreements. Since the issues involved are incapable of being adjudicated by four weeks of dialectic, the conference was in no position to write a prescriptive document in any case; our only reasonable hope was to help the future work of development and experiment; and we simply could not tell which of our conflicting ideas would be most helpful. We believe that all approaches discussed here are substantial enough to warrant further thought and experimentation.

In the second place, it should be understood that at the points where we report agreement, the agreements are genuine. One of the most striking developments, in the first two weeks of the conference, was the disintegration of party lines. As the list of members indicates, the composition of the conference was varied in the extreme: it included "pure mathematicians" of several kinds, applied mathematicians in both the conventional and various unconventional senses, statisticians, several physicists, and a chemist. After the first week, however, the views expressed bore no reliable relation, and sometimes no recognizable relation at all, to the professional backgrounds of the speakers; most of the time, alignments on controversial questions were orthogonal to scientific specialties. For example, some of the most vigorous remarks in favor of polynomial functions (versus formal polynomials) were delivered by a topologist; and one of the prominent advocates of a solid course in deductive synthetic geometry was a physicist, who argued that such a course was a prototype of the later study of physics by mathematical models. We believe that these developments are extremely encouraging for the future: they indicate that when scholars work together on the curriculum long enough to understand each other, they lose the urge to rescue the public from each other's predilections.

We also ignored the whole problem of designing, or even describing in general terms, valid tests for our program. It is argued later in this report that this problem is very serious.

To sum up: the direction of our efforts was governed by our conception of our own qualifications, and by the limits of time in which we were working. All our curricular ideas should be regarded as exploratory; none of them should be taken as prescriptive.

The Proposals of the Conference

The conference convened on June 18, 1963, in Cambridge, Massachusetts. After two days of general discussions the conference broke into three groups, each of which discussed the program for grades K through 6. After a few days of conversation the reports of these groups were discussed in plenary session. Again the conference broke into groups, this time to try to set operational criteria for the well-educated sixth grader. After these reports were discussed at a general meeting, we turned our attention to the problems of grades 7 through 12.

The conference operated quite differently in making recommendations for the upper grades. There was always at least one discussion afoot concerning the overall organization of some broad topic such as algebra or geometry. At the same time several conference members busied themselves with setting down detailed plans for smaller portions of the curriculum. One group studied the abstract concepts of modern pure mathematics in an effort to see just which ones might profitably be introduced into school mathematics.

Physically adjacent to the conference, Educational Services Incorporated was conducting an experimental summer school in connection with the teaching of science in the elementary schools. This made it possible for some of our members to try some of our ideas with children of various ages. While this experience may have given us a little more feeling for how children respond, we are under no illusions that we understand all the problems.

It was realized from the beginning that the feasibility of the program designed for the upper grades would depend on the success of the elementary school program. Since no one has had any experience with a class that has had seven years of well-organized pre-mathematical training, we simply had to guess what level of sophistication would in fact be appropriate. In view of this fact, it must be understood that these proposals are offered in a tentative spirit. We propose an ambitious program, aware that it may be impossible, but still convinced that it is worth shooting toward.

* * *

This report was drafted in the two weeks after the close of the conference by Davis, Gleason, Lomon, Moise, and Springer. A brief summary of the report was drafted by Gleason.

Late in August, copies of the summary were made available to all those attending the various mathematics meetings in Boulder, Colorado, and 75 copies of the full report were distributed, somewhat haphazardly. The response was remarkably heartening. The report came to be one of the most popular topics of discussion throughout the week. An open meeting called to discuss the report had to be transferred to the largest available auditorium, and more than 400 mathematicians attended.

At the end of the week, a sub-group of the Cambridge Conference met for two days and subjected the draft report to a page-by-page scrutiny. Frank Allen, Max Beberman, and Walter Prenowitz were invited to participate, and were present during both days. The present version of the report is a result of that meeting. In addition, a new summary has been prepared by Buck, Hilton, and Pollak.

Section 2

BROAD GOALS OF THE SCHOOL MATHEMATICS CURRICULUM

The subject matter which we are proposing can be roughly described by saying that a student who has worked through the full thirteen years of mathematics in grades K to 12 should have a level of training comparable to three years of top-level college training today; that is, we shall expect him to have the equivalent of two years of calculus, and one semester each of modern algebra and probability theory. At first glance this seems to be totally unrealistic; yet we must remember that, since the beginning of this century, there has been about a three-year speed-up in the teaching of mathematics. Of course, one cannot argue that such steps can be taken indefinitely, but it is comforting to realize that the proposed changes are no more radical on their face than changes which have actually taken place within the memory of many.

Acquisition of Skills

Since the amount of time to be spent on mathematics will certainly not increase in the face of the additional effort now being focused on the sciences in elementary schools, and the mean level of native ability of students probably does not change appreciably in periods shorter than geological, it is clear that the inclusion of more content at the top must be compensated by the omission of something else. There are a few topics whose omission has been frequently signaled over the recent past, the most obvious being the numerical solution of triangles. Dropping these will not release three years, however. We propose to gain three years through a new organization of the subject matter and the virtually total abandonment of drill for drill's sake, replacing the unmotivated drill of classical arithmetic by problems which illustrate new mathematical concepts.

Lest there be any misunderstanding concerning our viewpoint, let it be stated that reasonable proficiency in arithmetic calculation and algebraic manipulation is essential to the study of mathematics. However, the means of imparting such skill need not rest on methodical drill. We believe that entirely adequate technical practice can be woven into the acquisition of new concepts. But our belief goes farther. It is not merely that adequate practice *can* be given along with more mathematics; we believe that this is the only truly effective way to impart technical skills. Pages of drill sums and repetitious "real-life" problems have less than no merit; they impede the learning process. We believe that arithmetic as it has been taught in grade schools until quite recently has such a meagre intellectual content that the oft-noted reaction against the subject is not an unfortunate rebellion against a difficult subject, but a perfectly proper response to a preoccupation with triviality.

We are not saying that some drill problems may not be appropriate for the individual student whose technical skill is behind, but we do believe that this should be the exception, not the rule. We are definitely opposed to the view that the main objective is arithmetic proficiency and that new, interesting concepts are being introduced primarily to sugar-coat the bitter pill of computational practice.

Familiarity with Mathematics

The reorganization which we refer to above has as its principal aspect the parallel development of geometry and arithmetic (or algebra in later years) from kindergarten on. However, a mere recital of the topics proposed for the future curriculum does scant justice to our goals. Familiarity is our real objective.

We hope to make each student in the early grades truly familiar with the structure of the real number system and the basic ideas of geometry, both synthetic and analytic. In particular we urge that considerable attention should be paid to inequalities starting in the earliest grades immediately after learning to count. It is a matter of common experience that college students are often bewildered by the importance of inequalities in the study of calculus. This, of course, is the result of the almost complete preoccupation of the classical curriculum with problems of equality.

Moreover, we want to make students familiar with part of the global structure of mathematics. This we hope to accomplish by the "spiral" curriculum which repeatedly returns to each topic, always expanding it and showing more connections with other topics.

On this firm foundation we believe a very solid mathematical superstructure can be erected which will make the pupils familiar with the ideas of calculus, algebra, and probability. The elementary school program (K-6) should be understandable by virtually all students; it should lead to a level of competence well above that of the general population today. As students advance through junior and senior high school we must expect that fewer and fewer will elect mathematics; consequently we have attempted to move first in the directions most suitable for those who take mathematics for only a few years after grade school. Of particular importance is an elementary feeling for probability and statistics. Although there was considerable difference of opinion on this point (see Section 6) many felt that a nodding acquaintance with the calculus had the next priority.

Mathematics in Liberal Education

The conference felt that mathematics is a subject of great humanistic value: its importance to the educated man is almost as great as its importance to many technical specialists. The strongest argument for the early inclusion of the calculus was one of general education: liberal education requires the contemplation of the works of genius, and the calculus is one of the grandest edifices constructed by mankind.

Many poor patterns of thought common in ordinary life may be modified by the study of mathematics. The misleading, but almost universal, assumption that things must be ordered linearly might be dispelled by the study of partial orderings. Just a little experience with logic and inference can do away with some of the unfortunate reasoning we meet all too often. Even a nodding acquaintance with probability can clarify the "law of averages."

Mathematics is a growing subject and all students should be made aware of this fact. This recommendation is not made merely because we feel that every educated person should know the fact, but also because the knowledge that there are unsolved problems and that they are gradually being solved puts mathematics in a new light, strips away some of its mystique, and serves to undermine the authoritarianism which has long dominated elementary teaching in the area.

Building Self-Confidence

The authoritarian methods which have been used in elementary mathematics teaching have already received blows enough, but one point which has not often been stated is that, beyond the stifling of creativity of students, they carry the suggestion that one is helpless if he forgets the "formula." Actually the reverse is true. Even modestly endowed students can recreate large parts of mathematics if they can remember just a few basic ideas. This fact repeatedly has been demonstrated in the classroom by proponents of the so-called discovery method. The building of confidence in one's own analytical powers is another major goal of mathematics education.

Mathematics is the discipline par excellence in which intellectual effort can be a real labor-saving device. Multiplication of integers, which is basically a problem of counting rectangular arrays, becomes easier when the associative, commutative, and distributive laws are taken into account and a short algorithm is found which effects in a few seconds a calculation which would take a lifetime of simple counting. Problems which could be solved by guesswork methods are solved quickly through the introduction and algebraic manipulation of a literal unknown. The approximate calculation of π, which was for Archimedes an arithmetic problem of great length, is reduced through the study of functions to a short homework problem for calculus students. These examples should be brought home to the student in these terms and at the same time he should be shown that similar, if not such spectacular, short-cuts are within his own powers. Moreover, he should be convinced that he can rely on the results of analytical thinking.

The Role of "Modern" Mathematics

Some have argued that the mathematics curriculum should be organized to provide the quickest possible introduction to contemporary mathematical research. This view we reject. Contemporary mathematical research has given us many new concepts with which to organize our mathematical thinking; it is typical of the subject that some of the most important of these are very simple. Concepts like set, function, transformation group, and isomorphism can be introduced in rudimentary form to very young children, and repeatedly applied until a sophisticated comprehension is built up. We believe that these concepts belong in the curriculum not because they are modern but because they are useful in organizing the material we want to present.

Technical Vocabulary and Symbolism

Similarly we view the problems of language, notation, and symbolism. It is unquestionably possible to obscure a subject by introducing too much special terminology and symbolism; but we feel that most errors of this sort in fact cover an inadequate understanding of the subject matter. The function of language is to communicate. In mathematics its function is to communicate with extraordinary precision; it is inevitable therefore that mathematics requires some special terminology. Special terms are good or bad exactly according to their effectiveness in communication, and the same applies to special notations and symbols.

This principle must not only guide textbook writers, it must be brought home to the student. Mathematics is, to a large extent, a process of organizing data. Through symbolization and the precise formulation of new concepts, large blocks of information are brought within the grasp of the mind. One of our members related that a professor lecturing on currents (in the sense of manifold theory) had defined the concept of continuity for currents and developed a number of facts before he ran into a snag and announced, "Our definition of continuity for currents is inappropriate," and started over. Our panelist recalled the deep impression this made on him because it showed so clearly the sense in which mathematical definitions are arbitrary and at the same time that they must stand judgment on the question of their utility in organizing the subject.

Pure and Applied Mathematics

We hold no brief for a dichotomy between "pure" and "applied" mathematics, yet we must recognize as inevitable that there will be more users of mathematics than makers. One of the oft-expressed fears concerning the early introduction of rigorous mathematics is that students will become so enamoured of the logical precision possible in this field that they will lose interest in the less precise disciplines. A slight variation holds that a student required to prove every small point in his mathematics course will become so rigid that he can do nothing in a typical applied situation. Both of these fears are real and justified in part by experience. Once again we believe that the remedy lies not in flight but in intellectual honesty. If the nature and limitations of the mathematical models used in science are carefully described and if the intuitive steps which go into their construction are fairly presented, then the inherent attraction of unlocking the secrets of nature or solving a practical problem should easily balance the attraction of logical certitude.

To foster the proper attitude toward both pure and applied mathematics we recommend that each topic should be approached intuitively, indeed through as many different intuitive considerations as possible. In such a program, the student must be kept informed of where he stands. A curriculum which oscillates between logical rigor and guesswork can be confusing unless the student knows the level at all times. To present mathematics entirely in the rigorous deductive spirit not only precludes any possibility of applying mathematics, it is dishonest, even as a picture of contemporary pure mathematics. We hope that many problems can be found (we know a few) that read, "Here is a situation — think about it — what can you say?"

The Power of Mathematics

Another goal of our program is the inculcation of an understanding of what mathematics is (and what it is not). We need not here belabor the point that the man in the street has considerable misinformation on this point; suffice it to say that this misunderstanding frequently seems to take the paradoxical form of ascribing both too much and too little power to mathematics.

Concentration on equality is probably the reason why so many people are convinced that mathematics deals only with "exact answers." This feeling was nicely illustrated at the conference by a statement made by a fifth-grader during a Socratic session on probability theory. Showing a firm belief in the "law of averages," but somewhat shaken by the question "How does the coin remember that it has just come up heads?" he stated, "The law of averages says it will even up and mathematics only deals with exact laws." The fact that mathematics is able to deal effectively with both qualitative and uncertain relationships has only recently been known to any but professional mathematicians. Today, however, qualitative and probabilistic mathematical techniques are playing an ever larger role in disciplines ranging from the exact sciences to the practical field of management. We believe that a high school student can and should learn something of the mathematics which underlies these methods.

Understanding the Limitations of Mathematics

While everyone should know about the wide range of topics suitable for mathematical analysis, it is almost equally important to understand the limitations of mathematics. The success of mathematics in one area often conjures up an inflated image of what it can do in another. It cannot solve the fundamental problems of politics, economics, or social relations. Operations analysis, one of the glamorous new tools of management, can be described as the process of organizing the available evidence to help predict the consequences of various decisions. But it is essential to realize that mathematics does not provide new evidence; a decision based on poor evidence is likely to be bad no matter how careful the analysis.

The limitations of mathematics are by no means confined to its newer applications. Mathematics per se does nothing directly for even the classical, exact disciplines of physics and astronomy. Only after a model of the real world has been formulated does mathematics enter the picture. Every application of mathematics depends on a model, and the value of a deduction is more an attribute of the model than it is of mathematics. We believe that students can be made aware of the distinction between the real world and its various mathematical models; in this we can look forward to cooperation from the sciences.

Section 3

PEDAGOGICAL PRINCIPLES AND TECHNIQUES

In reaching the goals discussed above, the selection of germane material and the method of presentation are of prime importance. An omission of a subject from a student's curriculum can be made up readily later, in college or adult education courses, if the student has previously developed a sound approach to mathematics. On the other hand, improperly taught material may confuse the student's understanding of the facts, inhibit good mathematical reasoning, and lead to dislike of the whole subject.

The conference agreed on several principles of instruction and on basic content important to the realization of these principles. We believe that sufficient implementation of the majority of the proposals discussed below is very rare in contemporary curricula. Often the contrary occurs.

Degrees of Rigor

The importance of a suitable background of experience, involving diversified sensory input, in developing clear mathematical concepts suggests that full use be made of general heuristic cognitive patterns to be called henceforth "pre-mathematics" to introduce each new topic of study. In explicitly introducing "pre-mathematics" as distinguished from the "formal study" of mathematics we recognize the existence of various levels of rigor which naturally supplement and in part complement each other in the course of development of the student's mathematical maturity.

The use of a spiral curriculum, in which the same subject arises at different times with increasing degrees of complexity and rigor, offers many advantages. At the first stage an intuitive or pre-mathematics approach offers the opportunity of an early introduction of important concepts. There is time for each of these concepts, first drawn from the student's general experience, to be made more familiar and more precise, and time to develop the concept further. The concept can be used by the student from the beginning in appropriate simple contexts.

The intuitive discussion should not be wrong or misleading, as it often is, but incomplete structurally. Frankly stated assumptions should replace many theorems later proved in terms of a set of axioms. These assumptions are to be made reasonable in terms of previous experience. When possible the limitations involved are to be described and counter examples established. The student is given the confidence to use the results of an intuitive discussion, but with care. He should have the expectation that the result is powerful enough to be usable in situations he is likely to meet soon, but that he may later meet problems where the result needs modification.

After an interval in which the concepts have become more concrete in the student's mind through use in other contexts, the subject is brought back to the student's attention with a greater degree of formal organization. It is important that this more difficult approach should immediately yield more power and a greater scope of accomplishment. The boredom of pure review and reiteration must be avoided.

The rules of logic will have to be evolved to a sufficient level before this second stage is reached. The cohesiveness of large areas of mathematics now becomes apparent and this is a reward in addition to the greater power achieved.

The first stage of pre-mathematics training will here enable the student to see through the formal discussion to the structure and meaning of the proof. Sighting the goal will alleviate the danger of tedium in proving the more difficult theorems. When an important theorem is proved, its implications and meaning must be brought out by discussion and by application.

Later the learning of other material may enable the student to handle problems that require several techniques and to extend the level of the first subject matter once again. Then that subject will arise again in a major way. For instance, the development of analysis and calculus will allow the treatment of continuous probability distributions. The course will then be able to return to probability for a new, more advanced treatment with the proofs of theorems concerning continuous distributions.

The frequent use of older material throughout the course is related to the spiral technique of teaching. It also develops the theme of the unity and interdependence of mathematics.

Although more and more rigor is introduced for older topics as the course proceeds, at the same time new topics and advanced concepts are being introduced pre-mathematically. The student should not obtain the impression that rigor will eventually replace intuition in all of his mathematical endeavour. That would kill his creative potential. He should realize that he can think ahead and even apply operationally in advance of "certain knowledge." Hypotheses usually come before proofs in creative mathematics. This is an historic fact, but even more, conjectures such as Riemann's and Fermat's have influenced mathematics in important ways, although they are still unproved.

Nevertheless the importance of proceeding to a rigorous development is made clear. The student discovers the qualifications of his original concepts and finds out which axioms are required. Experience indicates that many students obtain a deep satisfaction from the logical connections realized, and a newly clear idea of what they have been talking about. The precise viewpoint improves their "feel" for the material, filling in parts of the picture for which their intuition was inadequate. Their appreciation of the whole concept is advanced immensely.

Logical development should often follow the intuitive one rapidly in the later years of the curriculum, sometimes arising in the same series of lessons. At other times it may have to await the interval of one or more semesters in order that the student be prepared with all the required concepts for the rigorous development. At the elementary school level the amount of logical or inductive reasoning that will be appreciated is uncertain. More information through experiment is needed. However, there is sufficient information from experimental courses now operating (such as the R. Davis and the P. Suppes projects) to establish that a nonnegligible amount of such work can be included from the fourth grade on. A tentative list of proofs that should be included is discussed for the elementary school curriculum.

The Use of Several Approaches

The spiral method proposes that the same concept or theorem be dealt with upon several occasions in the curriculum, separated by varying intervals of time. Upon each of those several occasions, however, several approaches should be employed, all of them on approximately the same level of rigor. The advantages to be gained on occasion by this multiple discussion of a topic are several. A student who does not "see" one approach may find another understandable. Fewer students are left behind this way. Also different aspects of a concept or proof are emphasized by each approach, each approach may show the relevance of different topics to the one at hand, and a larger range of ramifications comes into view. The use of different notations, most appropriate to each approach, may come in here.

On a large scale, mathematics is a unified subject in which each part may benefit from systematic investigations with different starting points. Geometry and algebra or function theory are unified by analytic geometry. Nevertheless geometry and algebra are different in the intuitions and experiences they are based on. This is reflected in their different axiom systems. They lead to the same results through different paths often of different lengths. They illuminate different aspects of the results. The coordinated development of these two main streams of mathematics is a central theme of the curriculum proposed here. We believe that, if the relations between arithmetic and geometry are brought out so that arithmetic ideas can be interpreted geometrically and vice versa, this will contribute to the student's understanding of both.

Often in advanced topics there is a choice between a constructive and a non-constructive proof. If both cannot be given, then the constructive proof is usually to be preferred because it develops a concrete feeling for the result and because it may be operationally useful.

Development of Skills

The conventional program spends too much of its time (certainly in K through 6) on the development of manipulative skills, supposedly leading to speed and mechanical accuracy. The traditional curriculum has stressed arithmetic drill throughout elementary school. This is natural if the only use of arithmetic by the student is in the drill. The time spent on drill then prevents the teaching of new material, and a vicious circle has been established. By going forward to new work utilizing arithmetic and other skills, these skills are improved and more interesting developments are obtained at the same time.

The conference felt strongly that the understanding of the algorithms justifying the manipulations will in the end lead to better skills while opening the door to deeper and more advanced mathematics. The last allows more advanced skills to evolve.

When skills are being required of the student it is proposed that accuracy is more important than speed. For this reason adequate time for checking should be allowed and methods of checking discussed. These methods have their own mathematical interest as they make use of the laws and structure of the problem. The finding of rapid checks also develops ingenuity.

It should be emphasized that the student is not necessarily wrong if he uses a clumsy operational procedure. Counting on fingers goes back to fundamentals in calculating, and the student who does this occasionally may learn better what the operations of arithmetic involve than the good memorizer of tables. We do not want a student who is restricted to calculating in this way; he should certainly be expected to learn a more powerful algorithm, but he should not be told he is wrong; rather he should be given approval for knowing how to go back to fundamentals.

Fostering Independent and Creative Thinking

1. Discovery approach. The discovery approach, in which the student is asked to explore a situation in his own way, is invaluable in developing creative and independent thinking in the individual. In this system memorizing a mechanical response does not help the student to advance. His innate interest and competitive nature force him to concentrate on the creative problem at hand.

2. Aiding discovery. It is obvious, however, that the discovery method is slow. It took mankind thousands of years to discover, collectively, the concepts we wish to teach. Cut off from communication with the knowledge of others the student can proceed but a little way along the path of wisdom in his allotted time. This is at least tacitly admitted by all proponents of discovery, but there is considerable variation in the amount of feedback and reinforcement used to guide the student. As a minimum the context and the very statement of the problem, or the equipment given to work with, is a guide to the student — a very important one.

We believe that usually one should go farther than this in aiding discovery: that the teacher should be prepared to introduce required ideas when they are not forthcoming from the class; that he should bring attention to misleading statements in the way of the discussion, and summarize results clearly as they come forward. He should not allow the "moments of triumph" to pass by unnoticed.

This must be done, however, with a minimum of authority. The student should never, for fear of being wrong, hesitate to state the results of his best efforts. Wrong statements are not to be embarassingly rejected, as is the common practice. Worse, correct statements that the teacher did not want are often rejected. Half-formed ideas should be used as steppingstones to true or more relevant statements.

This development of independent and creative habits of thought does not require the complete devotion of the curriculum to the discovery method. Discovery directed by a dialogue between teacher and class, and the direct teacher presentation of material, will be required to attain a reasonable rate of advancement. This is especially true in the later years when large numbers of concepts and proofs of theorems are to be taught. A good balance may be as follows, although experimentation is needed to make a final decision. In the earliest grades the discovery approach, teacher aided, should dominate. By grade 7 most of the classroom time will be occupied by more direct teaching procedures. However, in these later grades, creative thinking and independence should be fostered extensively by the exercises in school time and homework. A transition should take place in the intermediate years.

In utilizing discovery the class may be expected to find a yes or no answer to a stated question. Usually it will be more valuable to define the area the class is to explore and allow the results to arise from the flow of the conversation. Even in the later grades new concepts should often be introduced by asking the class to explore possibilities. This will help develop intuition equally with logical processes and keep alive a willingness to enter into a pre-mathematical approach to new areas of mathematics. Examples of this technique are furnished in appendix D.

3. Advantages of directed approach. The directed discussion offers some important advantages. Communication between a student and his peers opens to him a large pool of ideas. These ideas are fed in close to the student's own level of understanding. The young student is aware that the knowledge and facility of his peers covers a range closer to his own than the teacher's. He usually believes that he can be right on occasion when his peers disagree, but may well doubt this with respect to the teacher. Thus we can expect that he will think critically before accepting a suggestion of other students. Statements from them that he cannot understand or with which he disagrees will be rejected, whereas coming from the teacher they will be remembered as at least useful for mechanical response. In disagreeing he will defend his point of view against that of a peer, but may well avoid a confrontation with the teacher's argument. Thus, this type of communication does not interfere with the independent attitude desired and should be fostered. Even the confused statements of his peers are valuable to the student, as he is likely to share many of these confusions and they should be brought to light.

As some students will be superior and may begin to assert an authority that will make the others timorous, the teacher may consider partially separating them from the discussion. This could be done, for example, by giving them another point to work on at their desks or at the board. Or they may be designated explicitly as "experts" who will speak after the discussion by the rest of the class.

4. Motivation. The role of tricks deserves some discussion. It has been remarked that a mathematician's chief incentive is laziness; frequently a trick calculation may obviate much laborious computation. Some of these tricks should be presented, both for fun and as an aid to the acquisition of manipulative skill. It is essential that such tricks be discussed in theoretical terms. In this way they can illustrate the power of mathematical analysis in a general way and at the same time provide applications of specific theorems. The student should be encouraged to seek similar short cuts by continually asking what is the relevance of the theory to his problem. This will foster a truly mathematical spirit. Shorn of adequate explanation, tricks seem disconnected and arbitrary and frequently serve to discourage a student from any hope of being able to advance by his own intellect.

It is clear in this context that the exercises are of great importance. Indeed it is urged that they are the most important part of the prepared course material. The exercises should guide the student, and also the teacher, to the meaning and relevance of the theorems and concepts; they should train and develop the skills, stimulate creative thinking, and develop ingenuity. It is suggested that when the material for the curriculum is written that it is most likely to emerge with the proper perspective if the exercises for each section be composed immediately after the theme of the section is formulated. The textual discussion should then be written to illuminate the theme. Some extensions of the main theme will appear in the exercises and not need discussion beforehand. Alternate approaches to the same result can be developed in the problems and need not be stated in the discussion where they may give the impression of redundancy. The operational needs of the exercises will isolate points that must be made clear in the discussion. These advantages indicate that the text be written exercises first, discussion afterward, even though it may not appear in print in that order.

The historical background of a topic often makes clear the motivation for discussing it. This background stimulates many students by providing the human interest and showing the connection of mathematics with other important events. It is of value in the understanding of the processes of mathematical creativity: all known mathematics was discovered by somebody.

Symmetrically, looking into the future provides an encouragement to originality and will supply to some individuals the basic reason for becoming a mathematician. To show that mathematics is open-ended, unsolved problems should be discussed in the appropriate context.

Setting aside time for a mathematical laboratory is another way of stimulating interest and a creative approach. In elementary school several hours each month should be made available for mathematical games, special topics, experiments with apparatus such as needles and lines, thumbtacks, and computers, etc. This provides a means of reaching many students not responding well to the regular classroom instruction. It gives regular opportunity for progress through experimentation with the curriculum and with pedagogical techniques. As many such laboratory sessions cut across grade and ability levels several classes may be handled together. This gives opportunity for several mathematics teachers in the same school to pool their time and talents in the design of these sessions and in their supervision. On occasion one may use this opportunity to bring in a visiting teacher.

In secondary school enrichment and stimulation may be obtained practically by having available a series of pamphlets on interesting or advanced material growing out of classroom work. This helps solve the problem of what to do for the superior student, without requiring a separate curriculum.

5. Testing Achievement. When we take so much care to develop understanding and creativity in the student, it would be a pity to test his achievement only in terms of the mechanical skills and rote responses he has learned. Unfortunately it is extremely difficult to prevent these last elements from dominating any examination. The difficulty of formulating questions testing understanding and creative ability is magnified by the uncontrollable tendency of teachers and schools to "teach for the exams." Almost any question can be reduced to a rote response when the teacher knows the class of question the student will have to answer, and uses his (the teacher's) ingenuity to find a "formula" type of response.

It is possible that a continuing committee can formulate probing questions at a sufficient rate to overcome the skill of others at ferreting out expected examination questions. The developing of good questions in volume is in any case valuable and should be pursued. Another tactic, perhaps one of desperation, is to use only a few problems of this probing nature on an examination, or to use them in informal tests not used for grading. As neither the student nor the school will then gain much in public merit by the effort of reducing these questions to a rote response, they will concentrate their efforts on the other questions. In this way a natural response may be obtained to the important questions; but these responses will be usable only by curriculum designers, not by admissions offices.

Language and Notation

It is generally agreed that in the early years the student understands more than he can verbalize. Both the words and the habit of using them are not at first available. In the first instance the teacher compensates for this, or should, by patient and sympathetic listening to find the child's meaning. A too early insistence on the use of the "correct" word may well stifle the child's idea and will encourage parrotlike responses.

However, making a language available to the children early will help them in thinking about mathematics as well as talking about it. So while any mode of expression of a good idea should be accepted, useful words should be introduced at all times. As the students connect the words to the concepts, the concepts become better and more uniformly defined. The children can then talk to each other understandably, enabling collective thinking. Finally, rapid communication between the teacher and student is possible. It is therefore recommended that conventional and precise mathematical language be introduced to the extent suggested by the subject matter. Superfluous language is indeed to be avoided. But unusual or foreign words are no harder to learn than usual words, and the choice should be based on the considerations of clarity and utility.

The same points are relevant to mathematical notation; moreover, notation is often ambiguous. The student should be prepared for this by occasionally being shown alternate notation. Often the need for different notation is inherent in the subject matter, as when the number system is being written in different bases. There is no point in requiring a fixed notation from the student if he shows his ability to handle variants.

The Role of Applications

All mathematical ideas are motivated by applications of some sort: They enable us to solve new problems and to understand situations which we did not understand before. Many of these applications are to the physical sciences or to other aspects of the real world. But many of them are internal applications, to mathematics itself. Both the internal and the external applications should be taught, so that the student will understand both the power of mathematics as a scientific method and the unity and beauty of mathematics as a science in its own right.

When introducing or generalizing a mathematical concept, it is important to have applications. A concept should always be motivated. Its need in an application is a strong motive. Applications afford to the student the opportunity of discovering whether he has the appropriate ingenuity and flexibility in the use of the theorems that he knows.

Some conditions must be met if external applications are to play the proper role in the student's training. To be meaningful, external applications require a knowledge of another discipline. The added concepts required for any application compound the difficulty of understanding the mathematical material at hand, unless the student is already acquainted with them. It is useless and can be harmful to introduce applications whose context the student does not understand. At best it is then a relabeling of the student's mathematical entities. At worst it both confuses the new mathematical context and causes misunderstanding of the other subject matter. Applications must be chosen that minimize this difficulty. The student should have mastered the mathematical topic sufficiently so that the extra difficulties of the application are not overwhelming. Only then can an application serve to advantage.

This stricture applies equally to internal and external applications of mathematics, but its force is greater for external applications. For an external application to be valuable, the experimental background and the mathematical identifications of the model must be in the student's experience, taught to him previously in the curriculum of the other discipline, or supplied in the mathematics course. The last alternative offers formidable difficulties.

In fact, the use of mathematical models in describing the real world is a very delicate matter. If models were judged in terms of the dichotomy, right or wrong, then all would be wrong; the real world is far too complex to be represented accurately by anything but the analogue computer COSMOS. Models must be evaluated along a spectrum from good to bad according to the validity and usefulness of the predictions based on them. Every model has its limitations; in using it, one should know at least those limitations which are important operationally.

The design of mathematical models is a complicated process indeed, involving intuitions drawn from long experience with the object discipline and from mathematics itself. Usually models are not simply designed; they evolve from a long sequence of experimentation, guesswork, and logical inference. It is only when the model is fully formulated that the purely deductive methods of mathematics take over. We regard it as an open question just which parts of this process should be taught in mathematics courses and which should be taught in other science courses.

Besides the recognized applications in the sciences there are many of importance to the professional and business spheres of life. It is of critical importance in these spheres to have adequate means of estimating and synthesizing facts, and methods of making decisions based on those facts. Arithmetic together with elementary geometry and algebra suffices for many everyday operations; but a large and important part of the population frequently needs to make decisions requiring probability theory, computer methods, the calculus, or other advanced mathematics. Those so involved may hire or designate mathematicians to do the work, but must be aware of the concepts to control the goals wisely and know what personnel they require. Without blind faith in their professional advisers, they must understand the basic principles to be able to make decisions for which they are responsible.

Many such decisions do not require the deliberation of a professional mathematician but only a better understanding of important mathematical results than the populace now generally has. For instance, consider the relevance of making two independent quality control checks, each of which eliminates the same defect with an error of one in a thousand. The simple fact that the use of both tests reduces the error to one in a million is basic to deciding how many independent checks to institute. If an individual believed that the combined chance of error was one in two thousand in the above case, he might well decide that there was insufficient advantage to warrant a new checking system. If an error worse than one in a million is intolerable, he may erroneously decide that a thousand independent checks are required.

Topics such as game theory, queuing theory (and other operations analysis), and digital computer methods should be included so that the population becomes aware of the widespread uses of mathematics outside of scientific endeavors.

The applied aspects of mathematics go beyond the application of mathematical results to specific problems. The transfer of a mathematical way of thinking to the rest of our intellectual effort is, in a sense, an application of mathematics. Concepts such as probability distribution, limit, and instantaneous rate of change have a deep cultural significance beyond their immediate uses. They introduce important ways of thinking, involving the whole spectrum of thought, of which the nonmathematician is often not aware.

Among such concepts those of *order* and of *order of magnitude* can and should be brought into the curriculum at a very early stage. The concept of order, in the sense of less than or greater than, is at least as elementary as that of equality. It should be introduced in the earliest grades. The concept of order of magnitude is basic to approximation and estimation. These latter should be used throughout the course for quick understanding and checking of results, and also for sensible manipulation of applied calculations. The fact that a precise numerical answer is not usually required and may not even be consistent with the input and structure of the model is important. Reasonable "rounding off" procedures should be understood and used.

Section 4

SOME OVERALL OBSERVATIONS

The Children

Anyone who reads our curriculum proposals, particularly for Grades 7-12, will certainly realize that the conference showed a far greater respect than is customary for the intellectual competence of students in the schools. This respect rests primarily on the fact that almost everyone who has tried to teach what have been generally considered advanced topics to children reports success. Such success does not necessarily imply, however, that a curriculum heavily loaded with mathematical ideas can be successful. Most of the experiments represent quick thrusts into a restricted area of mathematics, and it is not at all clear that with very young students successful work in these accessible subjects will serve as a foundation for more advanced work. It is quite possible that the intercorrelation between the various topics in the student's mind will simply not develop sufficiently rapidly to make possible the schedule set forth below. If the observation of Piaget represented an intrinsic inability of young children, then this would indeed be the case, but Piaget himself does not claim that the difficulty is intrinsic, only that it is observable among children educated in the presently traditional manner. How much we can change is not a matter for debate but for experiment.

Any attempt to move mathematics teaching in the direction here proposed will raise another important consideration. With the present rather slow pace, all but a few have been able to keep to the minimum standard. Hence it has been possible to teach arithmetic on a grade-by-grade basis. Although we believe that, when properly taught, everyone will be able to go faster, it seems quite likely that in an environment containing so many more ideas, differences in native ability will become more significant. The intercalation of enrichment material to attract the attention of the more gifted student may no longer suffice to keep children of the same age together, in which case it will become totally impractical to teach mathematics on a grade basis. We have no solution to the problems this raises, but we note that many school systems are already experimenting with ungraded classes.

Another aspect of our respect for the child's intellect leads to the following suggestion. If a mathematical idea seems not to be understood by the children, then we should recheck our own ideas to be sure that we have in fact presented the ideas correctly. We believe no mathematical idea can be presented clearly unless it is also presented correctly. Various comments to this effect have been made concerning sophisticated topics like geometry and algebra, but even at the most elementary level, in the teaching of counting, we find inadequacies of presentation. Reviewing a number of the most recent books for kindergarten and first grade, we found that they often emphasize the notion of a one-to-one correspondence between two sets. Very good. But none of the books made any effort to bring out the point that the existence of a one-to-one correspondence between two sets is independent of the way in which the correspondence is set up. We have no evidence that children are in any doubt on this point, but it seems clear that counting must be an entirely meaningless process to any child who does not understand this fact. Even if it were certain that each child understands the principle, there would be much merit in making it verbally explicit. Of course, we do not argue that a formal mathematical proof should be given. But each child should be aware that this is the fact on which counting and ultimately the whole of the arithmetic rest.

As adults we have grown so accustomed to such basic facts that we fail to appreciate that they are not necessarily obvious. Indeed, some "geometrically obvious" facts are now doubted insofar as the real world is concerned. In many cases, it has required the greatest geniuses to see the significance of (or to question) the obvious. To avoid major errors of presentation, we believe that the serious thought of our best mathematical minds will be required in the design of even the kindergarten curriculum.

The Teacher

As we explained in the introduction of this report, our recommendations were developed without any regard to the question of staff resources or the problem of teacher training. Partly as a result of this, our program makes extremely heavy demands on the teacher.

This is obvious as early as the seventh grade, in which we propose algebra and geometry on a higher intellectual level than those now commonly taught in the ninth and tenth grades. It becomes more obvious in the later grades of high school: here our courses would include more content, on a higher level of sophistication, than most colleges now offer. (It should be recalled that the number of college mathematics teachers with Ph.D.'s is less than twice the number of colleges. *The majority of colleges have none.*) For this reason our high school program will not be feasible, as a part of mass education, until the number of students reaching the present M.A. level in mathematics has increased by an order of magnitude. This is, however, not an unreasonable hope. College mathematics departments are increasingly overloaded by sharply rising enrollments in advanced courses. If this tendency continues, then the overwork of the present generation of college teachers will contribute to the solution of the problems of the next generation.

In less obvious ways, our program for the first six grades is equally demanding. In the first place, the pre-mathematical topics range rather widely. It appears that many teachers do a better job than the state of their own knowledge would give anybody a right to expect: the craft of the schoolmaster, plus a reasonably good textbook that the schoolmaster can manage to keep up with, goes a long way in place of the basic knowledge which we would like to regard as normal. With due allowance for this principle, the gulf between the demands that we propose to make on teachers and the qualifications of the present generation of teachers is very wide. It is common knowledge that the average elementary teacher knows, at most, formal arithmetic narrowly construed, and some of those now entering the profession have a proficiency in arithmetic which is below the eighth-grade norm.

For a different and less obvious reason, our program makes even heavier demands on teachers than our remarks so far would suggest. Almost any drill material can be taught by almost any good drillmaster. But we do not propose to teach by drill. At every level we propose to present mathematics as the pursuit of the truth by a process of inquiry; we propose to elicit all the insight and all the creative responses that the student is capable of. Just how much they will turn out to be capable of we do not know; but the experience of a few bold experimenters amply proves that the present apparent limits on the insight and creativity of children are being set by the materials presented to them, and not by the native talent of the children.

Obviously the task of leadership in a Socratic inquiry is harder, as a matter of pedagogic technique, than the task of teaching by drill. It also requires a much deeper mastery, in a purely mathematical sense, of the subject matter. It requires that the teacher recognize, as quickly as possible, the validity of unexpected responses. It requires also that the teacher be able to tell when a response which is not correct as stated nevertheless includes a valid idea, so that the discussion can then be guided in the direction of the valid idea. Such work requires far deeper understanding than lecturing does. Without deep understanding, the only responses whose validity the teacher can recognize are the expected responses given in "the book." Often the result is that at moments when the students are most entitled to feel proud of themselves they get called down and marked wrong. The only way that we can see to eliminate such behaviour in well-intentioned teachers is to alleviate the purely intellectual incomprehension which forces them into it.

This means that an ambitious program will have to be developed, designed to provide an understanding not merely of the material to be presented in the classroom but also of its immediate mathematical context and of its surrounding folklore. Fragments of algebra and number theory are going to appear in the classroom; the teacher should therefore know more than fragments of both. The same is true of geometry and probability. We propose, as a rough criterion, that the teacher should know enough about the mathematical background to be pleased, instead of being embarrassed, by nearly all the questions that an eager and able student is likely to ask.

Thus the training of teachers involves a threshold phenomenon. The point is that we propose to teach ideas. The mechanical processes of arithmetic can be taught, after a fashion, by rote and drill. Ideas cannot be. If the elementary teachers in the next generation do not understand the ideas that they are supposed to be teaching, then the results may easily be worse than the results that we are getting now.

There is a special reason why the training of teachers should be rapid and highly coordinated. Probably the easiest part of our program to put into the classroom is its first part, in the first three grades. If this is done, then children will develop a set of expectations which will be disappointed by teachers in later grades unless the training of teachers has kept pace with the progress of the children themselves. This sort of problem is already arising in the weaker colleges. In most of these the mathematics staff is very poorly trained, and hardly competent to teach advanced courses, or to teach any course at all in a modern spirit. Meanwhile, high school programs are rapidly improving, quantitatively and qualitatively; and the result is that some freshmen arrive on campus already knowing a large portion of the material that their college teachers can teach them. For this reason, much of our work will be wasted unless curricular development and teacher training keep pace with each other.

The Problems

Problem material in a mathematics course takes a long time to read, and an even longer time to write. Mainly for this reason, superficial efforts by both authors and reviewers usually put heavy emphasis on the text material proper. We believe that this is the reverse of sound procedure in each case: It is the problem material that the student lives with. Obviously this is true in mere drill books, in which the text is all but vacuous. It remains true when the text is intellectually substantial. Psychologically speaking, mathematics is something which people *do;* it is not something that they receive in a passive sense. We believe that this principle holds even at very high maturity levels, where it may not seem to. To a mature mathematician, no learning process is passive: he is aware of the possibility of alternative treatments; he connects up the new material with things that he already knows; he translates what he hears into his private language; and so on.

Mass education in mathematics now depends on textbooks. Probably this will be true for a long time. If so, the problem material should be considered at least as important as the text proper; and it should get at least half of the time and attention of the authors. Even if textbooks become less important, as the qualifications of teachers rise, problem sequences will continue to be crucial; and the job of composing them is so time-consuming that classroom teachers cannot compose them, one at a time, as they go along.

We therefore believe that the composition of problem sequences is one of the largest and one of the most urgent tasks in curricular development.

Obviously problems should illustrate and reinforce the ideas in the corresponding portion of the text. They also should provide a continual review. In the earlier grades of our program, moreover, they have a special and less familiar purpose. It will be recalled that a basic strategy of our courses in grades K through six is to produce arithmetical skill as a side effect of the study of mathematical ideas. The problems at this stage, therefore, should not merely furnish practice in computation, but should also furnish the student with good reasons for wanting to know the answers to arithmetical problems.

It is possible that extensive use may be made of "discovery exercises" in the sense of Beberman and of "discovery problems" in a still more ambitious sense. In Beberman's discovery exercises, the student is provided with opportunities to observe mathematical principles (e.g., the commutative laws of addition and multiplication) in order to use them as short-cuts in dealing with problems which might otherwise be solved by brute force. More difficult discovery problems can be used to introduce mathematical ideas which will soon thereafter be explicitly presented in the text. We have hopes that a great deal can be accomplished by this device. The difficulty of teaching heterogeneous classes may be reduced if less able students do routine problems on the basis of ideas and techniques that have been taught to them, while the brilliant students discover (or at least *seem* to discover) most of the mathematics for themselves. If this happens, then the brilliant student will have a view of mathematics, as a field for creative activity, which is now very rare in the schools. Students in the middle range of ability, who solve some but not all of the discovery problems, would take a different view of the eventual expositions and have a new sort of respect for the ideas presented. Our hope would be to convey continually to the student that every mathematical idea appeared first as the solution of some problem by some person.

Testing, Testers, and Tests

It is an unfortunate fact that the difficulty of designing valid educational tests rises sharply with the intellectual level of the material being tested. Thus it is trivial to find out whether the student knows the date of the Pilgrims' landing on Plymouth Rock; but it is far more difficult to find out whether he understands what they did and why during the rest of the seventeenth century. Similarly, in mathematics, it is very easy to find out how fast and how accurately a student can multiply two three-digit numbers, but it is much harder to measure the extent and the depth of his grasp of mathematical ideas.

This means that the problem of testing is going to be more difficult for our program than for conventional ones, and that it may be hard to "prove" by testing that the program has the merits claimed for it. At present, the process of judging new programs by tests is a sort of a game: the new programs give odds, as it were, by showing that they produce just as good results on standard tests as the conventional programs that the tests were designed to fit; and their proponents then argue that the new content must be net profit. Ideally, we should do better than this by measuring the net profit as accurately as possible, and finding out which students have gotten how much of it.

For these reasons, we believe that the problem of devising subtle and valid tests is very hard and very urgent. Independent of the problem of testing, however, is the problem presented by some of the testers themselves. There is a school of thought which holds, in effect, that whatever tests have been devised at a given moment are *ipso facto* valid. At least, this is what people must be thinking when they say that educational objectives that can be tested "scientifically" are well-defined, and that the rest is mysticism. Opposing this view we insist that teaching comes first, and that testing follows it and measures it — or at least tries to. We are shocked by the callow empiricism which confers honorary validity on whatever measurement techniques it has managed to devise, and confers honorary nonexistence on all aspects of the human psyche that have not yet been explained to an IBM punching machine.

Moreover, even valid tests can be partially invalidated, and may distort the curriculum, if the schools attempt to coach students for them in a direct way. The Scholastic Aptitude Test (SAT), for example, includes a long vocabulary test. In fact, a person's vocabulary in his native language is a very good measure of his educational level, because it is normally a by-product of reading about things other than words. At present, however, the SAT is so important to students' futures that in some schools English vocabulary is abnormally the direct result of coaching in vocabulary itself, done simply in an effort to raise SAT scores. Similarly, there are schools which spend several months of the twelfth grade working through old mathematics tests. It is easy to understand the pressures which create such temptations. But the problems of test design become almost impossible when teachers who ought to be teaching devote themselves instead to a massive and systematic effort to falsify the results of the tests.

We are especially worried about this sort of thing in our own program. If the teachers turn their classes into cram sessions directed at the aspects of the program which most lend themselves to objective testing, then the real spirit of the program would be lost.

To sum up: we hope that in the future educational tests will deserve more respect, but will actually get less respect and less attention than they do now.

Section 5

CURRICULUM FOR ELEMENTARY SCHOOL (K-6)

As we have indicated above, the objective for mathematics instruction in the elementary grades is familiarity with the real number system and the main ideas of geometry. Familiarity requires in addition acquaintance with some of the principal applications of real numbers and geometry. We outline here in some detail several topics which may be suitable. We have divided the elementary school roughly in two parts comprising grades K through 2 and grades 3 through 6, but it must be understood that this is only to give an indication of the kind of performance we expect from very young children.

The brevity of the present discussion derives from a respect for the reader's time. The authors request that this brevity should not be mistaken either for dogmatism or for oversimplification. The evolutionary future of mathematics instruction in American schools is a complex matter indeed, which involves many uncertainties, difficulties, compromises, and points of disagreement. The present discussion is, we hope, a first approximation to an intelligent plan for guiding this evolution in a generally wise direction.

In particular, nothing in this report is intended to exclude any better ideas that may arise elsewhere. Where ideas are concerned, it seems clear that the evolution of the curriculum should be inclusive and receptive, rather than exclusive and dogmatic.

The topics below are organized along mathematical lines; it is not intended that they should necessarily be taken up in the order indicated.

The Earliest Grades, K through 2

The Real Number System

The child usually learns quite early and easily how to count. As soon as he is able to count, he can begin to get experience with the number line. This line can be regarded from the first as a representation for all real numbers, even though the child will not be immediately able to give sophisticated names for most of these numbers. Nonetheless, he *can* speak of "a little more than three" and "a little less than five," and he can give a temporary name, like ☆, to any number.

Early experiences in studying numbers should be designed to give insight into the mathematical properties of the real number system. *They probably should not focus on the learning of algorithms*, which will come considerably later in the curriculum.

Experiences with numbers using concrete objects which can be counted, measured, and arranged in various ways should have a prominent place in the first years of school.

These early experiences can be surprisingly creative, can involve the child actively, and can deal with matters of honest mathematical merit. All of the following appear to be possible very early in the child's career:

(1) Experiences with "grouping" that will establish the idea of place-value numerals to various bases, including base 10.

(2) Extensive use of zero as a number, not merely as a symbol.

(3) The idea of inequalities, and the symbols $<$ and $>$.

(4) The idea of transitivity of $<$. (This can be built into game situations where the child is asked to guess a "secret" number from a set of carefully devised clues, and so on.)

(5) The number line, including negatives from the beginning.

(6) Use of rulers with 0 at the center.

(7) Use of the number line in the "transitivity" games mentioned above.

(8) Use of fractions with small denominators to name additional points on the number line.

(9) Use of the idea of "the neighborhood of a point" on the number line; relation to inequalities.

(10) Use of the number line to introduce decimals by change of scale.

(11) The use of "crossed" number lines to form Cartesian coordinates; various games of strategy using Cartesian coordinates.

(12) Use of an additive slide rule, including both positive and negative numbers.

(13) Physical interpretations of addition and multiplication, including original interpretations made up by the children themselves (such as 2×4 represented by 4 washers on each of 2 pegs, or 2 stacks of 4 washers each, or a 2×4 rectangular array of washers [or dots, or pebbles, etc.], of 2 washers of each of 4 different colors, 4 washers of each of 2 different colors, and so on).

(14) Questions that lead the children to "discover" the commutative nature of addition and multiplication.

(15) Multiplication of a number "a little bit more than three" by a number "a little bit less than five."

(16) Division with remainder using, for example, the pattern: "$20 \div 8$" means
"If we have 20 dots, how many rows of 8 will there be?"

. . . .
. . . .
. . . .

Answer: "2 whole rows and 4 left over."

(17) Division with fractional answers. $20 \div 8 = 2\frac{1}{2}$

(18) Recognition of inverse operations.

(19) Use of $\square$ as a variable in simple algebraic problems.

(20) Experience with Cartesian coordinates, including both discrete and continuous cases, graphs of linear functions, graphs of functions obtained empirically, simple extrapolation ("When will the plant be seven inches tall?"), and so on. Various games of strategy played on Cartesian coordinates, etc. Graph of $\square + \triangle = 10$, in connection with learning "addition facts," etc.

Geometry

Geometry is to be studied together with arithmetic and algebra from kindergarten on. Some of the aims of this study are to develop the planar and spatial intuition of the pupil, to afford a source of visualization for arithmetic and algebra, and to serve as a model for that branch of natural science which investigates physical space by mathematical methods. The geometric portion of the curriculum seems to be the most difficult to design. Therefore the geometry discussed here for grades K, 1, and 2 represents a far more tentative groping than was the case for the work in real numbers described earlier.

The earliest grades should include topics and experiences like these:

(1) Identifying and naming various geometric configurations.

(2) Visualization, such as cutting out cardboard to construct 3-dimensional figures, where the child is shown the 3-dimensional figure and asked to find his own way to cut the 2-dimensional paper or cardboard.

(3) The additive property of area, closely integrated with the operation of multiplication.

(4) Symmetry and other transformations leaving geometrical figures invariant. The fact that a line or circle can be slid into itself. The symmetries of squares and rectangles, circles, ellipses, etc., and solid figures like

spheres, cubes, tetrahedra, etc. This study could be facilitated with mirrors, paper folding, etc.

(5) Possibly the explicit recognition of the group property in the preceding.

(6) Use of straightedge and compass to do the standard geometric constructions such as comparing segments or angles, bisecting a segment or angle, etc.

(7) Similar figures, both plane and solid, starting from small and enlarged photographs, etc.

Logic and Set Theory; Function

The logic of everyday life is often appallingly sloppy. While nothing approaching formal logic is presently recommended for the earliest grades, it does seem likely that general use of good logic by teachers will pay dividends in terms of the logic subsequently used by the children.

The ideas of *set* and *function* should be introduced as soon as possible. In the earliest grades:

(1) Number as a property of finite sets.

(2) The comparison of cardinals of finite sets with emphasis on the fact that the result is independent of which mapping function is used.

(3) Numerical functions determined by very simple formulas.

(4) The use of logical statements to determine certain sets. For example, games like *Twenty Questions* in which the set of possibilities is successively narrowed through the answers to yes-no questions.

(5) Familiarity with both true and false statements as a source of information.

Applications

The work with real numbers, described above, can be closely related to work in "science" and "applications," such as:

(1) Measurement and units, in cases of length, area, volume, weight, time, money, temperature, etc.

(2) Use of various measuring instruments, such as rulers, calipers, scales, etc.

(3) Physical interpretations of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$.

(4) Physical interpretations of negative numbers in relation to an arbitrary reference point (as 0° Centigrade, or altitude at sea level, or the lobby floor for an elevator, etc.).

(5) Physical embodiments of inequalities in length, weight, etc., again using games where the child must use the transitive property, or the fact that $a > c$ implies $a + b > c + b$.

(6) Estimating orders of magnitude, with applications related to physics, economics, history, sociology, etc.

(7) Visual display of data on Cartesian coordinates, such as recording growth of seedlings by daily measurement of height, or graph of temperature vs. time for hourly readings of a thermometer.

General Remarks

The concepts described above are uneven in difficulty. Some of them can probably be introduced in nursery school, and no doubt should be. Others may prove to be impossible by second grade, either because of their intrinsic difficulty, or because of the large amount of material to be covered.

In nursery school, K-2, and, indeed, at all elementary school levels, the present suggestions assume a general pattern of pre-mathematics to introduce each new topic, to be followed later by as much formal study as may be appropriate. The pre-mathematics at each level will serve to provide a background of experiences, and to help develop clear concepts for the work of the following months or following years. Nearly all the preceding suggestions for K-2 fall under the general heading of "pre-mathematics." (Probably "pre-mathematics" and "formal mathematical study" are not dichotomous categories, but extremes on a continuous interval, with increasingly detailed "informal" study leading gradually into "formal" study.)

A comment might be made on the role of *physical equipment* in the earliest grades. Whether one thinks in terms of the pre-mathematical experiences that are embodied in the manipulation of physical materials, whether one regards these physical objects as aids to effective communication between teacher and child, or whether one regards them as attractive objects that increase motivation, the conclusion is inescapable that children can study mathematics more satisfactorily when each child has abundant opportunity to manipulate suitable physical objects. Possible candidates include blocks of appropriate sizes, plastic washers and pegboards, rulers, compasses, French curves, circles divided into equal sections, graph paper, paper ruled into columns to help the child line up digits in column addition, geometric shapes cut out of wood or heavy cardboard, pebbles for counting, numerals cut out of wood or cardboard, circular protractors, and so on.

One important general principle appears to be this: wherever possible, the child should have some intrinsic criterion for deciding the correctness of answers, without requiring recourse to authority. In the present work, this bed-rock foundation is generally provided by the fundamental operation of *counting*. The child's slogan might well be: when in doubt, *count!*

In more advanced work in later grades, solving problems by several different methods, recognition of patterns, and even the use of simple logic will play the role of a foundation for deciding correctness without recourse to authority. Physical interpretations also help fill this role.

Clearly, *counting* and *guessing* both have important roles to play, and should be neither excluded from the curriculum nor unduly restricted. Rather, they should be put in their proper place. Counting, at this stage, should probably have a fundamental place. The place for guessing is more confined, but not nonexistent.

In the general area of problem solving, the primary emphasis should be on *understanding the problem*, with secondary emphasis on carrying out the calculations to get the "answer." For example, after the concept of multiplication has been studied, it is appropriate to consider problems involving multiplication of large numbers, even though the actual computations appearing are beyond the algorithmic skill of the pupil. (They would, presumably, be carried out by the teacher or with the aid of a desk calculator.) When computing machines of all sizes are widely available, surely it is more important to know when to multiply than how to multiply.

Grades 3 through 6

In these four grades we should continue pursuit of the main objective, familiarity with the real number system and geometry. At the same time we must start pre-mathematical experiences aiming towards the more sophisticated work in high school.

The Real Number System

(1) Commutative, associative, and distributive laws. The multiplicative property of 1. The additive and multiplicative properties of 0.

(2) Arithmetic of signed numbers.

(3) For comparison purposes

- (a) Modular arithmetic, based on primes and on non-primes.
- (b) Finite fields.
- (c) Study of 2×2 matrices; comparison with real numbers; isomorphism of a subset of 2×2 matrices with real numbers; divisors of zero; identities for matrices; simple matrix inverses (particularly in relation to the idea of inverse operations and the nonexistence of a multiplicative inverse for zero). Possible use of matrices to introduce complex numbers.

(4) Prime numbers and factoring. Euclidean algorithm, greatest common divisor.

(5) Elementary Diophantine problems.

(6) Integral exponents, both positive and negative.

(7) The arithmetic of inequalities.

(8) Absolute value.

(9) Explicit study of the decimal system of notation including comparison with other bases and mixed bases (e.g. miles, yards, feet, inches).

(10) Study of algorithms for adding, subtracting, multiplying, and dividing both integers and rational numbers, including "original" algorithms made up by the children themselves.

(11) Methods for checking and verifying correctness of answers without recourse to the teacher.

(12) Familiarity with certain "short cut" calculations that serve to illustrate basic properties of numbers or of numerals.

(13) The use of desk calculators, slide rules, and tables.

(14) Interpolation.

(15) Considerable experience in approximations, estimates, "scientific notation," and orders of magnitude.

(16) Effect of "round-off" and significant figures.

(17) Knowledge of the distinction between rational and irrational numbers.

(18) Study of decimals, for rational and irrational numbers.

(19) Square roots, inequalities such as $1.41 < \sqrt{2} < 1.42$.

(20) The Archimedean property and the density of the rational numbers including terminating decimals.

(21) Nested intervals.

(22) Computation with numbers given approximately. (e.g. find π^2 given π.)

(23) Simple algebraic equations and inequalities.

Perhaps no area of discussion brought out more viewpoints than the question of how the multiplication of signed numbers should be introduced. The simple route via the distributive law was considered, but a closely related approach was more popular. One observes that the definition of multiplication is ours to make but only one definition will have desirable properties. Others favored an experimental approach involving negative weights on balance boards, etc. Still others favored the "negative debt" approach. Even the immediate introduction of signed area was proposed. It seems quite likely that all approaches should be tried, since there will probably be much variation from student to student concerning what is convincing. The question is evidently not mathematical; it is purely pedagogic. The problem is to convey the "inner reasonableness" of $(-1) \times (-1) = +1$.

Geometry

In the later grades of elementary school, relatively little pure geometry would be introduced, but more experience with the topics from K-2 would be built up. The pictorial representation of sets with Venn diagrams and the graphing of elementary functions using Cartesian coordinates would be continued. In addition, there is much of value in the suggestions put forward by educators in Holland, and described by Freudenthal in an article in the *Mathematics Student* (1956, pp. 82-97), in which many geometrical questions are motivated by problems concerning solid bodies and the ways they fit together. New topics might include:

(1) Mensuration formulas for familiar figures.

(2) Approximate determination of π by measuring circles.

(3) Conic sections.

(4) Equation determining a straight line.

(5) Cartesian coordinates in 3 dimensions.

(6) Polar coordinates.

(7) Latitude and longitude.

(8) Symmetry of more sophisticated figures (e.g. wallpaper).

(9) Similar figures interpreted as scale models and problems of indirect measurement.

(10) Vectors, possibly including some statics and linear kinematics.

(11) Symmetry argument for the congruence of the base angles of an isosceles triangle.

Logic and Foundations

(1) The vocabulary of elementary logic: true, false, implication, double implication, contradiction.

(2) Truth tables for simplest connectives.

(3) The common schemes of inference:

$$\frac{P \rightarrow Q \text{ and } P}{Q} \qquad \frac{P \rightarrow Q \text{ and } \sim Q}{\sim P}$$

(4) Simple uses of mathematical induction.

(5) Preliminary recognition of the roles of axioms and theorems in relation to the real number system.

(6) Simple uses of logical implication or "derivations" in studying algorithms, more complicated identities, etc.

(7) Elements of flow charting.

(8) Simple uses of indirect proof, in studying inequalities, proving $\sqrt{2}$ irrational, and so on.

(9) Study of sets, relations, and functions. Graphs of relations and functions, both discrete and continuous. Graphs of empirically determined functions.

(10) Explicit study of the relation of open sentences and their truth sets.

(11) The concepts of isomorphism and transformation.

The common practice of traditional 9th-grade algebra has been to ignore truth values and questions of implication. True statements, false statements, and statements of unknown truth value are jumbled together on pages of writing, related by implication in an unspecified way. The authors apparently hoped that sooner or later something good would happen. As algebraic notions enter the elementary grades, truth values should be explicitly discussed.

As the child grows, he learns more and more fully what constitutes a mathematical proof. Specious proofs presented to him early in his education may tend to block his progress toward understanding what a proof really is. This must be avoided. If a discussion is *not* a proof, it should not masquerade under false colors.

On the other hand, experience in making honest proofs can and probably should begin in the elementary grades, especially in algebraic situations. While extensive formal study of logic in the elementary grades is not favored by most mathematicians, it is hardly possible to do anything in the direction of mathematical proofs without the vocabulary of logic and explicit recognition of the inference schemes. The feasibility of such study has already been demonstrated by classroom experimentation.

Work towards indirect proof will build on the experience with false statements started in the early grades. The study of inequalities can be particularly useful here. Exciting experience with implication, uniqueness, contradiction, etc., can be built into games that can be played in the classroom. Children in elementary school may be able to achieve some comprehension of mathematical induction, especially in relatively simple forms, such as the calculation of explicit terms of a sequence defined by recursion.

Theory of Real Functions

(1) Intuitive consideration of infinite sequences of real numbers.

(2) The logarithm function, built up by interpolation, from approximate equalities like $2^{10} \sim 10^3$ (see Appendix B).

* (3) Trigonometric functions.

(4) Partial and linear orderings, with applications.

(5) Linearity and convexity.

We have in mind an informal, experimental approach to the trigonometric functions similar to the approach to logarithms. We imagine defining the functions on the whole line using the intuitive concept of a point moving uniformly on the unit circle. We could then study many of the qualitative aspects of the functions, such as maxima, minima, and periods. We can relate them to problems in harmonic motion and the oscillation of pendulums, possibly even to wave motion if suitable equipment is available. Approximate values of the functions for acute angles could be obtained by measuring carefully-drawn right triangles. These tables could be extended by symmetry, periodicity, and interpolation (either by linear calculation or by applying a French curve to the graph). Applications to the usual problems in indirect measurement are immediate. The appropriate depth of penetration and level of sophistication for elementary school experiences with trigonometric functions is probably a matter to be determined on the basis of actual teaching experiences.

The treatment of trigonometric functions sketched here and the treatment of logarithms outlined in more detail in Appendix B was motivated by several considerations. Tables of these functions will be much more meaningful to a student who has worked hard to build his own table, even if the latter goes only to two decimals. The process of making the table will concentrate much attention on the definition of the functions. Moreover, the student who does the job conscientiously will acquire a good intuitive grip on their qualitative properties. In particular, the table of logarithms involves a good deal of arithmetic; this will afford plenty of practice in a context which many students will regard as worthwhile. Finally, experience at this level in calculating tables can only heighten the student's appreciation of the easier methods that become available through Taylor's series later. The same remarks apply to the empirical determination of π.

Applications

Because a good deal of science can, and probably will, be introduced into primary school, more applications of mathematics will be possible in the upper grades.

Some of the most important applications involve probability and statistics, which we conceive as purely empirical subjects at this level. The study should begin with

(1) Empirical investigation of many-times-repeated random events.

(2) Arithmetic study of how the ultimate stabilization of observed relative frequency occurs through "swamping."

These investigations should be applied to the problems of measurement in connection with all science experiments.

There will be many applications of geometry to problems of indirect measurement and to areas and volumes. The use of graphs, interpolation, and extrapolation should lead to the idea of rate of change. Scientific notation becomes a great convenience when dealing with the very large and the very small numbers which occur in astronomy or the atomic realm.

Even in these grades it seems desirable to emphasize the notion of a "model" which captures only a part, even an approximate part, of the real situation.

Longer Projects for Students

In addition to all the explicit topics mentioned above, it is important that each child get some experience with the more extended aspects of discussion. As the student progresses in mathematics, he will come increasingly to encounter long protracted discussions or solutions of problems. At some point in the future he will meet problems that take hours, days, or weeks for complete discussion, sometimes requiring a long sequence of lemmas or partial solutions. It is not clear *a priori* how one can best prepare for this at the elementary school level, but whatever preparation can be made would be worthwhile.

One possibility is that there be short topics to be studied entirely independently. These should probably be organized at the appropriate level and written up as pamphlets. In many respects, these pamphlets would be like enrichment material for the particularly able student, but every student should do some independent work requiring more extensive effort than the usual assigned problems. If a variety of topics were available, then students could have a choice of project; this in itself would probably increase the level of interest.

Section 6

CURRICULUM FOR GRADES 7-12

General Remarks

The child emerging from the sixth-grade program proposed in this report will have a thorough grounding in both arithmetic and intuitive geometry and will now be ready to begin algebra and a more deductive geometry. In a pre-mathematical form, he has learned to graph linear functions and to solve simple systems of linear equations. Experiments now being conducted in the eleventh and twelfth grades indicate success in teaching such subjects as calculus, linear algebra, and probability in high school. With the superior training students will receive in the first six grades, it should be possible to develop all three of these topics in addition to algebra and geometry.

It seems clear that the topics proposed for the high school have become the foundations upon which applications to the sciences, engineering, and mathematics itself are built. It was further felt that some of these topics have become part of what every person should know in order to understand the complex world in which he lives. In addition to the basic algebraic skills, an educated person should know about such things as the likelihood of an event, the reliability of statistical reports, rates of change and averages. The problem of students dropping out enters our considerations now and provisions are made to give those who do leave the mainstream the kind of mathematics that will be useful to them and which will develop in them an appreciation of the structure and the power of mathematics.

A modern development in mathematical science which is advancing rapidly at present is the electronic computer. Its impact in science and technology is already very great. Its continuing development may very well have implications for the high school mathematics curriculum which cannot be foreseen at present. We have, therefore, placed emphasis on the central concepts of iterative processes and flow diagramming.

Moreover, the concrete treatment that we propose and the avoidance of a loose use of symbolism ought to ease the transition to applied mathematics in general and to computing in particular. For example, loose calculus deals with "variables" (in a Leibnizian sense) rather than functions, while both rigorous analysis and computing deal with functions rather than variables: you cannot explain to transistors the meaning of the symbol dy/y. We believe that this principle applies rather broadly; significant applications of mathematics require, at least, that clear intuitive grasp of the concepts which is best gained from precise formulations.

Two somewhat different approaches to a curriculum for grades seven through twelve emerged from the deliberations of the two groups considering the problem. In the pages which follow, topical outlines of each of the two approaches will be presented. An attempt will then be made to point out some of the common features of these two programs, after which specific features of each will be discussed separately.

Topical Outline of the First Proposal for Grades 7–12

Grades 7 and 8. Algebra and Probability

Part I. Algebra

- a. Review of properties of numbers.
- * b. Ring of polynomials over a field, polynomial functions.
- * c. Rational forms and functions.
- d. Quadratic equations, iterative procedures, difference polynomials.
- e. Euclidean algorithm, Diophantine equations, modular arithmetic.
- * f. Complex numbers as residue classes of polynomials mod $x^2 + 1$.
- * g. Derivative of a polynomial.

Part II. Probability

- a. Review of earlier experience with probability, basic definitions in probability theory for finite sample spaces.
- b. Sampling from a finite population, unordered sampling, ordered sampling without and with replacement.
- c. Conditional probability, independence.
- d. Random variables and their distributions.
- e. Expectation and variance, Chebychev's inequality.
- f. Joint distribution of random variables and independent variables.
- * g. Poisson distribution.
- * h. Statistical estimation and hypothesis testing.

Grade 9. Geometry

a. Intuitive and synthetic geometry to the Pythagorean theorem.
b. Cartesian plane and space, lines, planes, circles, and spheres.
c. Motions in Euclidean space, groups of motions, matrices and linear transformations, vectors, linear independence.
d. Rotations in the plane and in space.
e. Complex numbers and rotations in the plane, trigonometry.
f. Vector space of n dimensions.
g. Conics and quadrics, projective geometry.
h. Transformation laws, tensors.

Grade 10. Geometry, Topology, and Algebra

Part I. Geometry and Topology of the Complex Plane

a. Geometry of complex numbers, linear fractional transformations, mappings by elementary functions, stereographic projection.
b. Neighborhoods, continuous functions.
c. Fundamental theorem of algebra, winding number, location of roots.

Part II. Linear Algebra

a. Simultaneous linear equations, linear mappings, matrices.
b. Subspaces and factor spaces.
c. Equivalences of matrices, change of bases, and matrices of a transformation.
d. Triangular form of matrices, invariant subspaces, diagonal form of symmetric matrices and quadratic forms.
e. Determinants.
f. Cayley-Hamilton theorem.
g. Inner products and orthogonal transformations.

Grades 11 and 12. Analysis

a. Real numbers.
b. Sequences and series.
c. Probability for countable sample spaces.
d. Limits of functions, continuous functions.
e. Derivatives, Mean Value theorem, antiderivatives, simple differential equations.
f. Exponential and logarithmic functions, trigonometric functions.
g. Linear differential equations with constant coefficients.
h. Differential geometry of curves.
i. Definite and indefinite integrals, areas.

j. Taylor series, indeterminate forms.
k. Probability for continuous distributions.
l. Calculus for functions of several variables.

Topical Outline of the Second Proposal for Grades 7–12

Grades 7 and 8. Algebra, Geometry, and Probability

Part I. Algebra and Geometry

a. Review of properties of numbers.
b. Logic of open statements and quantifiers, linear equations and inequalities, systems of n linear equations in m variables, flow charts.
c. Logic of formal proofs discussed, axiomatic development of Euclidean geometry of two and three dimensions.
d. Analytic geometry, lines, circles, parabolas, quadratic equations.
e. Functions — composite, inverse; functional equations.
f. Polynomial functions.
g. Geometry of circles and spheres, trigonometric functions.
h. Vectors in two and three dimensions.
i. Complex numbers, possible introduction to logarithms.

Part II. Probability

a. Binomial theorem, combinatorial problems.
b. Review of earlier experience with probability, basic definitions in probability theory for finite sample spaces.
c. Sampling from a finite population, unordered sampling, ordered sampling with and without replacement.
d. Conditional probability, independence.
e. Random variables and their distributions.
f. Expectation and variance, Chebychev's inequality.
g. Joint distribution of random variables and independent variables.
h. Poisson distribution.
i. Statistical estimation and hypothesis testing.

Grade 9. Algebra, Geometry, and Calculus

Part I. Introductory Calculus

a. Limits of functions and continuity (lightly).
b. Derivative, slope of tangent line, velocity.

c. Derivatives of polynomials, sines and cosines, sums and products.
d. Applications, curve tracing, maxima and minima, rate problems, Newton's method for finding roots of polynomials.
e. Antiderivatives, definite integral and area.
f. The Mean Value theorem, Fundamental Theorem of Calculus, applications.

Part II. Algebra and Geometry

a. Volumes of figures (prisms, pyramids, cylinders, cones, spheres).
b. Linear equations and planes.
c. Rigid motions of space, linear and affine transformations, matrices, determinants, solutions of linear systems.
d. Quadratic forms, diagonalization, conics, and quadrics.
e. Numerical methods.

Grade 10. Analysis, Probability, and Algebra

a. Infinite sequences and series of real and complex numbers, absolute and unconditional convergence, power series.
b. Probability for countable sample spaces.
c. Linear algebra, subspaces, bases, dimension, coordinates, linear transformations and matrices, systems of equations, determinants, quadratic forms, diagonalization.

Grades 11 and 12. Analysis

a. Limits of functions, continuity.
b. Rules for differentiation.
c. Mean Value theorem and its consequences.
d. Definite integral, its existence for continuous functions.
e. Logarithmic and exponential functions, trigonometric functions, hyperbolic functions, applications.
f. Techniques of integration.
g. Taylor series, indeterminate forms, interpolation, difference methods.
h. Differential equations.
i. Probability for continuous distributions.
j. Differential geometry of curves in space.
k. Multidimensional differential and integral calculus.
l. Boundary value problems, Fourier series.
m. Integral equations, Green's functions, variational and iterational methods.

Common Features of the Two Programs

Geometry

It was recognized that there are many different routes to follow in teaching geometry and that each has its advantages. One possibility that was often mentioned and which received considerable support was via a pseudosynthetic approach until the Pythagorean theorem and similarity are reached. The term "pseudosynthetic" refers to an axiom system which includes an axiom endowing lines with a coordinate system and angles with measure. It thus presupposes the properties of the real numbers, with which the student became familiar in his first six years. The student's knowledge of the Pythagorean theorem and similarity lays the foundation on which to introduce coordinate systems in the plane and in space, allowing one to use the methods of analytic geometry both in developing algebra and analysis, and in the further study of geometry itself. In view of the fact that the student will have been studying facts about geometrical figures and constructions parallel to his study of arithmetic in grades one to six, it should be possible to reach a readiness for analytic geometry in about one quarter of a school year.

In the further development of geometry, the motions of Euclidean space are to be treated, leading to the introduction of linear transformations and matrices and the eventual study of linear algebra. This led to the suggestion that the study of geometry could be based on transformations of the plane or of space. It was felt that this suggestion warranted further study to see if an approach could be written up and whether it was appropriate at this level.

The student entering the seventh grade has been exposed to very few formal proofs in his first six years. He has probably seen arguments to convince him that there is no largest integer, no largest prime, that $\sqrt{2}$ is irrational, etc. From the seventh grade on, increasingly more proofs will be encountered and parts of the curriculum, such as the geometry, will be developed deductively from some axiom system. Where will he learn how to construct a logical proof? Several views were expressed but all felt that some attention should be given to guiding the student towards an understanding of what constitutes a proof and the usual rules of logical reasoning used in mathematics.

It will have been observed in Section 5, under *Logic and Foundations*, that the treatment of formal logic is very meagre. We do not know how thorough the treatment of logic should be. Since we do not propose to teach logic as a subject in its own right, the problem involved is pedagogic and hence pragmatic. Prolonged experimentation might lead to any of the following conclusions: (1) From the beginning, in the first grade, quantifications would be made explicit, first in colloquial English and later in the familiar shorthand. This may lead the student to such a good grasp of logic, on an informal basis, that a formal treatment would never be necessary. (2) It is possible, on the other hand, that once the students became aware of logical processes, they would insist on the sort of clarification that only a formal treatment can furnish. (3) It is also possible that refined logic can be conveyed inductively, by example, in the treatment of the substance of algebra and geometry.

The resolution of these questions depends, we believe, on student reactions; and our reservations of judgment spring from our inability to predict these. If our program in the first six grades achieves its objectives, then in the seventh and eighth grades we shall be dealing with a student mentality which is at present very rare indeed.

Algebra

The study of algebra in the seventh grade could begin with a review of the rules (axioms) governing the real numbers. These would naturally include the commutative, associative, distributive, and cancellation laws, the properties of 0 and 1, and the Archimedean property. The student should be aware that his "familiar numbers" do satisfy these axioms. The members of the conference were divided in their views about how algebra should be developed from this point. The contention centered around the question of whether to introduce general algebraic structures from which the special results can be drawn. This led to the construction of two curricula based upon different philosophies. The curriculum which is presented first was formulated by a committee assigned to this task. When their report was presented to the group as a whole, there was widespread disagreement and the curriculum which is presented second was drawn up by another committee to bring out some of the areas of disagreement.

The first program begins with the study of polynomial forms over a field. After the algebraic properties of this ring of polynomials is studied, applications are made to the study of polynomial functions, difference operators, and complex numbers. To anticipate the calculus, the idea of tangent of a polynomial graph is introduced as the line with second degree contact.

The members of the group framing the second plan differed from the first group on their approach to the study of polynomials. The emphasis in their study was placed on polynomial functions instead of the algebra of rings of polynomial forms. The trigonometric functions are studied earlier and the complex numbers are introduced as ordered pairs of real numbers instead of as residue classes in the ring of polynomials mod $x^2 + 1$.

Probability

Both programs include several rounds of probability. It is presupposed that the student will have as a first round an intuitive concept of the probability of an event from the pre-mathematical material in the lower grades. Based upon this, the second round studies finite event sets using the techniques of algebra. The third round is tied in with the study of infinite sequences and deals with countable event sets, while the fourth round treats the continuous case using calculus.

The Calculus

The approach to the calculus differs somewhat in the two programs. The first waits until the last two years to present a logically complete course, based on a thorough understanding of the limit concept. The second proposal gives a heuristic and brief introduction to the calculus in the 9th grade, and returns to a more complete study of the calculus in the last two years. The justification presented for each approach can be summarized as follows:

The student, who has already developed some taste for mathematical rigor, will be dissatisfied with only half the story in calculus when the fundamental concepts are not carefully defined and precisely used. Because he cannot carry his arguments back to well-defined concepts, he will not fully understand what calculus is about. Finally, one often forms wrong impressions in an intuitive approach which are hard to "unlearn" later, and the lustre is worn off the subject when one has to return to it later to tie together loose ends.

On the other side, it was felt that such concepts as rate of change and average of a function should reach a larger number of people than those who complete all twelve years of mathematics. From the point of view of general culture, the calculus was one of the greatest achievements of human endeavor, and an appreciation of the power of the calculus can be gained from a heuristic introduction. This experience should be made as widespread as possible by an early introductory course. Calculus has traditionally been taught successfully first on a heuristic level where an appreciation of the scope of the subject is developed. Efforts to base such a development on a thorough understanding of limits for all students may be too confining. It may confuse the student if he must immediately apply the newly learned concept of limit to the other topics in the calculus. A second round on limits and the ultimate union of the two subjects in the third round may be more satisfactory. Finally, it may well be that the student will be less inhibited in his use of the calculus (and, perhaps, mathematics as a whole) if he does not see everything presented in its ultimate refinement but also makes some bold leaps ahead without worrying about details and then later looks back to polish his efforts.

Linear Algebra

Linear spaces are studied in both programs, and in both, they are encountered in two rounds. The first encounter is in connection with motions of Euclidean space and the presentation is restricted to finite dimensional linear spaces composed of pairs, triples, or perhaps n-tuples of real numbers. The second round takes up the general study of linear spaces.

In both programs, no presumption is made concerning the amount of time necessary for each unit in the curriculum. This can only be determined after some experimentation is carried out. Perhaps it will later be determined that certain topics should be reordered to adjust to pedagogical problems this conference did not consider. The programs presented here are two samples containing the kind of mathematics which the conference believed suitable as a goal toward which our present school curriculum should aim.

Some Specific Features of the Two Programs

First Proposal for Grades 7 and 8

Part I. Polynomial algebra

1. A formal statement of the C-A-D (commutative, associative, and distributive) laws and the recognition that the numbers so far encountered satisfy them.

2. The use of C-A-D justifies formulas such as $(x + y)(x - y) = x^2 - y^2$ if x and y are numbers. Emphasize that these formulas are valid for any objects which satisfy the C-A-D laws. For example, $(x + y)(z + w) = xz + yz + xw + yw$ is valid if x, y, z, and w are sets and $+$ denotes union and multiplication denotes intersection.

3. The four fundamental operations on specific polynomials (the term "polynomial" not completely defined at present) with rational coefficients as if the symbols represent numbers. Manipulation of parentheses. Introduction of rational functions. Fundamental algorithms for fractions.

4. Discussion of the fact that the algorithms for polynomials in one letter may be carried out by using the coefficients only and omitting the letter. Treatment of synthetic division for a linear factor. Comparison with numerical algorithms where carrying is unnecessary.

5. It is sometimes convenient to consider a polynomial such as $1 + 5x - 3x^2$ as the polynomial $1 + 5x - 3x^2 + 0x^3 + 0x^4 \ldots$. Because of statement 4, all manipulations on polynomials are equivalent to manipulations on infinite sequences such as $(1, 5, -3, 0, 0, \ldots)$.

6. To understand polynomials better, the numbers must be understood. Point out that the integers form a *ring* (define and discuss). This particular ring has no zero divisors and is an *integral domain* (define). The integral domain can be extended to the set of rationals. This set forms a *field* (define and discuss). The real numbers form a field larger than the rationals because $\sqrt{2}$ is irrational (prove). The numbers of the form $a + b\sqrt{2}$, a and b rational, form a field between the rationals and the reals, and this field has the automorphism $\sqrt{2} \to -\sqrt{2}$. Point out that a number in this field is rational if, and only if, it is a fixed point of the automorphism; consequently, $(a + b\sqrt{2})(a - b\sqrt{2})$ must be rational (no calculation necessary) and this

is the basis for the method of rationalizing the denominator. Discuss the smallest field containing $a + b\sqrt{2} + c\sqrt{3}$, a, b, c, rational, and find the automorphisms. Show that the automorphisms form a group (discuss *very* lightly). Characterize fixed points and use to rationalize $(a + b\sqrt{2} + c\sqrt{3})^{-1}$.

7. Define a polynomial as an infinite sequence of elements from an integral domain with only a finite number of them being different from zero (there is a possibility of including formal power series). Define addition of two polynomials as addition of corresponding elements in the sequences, multiplication by an element of the integral domain as multiplication of every element in the sequence by that element. For convenience, the sequence $(1, 0, 0, \ldots)$ is identified with the number one, and the sequence $(0, 1, 0, 0, \ldots)$ is denoted by x. Point out that the original integral domain is embedded in the set of constant polynomials. Define multiplication of polynomials. Therefore, a polynomial may be written as $a_0 + a_1x + a_2x^2 \ldots$. The set of polynomials forms a ring. (The proofs of commutativity and associativity require the use of the Σ-notation. There is some feeling that this should be avoided; in which case, prove the properties for the first few powers of x, enough to make the theorem plausible, and warn the students that a proof is lacking.) Introduce the degree of the polynomial and prove the theorems about the degree of the sum and the degree of the product. Prove that the polynomial ring has no zero divisors.

8. To every polynomial Σa_kx^k and to every domain of numbers (this domain may be larger than the integral domain of coefficients) is associated a function f as follows: if r is in the domain, then $f(r) = \Sigma a_kr^k$. Graph the functions x^n, $(n = 0, 1, 2, 3, 4, 5)$ defined for all real numbers. Discuss the order of magnitude of a function and show that if $n > m$, $x^n = o(x^m)$ at $x = 0$, but $x^m = o(x^n)$ for very large values of x. Graph the function ax^n for $a > 0$ and $a < 0$, the function $ax + b$ and the function $ax^2 + bx + c$. Locate maxima or minima by completing the square and discuss applications.

9. Because the polynomial ring has no zero divisors, it can be extended to the ring of rational forms $p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials. A rational form defines a rational function over the domain of the real numbers excluding the zeros of the polynomial function $q(x)$. Graph the functions ax^{-1}, $(x - a)^{-2}$, $x(x - a)^{-1}$, $(ax + b)(cx + d)^{-1}$ for different rational values of a, b, c, d.

10. The problem of finding the zeros of a polynomial function in a given domain is discussed. Solve $ax + b = cx + d$ and two and three simultaneous linear equations. Word problems.

★11. Solve $ax^2 + bx + c = 0$ by factoring. Discuss factoring over different rings. Factor polynomials of higher degree by synthetic division, using the fact that a root must divide the constant term. Solve $ax^2 + bx + c = 0$ by completing the square — no formula. Point out that if the discriminant is negative, the equation has no real root.

12. The need for a square-root algorithm leads to a discussion of iterative processes. The solution of $ax + b = cx + d$ can be found graphically by determining the intersection of the graphs of the functions $ax + b$ and $cx + d$. An iterative process (not really necessary, but very instructive) may be set up by writing $x_{n+1} = a^{-1}((cx_n + d) - b)$ or by using $x_{n+1} = c^{-1}((ax_n + b) - d)$. A graphical discussion indicates convergence of one or the other according as $a^{-1}c$ or $c^{-1}a$ is less than one. Try iteration on quadratics; e.g., $x_{n+1} = -b^{-1}(ax_n^2 + c)$ or $x_{n+1} = -a^{-1}(b + cx_n^{-1})$, etc. Discuss the convergence graphically. Introduce the square root algorithm $x_{n+1} = (x_n + ax_n^{-1})/2$ and discuss graphically. Use inequalities and bounds to estimate errors and give a plausible proof of convergence. Discuss infinite decimals and nested intervals determining real numbers. Give sum of geometric series and show that periodic decimals define rational numbers. Discuss the effect of round-off procedures on iterative methods.

13. Iterative methods suggest the study of sequences u_n given *a priori* or defined by recurrence relations. Introduce the operator E such that $Eu_n = u_{n+1}$ and Δ such that $\Delta u_n = u_{n+1} - u_n$. Show that if u_n is a polynomial function of degree k over the integers, then Δu_n is a polynomial function of degree $k - 1$. Introduce polynomials in Δ and E. Since $E = 1 + \Delta$, obtain $E^k = (1 + \Delta)^k$. Use this to find Σn, Σn^2, Σn^3, $\Sigma n(n - 1)$, etc. Suggest use in interpolation. Consider $\Delta^n = (E - 1)^n$. Show that $E - r$ annihilates the function r^n. Solve $au_{n+2} + bu_{n+1} + cu_n = 0$, i.e., $(aE^2 + bE + c)u_n = 0$, by factoring $aE^2 + bE + c$.

14. Review the Euclidean algorithm for integers and its connection with the g. c. d. (greatest common divisor). Show how this implies a solution for the Diophantine equation $ax + by = c$. This leads to considerations of residue classes mod a or mod b. Show that the residue classes mod p, p prime, form a field, but mod n, n composite, form a ring with zero divisors. Show that the polynomial $x^p - x$ over a p-field corresponds to the zero function; consequently the same function may have distinct polynomial representations. Discuss the invertible elements in the ring of integers mod n, and the Euler ϕ-function.

15. Discuss the Euclidean algorithm for polynomials over a field. Obtain the remainder theorem and use it to factor polynomials. Consider congruences mod $(x^2 - 2)$ and show that over the rationals, a field isomorphic to the previous field $a + b\sqrt{2}$ is obtained. Use this idea to solve $x^2 + 1 = 0$ by considering congruences mod $(x^2 + 1)$. Show that a field is obtained with the automorphism $i \to -i$. Do the algorithms for complex numbers. Prove that a polynomial over the reals must have complex-conjugate roots. State the fundamental theorem of algebra. Prove that a polynomial function has at most n zeros and that there is a one-to-one correspondence between polynomials and polynomial functions. Prove the unique factorization theorem.

Discuss the partial fraction expansion for the case of real factors (linear and quadratic).

16. Consider the order of contact of a polynomial $p(x) = \Sigma a_k x^k$ and a linear function $ax + b$. If they intersect at $x = \alpha$, then $p(\alpha) = a\alpha + b$ and $p(x) - (ax + b) = (x - \alpha)q(x)$ where $q(x)$ is a polynomial. Determine the value of a for second-order contact, i.e., such that $p(x) - (ax + b) = (x - \alpha)^2 r(x)$ where $r(x)$ is a polynomial. Define $p'(x) = a$. The graph of the linear function $ax + b$ is tangent to the polynomial function $p(x)$. Obtain a standard formula for the derivative of a polynomial. Show that at a local minimum or maximum of the function $p(x)$, the derivative is equal to zero.

Part II. Probability

1. Review briefly the experience of the student with random sampling in the first six years. State the formal definition of a finite probability model. It is a finite nonempty set $\mathcal{E}$ (the event set or sample space) to each element e (sample event) of which is assigned a non-negative number $p(e)$ (the probability of e) with $\Sigma p(e) = 1$. The model represents a random experiment: each simple event e representing one of the simple results of the experiment and $p(e)$ representing the long-run frequency of occurrence of that result when the experiment is repeated under controlled conditions. Subsets E of $\mathcal{E}$ are called events and we define the probability $P(E) = \sum_{e \in E} p(e)$.

2. The notions of sampling from a finite population provide good illustrations that are conceptually simple, applicationally important, and subject to classroom experiments.

(i) Unordered sampling: from a well-mixed box of N similar marbles, labelled in some way, s are drawn at a grab. The possible results are the different samples, the number of which is denoted by $\binom{N}{s}$, each represented by one point in $\mathcal{E}$. Symmetry suggests that we assign to each sample the same probability $1\Big/\binom{N}{s}$. The student should construct a table for $\binom{N}{s}$ by Pascal's triangle. Sampling experiments with small N's should be conducted to check the approximate equality of frequencies.

(ii) Ordered sampling without replacement: now the s marbles are drawn out one at a time and the identity of the marble obtained on each draw is noted. Symmetry again attributes to each of the $N^{(s)}$ samples the same probability. If order is ignored, we obtain a sample of type (i), again with equal probabilities. Each unordered sample may be placed in $s!$ orders, and hence $N^{(s)} = s!\binom{N}{s} = N(N - 1)\ldots(N - s + 1)$.

The basic theorem is the equivalence law: if $1 \leq i_1 < \ldots < i_k \leq s$, the marbles in orders $i_1, \ldots, i_k$ behave like an ordered sample of size k. The special case $s = N$ gives interesting material (matching, etc.).

(iii) Ordered sampling with replacement: s marbles are again drawn one at a time, but after each draw the marble is replaced and the box remixed before the next draw. N^s results are distinguishable, with equal probabilities again assigned. This is perhaps the simplest context in which to discuss independence and product models, and experimental verification for at least $s = 2$ is essential here. The identity of this model when $N = 2$ and $N = 6$ with that discussed for fair coins and dice is noted. $N = 10$ leads to a discussion of random digits, a table of which should be constructed by the class and used in drawing random samples.

(iv) Observe the similarity of (ii) and (iii) for small samples and a large population.

3. Conditional models are introduced by considering experiments whose results are partially revealed. Formally, with each subset E of $\mathcal{E}$ having positive probability is associated a model $(\mathcal{E}, q)$ defined by $q(e) = 0$ if $e \notin E$ and $q(e)$ is proportional to $p(e)$ if $e \in E$. This is a convenient place to discuss the multiplication law and the formal notion of independence.

4. Often we are interested in the value of some quantity determined by the result of a random experiment. This leads to the definition of a random variable as a real-valued function whose domain is $\mathcal{E}$. The distribution of the random variable Z consists of its value-set $\mathfrak{z}$, each number z in which has associated with it the probability $q(z) = P[e\colon Z(e) = z]$ or $P(Z = z)$ for short; formally $(\mathfrak{z}, q)$ is another probability model. Interesting examples of random variables and their distributions are provided by having marbles of two colors, say r red ones, and considering the number of red marbles in the sample. This gives the hypergeometric random variable D in (i) and (ii) (these are both important) and the binomial random variable B (with rational success probability $p = r/N$) in (iii). The formulas $p(D = d) = \dfrac{\binom{r}{d}\binom{N-r}{s-d}}{\binom{N}{s}}$ and $P(B = b) = \binom{s}{b} p^b(1-p)^{s-b}$ are derived. The notion of histogram is introduced and several of these distributions are graphed. Other important random variables are indicators I, where $\mathfrak{z} = [0, 1]$ and constants, where $Z(e) = c$ for all e. Study distributions of simple functions of random variables.

5. Expectation is motivated by long-run average value, analogous to probability as long-run frequency. Formally $E(Z) = \sum_{e \in E} Z(e)p(e)$. Prove that $E(Z) = \Sigma z P(Z = z)$. The algebra of expectation is developed. The expectation of D and of B are derived. The relation to center of gravity is used to help motivate the concept of $E(Z)$ as a center of the distribution of Z.

6. Joint distribution of random variables and the notion of independent variables may now come in. Theorem: If Z and W are defined on the factors of a product model, then Z and W are independent.

7. Variance is introduced as a measure of the spread of a distribution about its expectation, and its laws developed, i.e., $\text{Var}(Z) = E([Z - E(Z)]^2)$. This leads to $\text{Var}(B) = sp(1 - p)$. (Covariance and its laws, leading to $\text{Var}(D) = \frac{N - s}{N - 1} sp(1 - p)$ is optional.) Derive Chebychev's inequality and interpret it as a weak law of large numbers.

8. The Poisson distribution can be introduced as an approximation to the binomial when p is small, rather than as a distribution in its own right (which involves the discretely infinite case). The student can compute Poisson tables by taking $P(0)$ as unknown, using the ratio of successive terms and finding $P(0)$ from the requirement of total probability 1. Relate the Poisson distribution to the matching problem. The normal approximation is presented without proof as a computing device; all probabilities obtained approximately with its aid could in principle be obtained exactly with the formula given.

9. Statistical ideas may be simply introduced in the context of point estimation. In model (ii) for sampling, assign to each marble a real value: $\nu_1, \nu_2, \ldots, \nu_n$. Then if $Y_1, Y_2, \ldots, Y_s$ are the values on the s marbles drawn, $E(Y_1) = \ldots E(Y_s) = \bar{\nu}$ and $\bar{y}$ is a reasonable estimate for $\bar{\nu}$. Notion of an unbiased estimate. B/s is an unbiased estimate for p. $\text{Var}(B/s) = p(1 - p)/s$. Discuss the relation with the original concept of probability as long-run frequency. (Variance of $\bar{Y}$ and D depend upon the covariance option.) Stratified sampling provides important and interesting material at this point, leading to notions of experimental design, optimum design, etc. Hypothesis testing may be presented, not only using the binomial and hypergeometric examples, but also notions of comparative experiments and the sign and Wilcoxon tests. Matched pairs and other instances of blocking are tied in naturally with stratified sampling.

First Proposal for Grade 9. Geometry

1. Intuitive geometry and synthetic geometry leading up to the Pythagorean theorem. Elementary facts about triangles, circles, and planes. Theorems in congruence and similarity. Euclidean space is described as a set of points with a notion of distance. Facts about congruence are interpreted as homogeneity and isotropy of space.

2. Cartesian coordinates are introduced as mappings from points in space to number pairs (x_1, x_2) or number triples (x_1, x_2, x_3). Distance is expressed by means of the Pythagorean theorem. Theorems about straight lines, planes, circles, and spheres are proved by analytic methods.

3. Motions in Euclidean space are interpreted as linear nonhomogeneous transformations of the coordinates. Treat the two- and three-dimensional case in detail and point out the formal extension to higher dimensions. Symmetric treatment of motions in space relative to fixed coordinate axes and

change of coordinates due to motions of the reference system. Introduce matrices to describe homogeneous linear transformations.

4. The motions are shown to form a group which is generated by special motions: rotations, translations, and reflections. Translations are shown to satisfy the axioms of a linear space. Translations are related to free vectors in Euclidean space. Given one fixed point, the attached vector space is identified with Euclidean space. Position vectors are fixed vectors attached to the origin; free vectors are generated by translations. Solve geometric problems (like centroids in triangles and tetrahedra) by direct vector methods. Concept of linear independence. Equations of lines and planes in vector form.

5. Describe rotations in the plane as a one-parameter group and exhibit the form of the rotation matrix

$$\begin{pmatrix} r & \sqrt{1 - r^2} \\ -\sqrt{1 - r^2} & r \end{pmatrix}$$

Rotations in space form a three-parameter group. Eulerian angles and reduction of general solution to two plane rotations. Prove the s.s.s. congruence theorem again analytically by rigid motions.

6. Show one-to-one correspondence between complex numbers and number pairs. Give the geometric interpretation as points in the plane and of multiplication of complex numbers as a linear transformation. Isomorphism between rotations and multiplication by complex numbers of absolute value one. Complex numbers and matrices. Study of the trigonometric functions and addition theorems by use of $e^{i\theta}$ and its geometry. Roots of unity. Powers of roots of unity and the connection with congruences modulo n. Orthogonality relations $\frac{1}{n} \sum_{k=1}^{n} \zeta_p^{-k} \zeta_q^{k} = \delta_{pq}$, where $\zeta_1, \ldots, \zeta_n$ are the nth roots of unity. Trigonometric interpolation.

7. Study vector spaces in n dimensions. Prove invariance of dimension under linear transformation and change of basis. Discuss the form of affine transformations and properties invariant under them, like concurrence, etc. Point out that affine geometry can be carried out without a distance concept. Introduce a norm in the vector space by a definite quadratic form and the scalar product of two vectors by the corresponding polar form. Use the Gram-Schmidt process to find orthonormal basis vectors. Now scalar products, lengths, angles, and orthogonality can be expressed in usual form and the connection with the Pythagorean theorem is made. Discuss the scalar product also in oblique coordinate systems. Define orthogonal transformations as the subgroup of the affine transformations which leaves the metric form unchanged.

8. Discuss the conics and quadrics by means of principal axes. Maximum-minimum properties of various axes. Reflection properties of conics and

significance of foci. Tangent lines and tangent planes. Homogeneous coordinates and some basic projective geometry. Concept of duality. Polarity and dual interpretation of geometric laws.

9. Transformations of dual vectors and quadratic forms which describe fixed quadrics under change of coordinates. Covariance and contravariance, invariants, and tensors. Covariant and contravariant coordinates for oblique axes.

First Proposal for Grade 10

Part I. The Geometry and Topology of the Complex Plane

1. The geometric interpretation of the algebraic manipulations with complex numbers and some geometry of the plane.
   (i) Geometric interpretation of various elementary functions such as $f(z) = z^n, f(z) = \frac{1}{z^n}$, etc. The complex conjugate.
   (ii) Linear fractional transformations.
   (iii) Stereographic projection, projectivities, and possible connections with projective geometry.
2. Topology of the plane.
   (i) Nearness, neighborhoods, and the definition of continuous function using inverse images of neighborhoods rather than ϵ, δ definition. Examples of continuous functions such as addition, multiplication, absolute value, and complex conjugate.
   (ii) Heuristic argument to show that a continuous function on a compact set into the real numbers has a minimum.
3. A combination of some topological notions and algebra.
   (i) The "fundamental" theorem of algebra: using the fact that a polynomial function $p(x)$ is a continuous function, that $|p(x)|$ takes on a minimum value, and certain algebraic manipulations with polynomials to show that if $|p(z_0)| \neq 0$, then $|p(z)| < |p(z_0)|$ for some z near z_0.
   (ii) Winding number.
   (iii) Location of roots of a polynomial.

Part II. Linear Algebra

1. Simultaneous linear equations as motivation for discussing linear mappings from one space to another, kernels, subspaces, matrices, rank of linear mappings and matrices, etc.

2. Factor spaces. The difference between the concepts of complementary subspace and factor space.

3. Discussion of various equivalences on matrices such as PAQ, PAP^{-1}, etc., in terms of the connections between choosing a basis for the vector space and the matrices of transformations.

4. The triangular form of matrices by elementary transformations.

5. Subspaces invariant under a transformation.

6. Reduction to diagonal form of symmetric matrices (associated with quadratic forms). This can be demonstrated for instance by using a sequence of two-dimensional rotations (a la Jacobi).

7. Determinants (either axiomatically or with exterior algebra).

8. Characteristic equations and the Cayley-Hamilton theorem.

9. Inner products and the Gram-Schmidt orthogonalization process, orthogonal transformations.

First Proposal for Grades 11 and 12. Analysis

1. Describe the set of real numbers as an ordered field which has the property that every bounded monotone sequence has a limit. Prove that the real numbers are Archimedean.

2. Introduce the notion of a sequence having a property ultimately, e.g., a sequence is ultimately bounded by M if all the terms of the sequence except for a finite number are less than or equal to M. A sequence converges to a limit L if ultimately the sequence is contained in any neighborhood of L. Point out that the space of convergent sequences is a vector space over the reals and that the limit of the sequence is a linear functional on the space. Consider series of positive terms and standard tests for convergence such as comparison, rates, grouping. Discuss series Σn^{-p} for $p > 0$. Emphasize order of magnitude ideas in the use of comparison tests. Discuss alternating series and show that partial sums give estimates of error. Introduce absolute convergence and show that an absolutely convergent series is convergent. Consider convergence of sequences and series of complex numbers. Emphasize that the notion of convergence depends on the topology only and not on the metric. Discuss the convergence of power series both in the real and the complex case. Point out that the power series absolutely convergent in a fixed circle form a ring.

3. As an application of sequences and series, study probability for countably infinite event sets. Discuss countably additive measures and check the validity of theorems proved in the previous round for finite event sets. Show, by considering the uniform distribution of the first n points, that a sequence of distributions can converge pointwise but still not converge to a distribution. Nevertheless, show that if $\frac{a_n}{n} \to \alpha$ and $\frac{b_n}{n} \to \beta$, then $P(a_n < x \leq b_n) \to \beta - \alpha$. Discuss the Petersburg paradox. Define the expectation of a random variable Z to exist if the series $\sum_{\nu=1}^{\infty} Z(e_\nu)p(e_\nu)$ converges absolutely, and similarly define the variance. Give examples where the expectation exists but the vari-

ance does not. Waiting-time problems lead to the negative binomial distribution; e.g., how many times must one toss a coin until the second head comes up. Markov chain processes (as time permits) including use of matrices, and possibly proving some fixed-point and limit theorems.

4. Define the limit of $f(x)$ as x approaches a by using deleted neighborhoods. Prove the theorems about limits of sums, products, and quotients. Define a continuous function of both real and complex variables. Emphasize that this concept is topological and not metrical by considering mappings from one neighborhood space to another. Illustrate by mappings from the plane and space into R and into vector spaces of 2 and 3 dimensions. Use bisection of intervals and the nested-interval theorem to prove that a continuous function has the intermediate-value property and a maximum in a closed interval. Also a continuous function maps a bounded closed interval onto a bounded closed interval. Prove the usual facts about sums, products, quotients, and compositions of continuous functions.

5. Review the derivatives of polynomials as obtained in earlier grades. Indicate that *this* derivative can be obtained by the limit of difference quotients and can be interpreted as the slope of the tangent to the graph of the function. Define the derivative of a function at a point. Obtain the algebraic properties of the derivation operator and the chain rule. Discuss derivatives of rational functions. Do the usual problems on tangents, velocity, and acceleration. Prove Rolle's theorem and the mean value theorem. Show that the derivative is zero at interior local extreme points and apply to maximum-minimum problems for rational functions. Define the antiderivative of a function and show that it is unique up to an additive constant. Use this to find distance as a function of time if velocity or acceleration is given. Prove the inverse function theorem for C^1 function f at points x such that $f'(x) \neq 0$. Use this to differentiate algebraic functions. Discuss the differential equations $y' = px^{-1}y$ for rational p and $y' = y^p$ for rational $p \neq 1$.

6. Consider $y' = y$ to obtain the exponential function. Show that the differential equation $f' = f$ with the side condition $f(0) = 1$ determines a unique function $f(x)$. Show that if g satisfies only $g' = g$, then $g(x) = g(0)f(x)$. Since $g(x) = f(x + a)$ satisfies $g' = g$, one has $f(x + a) = f(a)f(x)$. Conclude also that if such an f exists, it cannot vanish for any x. Use this method of successive approximations to construct a solution of $f' = f$, $f(0) = 1$, by introducing the recurrence relation $f_n'(x) = f_{n-1}(x)$, $f_n(0) = 1$, starting with the initial function $f_0(x) = 1$. This leads to $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ and to the study of the series $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. Study the convergence of this series by the ratio test and verify that $f(x)f(a) = f(x + a)$. Verify that for this series $f'(x) = f(x)$,

proving the existence of the solution to $f' = f$ with $f(0) = 1$. Invert to get the logarithmic function. Show that the functional equation for these functions indicates an isomorphism between the multiplicative group of positive real numbers and the additive group of real numbers. Obtain the antiderivative of x^{-1}.

7. The series for e^z obtained in (6) converges for complex values of the argument z and enables us to define $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$. Then for the real variable θ, $\cos \theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin \theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, and the usual differentiation formulas are easily established. Properties of e^z establish the basic trigonometric identities and addition formulas. Define π formally so that $\theta \to (\cos \theta, \sin \theta)$ is a continuous 1-1 mapping of $0 \leq \theta < 2\pi$ onto the unit circle.

8. Show that the trigonometric functions satisfy $y'' + y = 0$. Introduce the shift operator: for a polynomial $p(D)$ in the differential operator D and α a real number, $p(D)e^{\alpha x}y = e^{\alpha x}p(D + \alpha)y$. Use the shift operator to solve linear differential equations with constant coefficients, and some linear first-order and second-order systems with constant coefficients by using matrices.

9. Discuss the parametric equations of the motions of a particle in both two and three dimensions. The support of a motion is the point set through which the motion is made. A curve is an equivalence class under diffeomorphism of all motions having the same support. Discuss the tangent space and the differentials as elements of the dual space. Study the principal normal, the binormal, curvature, and torsion of a space curve and obtain the Frenet formulas. Investigate vector velocity and acceleration and normal and tangential acceleration. Prove that a planar curve with constant curvature is a circle and a space curve with zero torsion is planar.

10. Motivated by the area under a curve, prove that for a continuous function, the upper and lower sums converge to the integral. Show that the derivative of the indefinite integral is the integrand. Discuss additive set functions and their connections with integrals. Show heuristically that area and volume are additive set functions and use integrals to evaluate them. Do enough techniques of integration for definite and indefinite integrals to enable the student to use a table of integrals; e.g., substitution, partial fractions, integration by parts, etc. Obtain the mean value theorem for integrals. Define arc length and show that the circumference of a circle is π times the diameter. Define the area of a circle as its content and show that the circumference divided by the diameter is equal to the area divided by the square of the radius. Prove the integral test for convergence of series.

11. The fourth round of probability provides a nice application of the integral calculus. Define the distribution function of a random variable and

derive its basic properties. Review discrete distributions; e.g., binomial and Poisson distributions. Introduce moment-generating functions. Define density functions of continuous distributions. Give as examples the uniform, the Cauchy, the exponential, and the normal distributions. Distributions of two-dimensional random variables. Distributions of functions of random variables. Expectation and variance of a random variable. Covariance. The strong law of large numbers. Standardized random variables and a discussion of the central limit theorem. (It would be a great service to future generations if a simple proof of the central limit theorem, at least for some special distributions such as the binomial, were found that could be included at this level.)

12. Give Taylor's theorem with integral and derivative forms of the remainder. Consider Taylor's series and obtain the series for the standard functions such as $\log(1 + x)$ and $(1 + x)^p$. Apply to the study of maxima and minima and evaluation of indeterminate forms.

13. Consider directional derivatives of functions of several variables. Show that the directional derivative is the scalar product of the gradient and the unit direction vector. Partial derivatives and their use to solve maximum and minimum problems. Consider constrained maxima and minima and the use of Lagrange multipliers. Prove the implicit function theorem. Show that the only solution of the wave equation $u_{xx} = u_{yy}$ is $u = f(x + y) + g(x - y)$. Show that this solution represents a wave moving left and a wave moving right.

Second Proposal for Grades 7 and 8

1. Review the rules governing arithmetic, stressing again the C-A-D laws, the roles of 0 and 1, the absence of zero divisors, and the Archimedean property. State the axioms for an ordered field, and show that the "familiar" numbers satisfy these axioms.

2. Discuss open statements and solution sets. Clarify the use of the quantifiers "for all" and "for some" and their negation (some panelists felt that the quantifier "for all" and the question "for which?" were adequate for algebra). Solution of linear equations and inequalities using graphs as a visual aid (not in the spirit of analytic geometry yet). Solutions of systems of two linear equations in two unknowns, with graphical interpretation. Solution of systems of n equations in m unknowns (m and n specify small integers) by the Gaussian elimination method. This is perhaps a good place to bring in flow charts as a method for organizing material for computation. A discussion of the capabilities of a digital computer could give the student the justification for breaking a large computation into small components. In the remaining years of the curriculum, the idea of flow charts can be used whenever it is deemed helpful in organizing material for calculations.

3. As preparation for the axiomatic geometry that is to follow, the logic of formal proofs should be discussed. Some of the laws of inference should be set forth and illustrated either with the theorems of geometry as they come up in the subsequent development or by studying logic *per se*. Both points of view received support, and experimentation will determine what balance of the two is most satisfactory. It was felt that the amount of time devoted to the study of the logic *per se* should be limited to at most two or three weeks. The student should also be made aware of the different levels of formality on which proofs are presented.

4. An axiomatic development of Euclidean geometry of two and three dimensions. To save time in achieving the goal of laying a solid foundation for analytic geometry, one can here appeal to the student's familiarity with the real number line to introduce as an axiom that every line has a coordinate system. Include such topics as: lines, planes, separation, betweenness, angles, triangles, congruence, perpendicularity, parallelism, areas of polygonal regions, the Pythagorean theorem, similarity.

5. Elements of analytic geometry. Equations of lines, circles, and parabolas with horizontal and vertical axes. Properties of the parabola. Solution of quadratic equations by factoring and completion of square (no complex roots yet). Intersections of lines and parabolas. Cubics.

6. Define function from one set to another in various ways and illustrate (correspondence between sets, subset of Cartesian product, graph, table, formula, etc.). Domain and range of a function. The constant function, characteristic function, $[x]$, $|x|$, etc. Composition of functions, implicitly defined functions, inverse functions.

7. Polynomial functions. Degree of a polynomial. Behavior of a polynomial function for large values of x. Addition and multiplication of polynomials. The division algorithm and the remainder theorem. Factoring, making it clear by example that one must specify the set in which the coefficients may lie. Synthetic division both for dividing by a linear factor and for evaluating a polynomial. Some methods of location of roots, such as bisection method. Prove that a polynomial of degree n has at most n roots. Rational functions and the division of polynomials.

8. Geometry of the circle and sphere. Tangent line to circle and tangent plane to sphere, using both synthetic and analytic methods. Area and circumference of a circle. Lengths of arc on the circle.

9. Study of the trigonometric functions. Define radian measure. Define the trigonometric functions as functions of the real variable giving the length of arc on a unit circle. Make clear the distinction between the three different sine functions: (a) $\sin x$ where x is in radians, (b) $\sin x$ where x is in degrees, and (c) $\sin x$ where x is a geometric angle (two rays with same endpoint). Stress that we shall study the function in (a). Obtain the trigonometric identi-

ties, using analytic geometry to prove the addition formulas. Prove that the trigonometric functions are not polynomials or rational functions by studying the zeros and growth. Discuss the solutions of trigonometric equations. Graphs of the trigonometric functions.

10. Vectors in the plane and in space as directed segments. Addition and multiplication by scalars. Inner product and angle between vectors. Projections and components. Use of coordinates.

11. Introduction of complex numbers as ordered pairs of real numbers and connection with vector arithmetic. Identification of $(a, 0)$ with the real number a. Observation that multiplication by $(-1, 0)$ rotates through an angle of π. Define multiplication by $(0, 1)$ as a rotation through an angle of $\frac{\pi}{2}$ about the origin. Write $(a, b) = a(1, 0) + b(0, 1)$ and extend multiplication to any pair of complex numbers. Show that the field axioms stated for real numbers are satisfied, but that we do not have an ordered field. Introduce absolute value and complex conjugate. Polar form and deMoivre's rule. Discuss factoring polynomials over the complex field.

12. Binomial theorem and binomial coefficients. Combinatorial problems. The study of probability for finite event sets as outlined in the second part of the 7-8 grades in the first proposal.

13. Some work with logarithms. If they were not developed arithmetically in the lower grades, one could now seek an isomorphism between the multiplication group of positive numbers and the addition group of reals: $L(xy) = L(x) + L(y)$. (See Appendix B.) The calculation can be done for different bases and the relations with exponents observed. To gain accuracy, one is led to consider powers of $(1 + t)$ for small t. This motivates the study of the limit $\left(1 + \frac{1}{n}\right)^n$, and using the binomial theorem one obtains a series for e. This is good pre-mathematical material to anticipate some of the later work on limits.

Second Proposal for Grade 9

1. This is to be a brief introduction to the calculus. Begin with a discussion of limits to give the student an intuitive feeling for the subject. The definitions of limit of a function and continuity can be formulated using boxes. The derivative as the limit of a difference quotient is immediately interpreted as a rate of change (velocity) and as the slope of a tangent line. This round of calculus will be limited to the study of the polynomials and the sine and cosine functions. (This leads to a convenient ring of functions closed under differentiation and integration.) Differentiation of sums and products (quotients, composite functions, and implicit functions need not be included at this stage to handle the functions designated above.) Applications to curve tracing,

maximum and minimum problems, rate problems. Newton's method for finding roots of polynomials. Point out that the value of a polynomial and its derivative can easily be found by synthetic division. Determination of distance from velocity, antiderivatives. The definition of area by rectangular approximations. The mean value theorem, pointing out what is being assumed in the proof. The fundamental theorem of calculus and its applications. Applications to finding areas and averages.

2. Further topics in plane and solid geometry, using analytic and synthetic methods. Volumes of figures, such as prisms, pyramids, cones, and spheres (use calculus here where possible). Linear equations and planes. Rigid motions of space. Linear and affine transformations. Matrix of a linear transformation using a given coordinate system. Algebra of matrices. Solutions of systems of 3 linear equations in 3 unknowns, using matrix inversion. Determinants. Quadratic forms in two or three variables. Reduction of a symmetric 2×2 or 3×3 matrix to diagonal form using orthogonal transformations and the geometrical interpretation. Study of conics and quadratic surfaces. Numerical methods for inversion and diagonalization of 2×2 and 3×3 matrices.

Second Proposal for Grade 10

1. Infinite sequences and series. Give definition of limit and prove basic theorems for sequences and series of complex numbers. It was generally agreed that the most suitable completeness axiom for the real numbers uses nested intervals. Study the convergence of series with positive terms, obtaining the usual tests for convergence. Alternating series. Absolute and conditional convergence. Power series in the complex plane, showing circle of convergence. Study geometric series; series for $\sin x$, $\cos x$ can be found by integration by parts and showing that the remainder tends to zero. Introduce the series for e^x and show from the series that e^x has the properties $e^x e^y = e^{x+y}$ and $\frac{d}{dx} e^x = e^x$. Obtain Euler's formula. Perhaps one could study the properties of e^x and $\log x$ more thoroughly at this point, taking up exponential growth problems.

2. As an application of infinite sequences and series, one should present the next round of probability theory, this time for countably infinite event sets. For a description of this material, see item 3 of the First Proposal for Grades 11 and 12 (p. 58).

3. The rest of this year is to be spent on the study of linear algebra. Based on the students' experience with 2- and 3-dimensional linear algebra in connection with solving linear equations and simplifying quadratic forms in the 9th grade, we proceed to consider the axioms for a vector space. As examples, consider 2- and 3-dimensional vector spaces, the spaces of n-tuples, of poly-

nomials, of trigonometric polynomials, of sequences, of bounded sequences, of convergent sequences, etc. Study subspaces, bases, and dimension. Restrict most of the remaining discussion to finite dimensional vector spaces. Coordinates. Linear transformations and their matrix representations. Algebra of matrices. Solutions of systems of linear equations. Determinants. Quadratic forms. Rank of a matrix. Invariant subspaces of a transformation. Orthogonal and unitary transformations (matrices). Diagonalization of symmetric or Hermitian matrices, associated with quadratic forms. Stress numerical methods for solving systems of equations and diagonalizing matrices.

Second Proposal for Grades 11 and 12

These two years are to be spent developing further the material in calculus, probability, and differential equations. This brief listing of the kind of topics that could be included indicates the scope of the course that is proposed. Recall that the student has already had approximately one semester of introductory calculus and about a half semester of infinite sequences, series, and their application to probability.

1. Limits of functions. If the notion of limit introduced in the 10th grade was presented properly (say, as suggested by McShane in his article "A Theory of Limits" in the MAA Studies in Analysis), one would not have to re-prove the limit theorems. Continuous functions and their basic properties. Recall that the completeness axiom for the real numbers was stated in terms of nested intervals. Prove that on closed bounded intervals, continuous functions assume a maximum and are uniformly continuous. Prove the intermediate value property.

2. Recall the definition of derivative. Rules of differentiating composite functions, implicit functions, inverse functions. Expand the class of functions which can be handled to include rational, algebraic, exponential, trigonometric functions and their inverses.

3. Mean Value Theorem and its consequences. Antiderivatives. Differentials. Solution of simple differential equations. Review application to extremal problems, curve tracing, and rate problems.

4. Define the definite integral as a limit of a sum. Obtain the properties of the definite integral. Prove the existence of the integral for monotonic functions and for continuous functions. Review the fundamental theorem of the calculus.

5. Study the logarithmic and exponential functions, starting from the definition of $\log x$ as an integral. Show the identity of $\log^{-1}$ with the function exp previously defined by the series. Hyperbolic functions. As an application of hyperbolic functions, one can study rotations in a space with an indefinite metric $dx^2 - dt^2$ (Lorentz transformations with one spatial and one time dimension).

Mention the connection with special relativity theory. Solve differential equations related to exponential growth.

6. Techniques of integration including substitution, integration by parts, partial fractions, and use of integral tables. Solve problems involving areas, arc lengths, simple volumes, and surface areas. Improper integrals.

7. Study the Taylor series of a function with remainder in derivative and integral forms. Indeterminate forms. Interpolation and difference methods, including Lagrange interpolation formula and Newton's formulas.

8. Linear differential equations of the first order. Some other special classes of first-order equations. Linear differential equations with constant coefficients can be studied by operator methods or by reducing to a system of first-order equations and then using matrix methods. Discuss the linear space of solutions using the methods developed in the linear algebra. Discuss general equations $y' = f(x, y)$ the iteration process, first for "solving" it numerically, and then to prove the existence and uniqueness of a solution. Discuss other numerical methods for solving differential equations and for evaluating integrals.

9. The fourth round of probability comes next. This is the same as the material in item 11 of grades 11 and 12 of the first proposal (pp. 60, 61).

10. Differential geometry of curves in 3-space. Tangent, principal normal, binomial vectors, curvature, torsion. Frenet-Serret formulas. Applications to problems in mechanics.

11. Directional derivatives, gradient vector, partial derivatives. Extremal problems without and with constraints. Lagrange multipliers, Jacobians, inverse and implicit functions. Elementary vector analysis.

12. Multiple integrals and iterated integrals. Areas and volumes. Multivariate analysis in probability as an example of the use of multiple integrals and moments.

13. Boundary value problems and orthogonal functions. Preliminary work with Fourier series. Partial differential equations solved by separation of variables. Use linear space interpretation of space of solutions. Physical applications.

14. Transform differential equations to integral equations. Green's function. Use of variational and iterational approximations.

Conclusion

A proper development of the school mathematics curriculum requires a constant dialogue in the whole mathematical community, with essential contributions from both research scholars and classroom teachers. Until recently, research scholars tended to neglect their part of this task. This is especially unfortunate because their natural province includes, at least in outline, the settings of goals in a philosophical sense and also includes various exploratory work which requires a research scholar's grasp of subject matter. It is true, of course, that any conception of goals needs to be adjusted to what turns out to be the limits of the possible, and that some exploratory ideas simply fail to work in the forms in which they are proposed. Nevertheless, the check of practicality should come second; and we believe that research mathematicians should continue the sort of work represented by this report.

The passage of time will, we hope, make this tentative statement of goals obsolete. We look forward to the continued work of our successors.

APPENDICES

The following appendices are a somewhat edited sample of the working papers that were produced at the conference. It is hoped that more of these will ultimately be made available to those interested.

Appendix A

PROBABILITY AND STATISTICS

Probability theory can be regarded simply as a branch of pure mathematics, consisting mainly of measure theory. In fact it is so regarded by many probabilists. But in the course recommended here it would be treated in a quite different spirit, as a branch of applied mathematics, or perhaps as an essentially extra-mathematical study, comparable to physics. It would be presented as an analysis of experience by mathematical methods, and the student would be encouraged to infer that events with very small probabilities are very unlikely to happen, and that hypotheses with very small probabilities are very likely to be false. In short, we propose that the subject be aimed toward the areas of statistical inference and decision.

By this we do not mean that fundamental concepts should be formulated in empirical terms. We mean merely that sample spaces should be regarded as descriptions of a reality external to pure mathematics.

We believe that probability theory, presented in this spirit, has a large contribution to make to liberal education. In the first place, it can raise the level of sophistication at which a person interprets what he sees in ordinary life, in which theorems are scarce and uncertainty is everywhere. In the second place, the fact that the problem of measuring degrees of certainty lends itself to mathematical analysis is in itself a matter of great philosophical import; and the moral is reinforced by the discovery that quite simple-looking problems have answers that startle common sense.

To achieve these objectives, a course in the junior high school need not use the more difficult mathematical techniques; finite sample spaces cover a lot of ground and permit the introduction of most of the fundamental ideas. We believe that this is the best introduction to the theory, even for a student who intends eventually to study the continuous case after calculus. The reason is that purely probabilistic concepts are deep and subtle in themselves, even when they are separated from the analytic techniques which will later be used in dealing with them. In a program of mass education, however, there is a different and even more compelling reason for introducing probability theory early. In every grade after elementary school a certain number of students are going to stop studying mathematics. For this reason, other things being equal, topics which have high value in liberal education deserve priority. In this case, other things are approximately equal: probability theory is not falsified by exposition in elementary terms; and its insertion does not disrupt the order of the curriculum, because it uses enough of the methods of the preceding courses to keep alive the skills that the student has already acquired.

We suggest that probability be taught in four doses through the curriculum:

1. In the elementary school, empirical study of the statistics of repeated chance events, coupled with some arithmetic study of the workings of the law of large numbers.

2. In junior high school, probability as an additive set function on finite sets. Conditional probability, independence, binomial distribution, expectation, variance, and some simple statistical tests.

3. In senior high school, after the first work on limits and series, probability as an additive set function on countable sets. Poisson distribution, law of large numbers, etc.

4. In senior high school, after integral calculus, probability as an additive set function of intervals on the line. Continuous distributions on the line and in several dimensions, normal distribution, limit theorems, etc.

The pre-mathematical experience in primary school is of fundamental importance. We imagine it might take shape along these lines:

1. A central position would be assigned to large-scale cooperative experiments by the entire class, as in recording 2,000 or more tosses of a thumbtack, irregular or several-sided solid (such as a short section not-too-carefully cut from a triangular ruler). We propose to begin with an asymmetric situation in which no "obvious" theory exists in hopes of avoiding a large inheritance of widely-believed fallacies and misconceptions. Later one might do experiments with the classical symmetrical situations: dice, coins, cards, etc.

2. By entirely empirical means, the class would observe that a single toss of a thumbtack is quite unpredictable; in a few tosses both the total number and the fractional proportion of "successes" are largely unpredictable; as we move to much larger sample sizes, the total number of successes becomes even more unstable, but the fractional proportion of successes gradually (indeed very gradually) assumes considerable stability. Why does this occur? Is it because of the popularly-assumed law of "compensation," or is it (as Feller has put it) because of the "swamping" effect of very large averages? What bases can we find, logically or empirically, for choosing between these alternative explanations?

3. At this level only rather simple measures of variability would be used, like range, interquartile range, average absolute deviation from the mean.

4. Application would be made immediately to the variability of the results of measurement. This would deepen the students' understanding of the nature of measurement. Moreover, it might serve as an antidote to the erroneous idea that "in mathematics there is always one *exactly* right answer."

5. Work on the statistical aspects of round-off error might be included or written up as a separate pamphlet for study by brighter students.

Appendix B

LOGARITHMS IN ELEMENTARY SCHOOL

We shall show how easy it is, using only techniques available in the elementary school, to construct a table of common logarithms accurate to 3D (i.e., 3 decimal places), indeed almost to 4D. Such a construction has several points to recommend it. The operations give an insight and sense of realism hard to obtain in other ways. The process provides useful experience in graphing, table construction and checking, and arithmetical practice in a simple context.

The motivation for seeking a logarithm function can be introduced through the study of exponents. Integral exponents show how to convert certain multiplication problems into addition. We can plausibly interpolate the function 2^x using linear or French curve interpolation and verify that the resulting inverse function does nearly satisfy

$$\text{(1)} \qquad L(xy) = L(x) + L(y)$$

We can also consider only the integral powers of 1.001 or of $\sqrt[10]{2}$ which are easily seen to be rather densely distributed along the interval [1, 10]. Such considerations make it entirely reasonable that there is an increasing function satisfying (1). We can readily see the advantage of such a function, so we set about calculating one.

There is, of course, a normalization factor to be chosen. We can proceed with base-2 logarithms, L(2) = 1, or go to common logarithms, L(10) = 1. There are advantages to either choice. Ultimately common logarithms are more useful, but the base 2 probably obviates the temptation to cheat by looking up the answers. In what follows we seek common logarithms, but there is no essential difference with another base.

Our immediate objective is to find the values of L for the integers from 1 to 20. It will be enough to discover the values for the prime integers: if we have L(2) and L(3), we can at once get L(6) = L(2) + L(3), etc. Thus the problem is reduced to finding L(2), L(3), L(5), L(7), L(11), L(13), L(17), and L(19). Indeed, since $2 \times 5 = 10$ and hence $L(5) = 1 - L(2)$, one of these drops out. Seven remain.

We know L for powers of 10. If we could find a power of 2, 3, . . . which was equal to a power of 10, we could find L for that prime. Is this possible? Even if the equality were only approximate, an approximate value would result. Thus $3^2 = 9$ is close to 10. If 3^2 were 10, we should have $L(3^2) = 2L(3)$ equal to $L(10) = 1$, or $L(3) = .5$. Since 9 is near 10, we may reasonably hope that $L(3) \sim .5$. Putting it another way, $3^2 = 10 \times .9$ or $L(3) = .5 + \frac{1}{2}L(.9)$. Since $L(1) = 0$ and .9 is near 1, L(.9) should be near 0 and $\frac{1}{2}L(.9)$ even nearer.

Can better combinations be found? The class should construct a table of low powers of small primes, and discover, for example, $2^{10} = 1024$, and $7^6 = 117{,}649$. Of these, the first is by far the better, not only because 1.024 is closer to 1 than 1.176 is but because we divide the error by 10 in the first case and by 6 in the second. We settle on

$$10L(2) = 3 + L(1.024)$$

as our starting point and get as first approximation $L(2) = .300$. This at once gives $L(4) = .600$, $L(5) = .700$, $L(8) = .900$, $L(16) = 1.200$, and $L(20) = 1.300$.

Once L(2) is available, we are in a much stronger position. Can we find a power of (say) 3 which is nearly equal to a product of powers of 2 and 10? Inspection of the prime-power table at once gives $3^4 = 81 \sim 2^3 \times 10$ so that $4L(3) = 1 + 3L(2) + L(1.0125)$. Ignoring $\frac{1}{4}L(1.0125)$ gives $L(3) = .475$. We now pick up L(6), L(9), L(12), L(15), and L(18).

When seeking L(7), we can use all values previously obtained. The relation $7^4 = 2401 = 2 \times 12 \times 100 \times 1.0004$ is excellent. Continuing, we may use $11^2 = 12 \times 10 \times 1.0083$; $13^3 = 2 \times 11 \times 10^2 \times .9986$; $17^2 = 2 \times 12^2 \times 1.0035$; $19^2 = 6^2 \times 10 \times 1.0028$. Thus we get the entries under "Preliminary L(x)" in the table on page 75.

These values should now be graphed, partly as a check, partly to see how L looks, and partly to stimulate improvements in the table. The table was derived under the assumption that when x is small, $L(1 + x)$ is near 0. Does that assumption appear to be justified? Since $11 = 1.1 \times 10$, $L(1.1) = .038$. Similarly $L(.9) = -.050$, $L(.8) = -.100$, $L(1.2) = .075$. Furthermore, since $95 = 19 \times 5$, $L(.95) = -.025$, and similarly, $L(.85) = -.075$, $L(1.05) = .019$. Inspection of the graph suggests that our assumption is reasonable. Further, it suggests that when x is small, $L(1 + x) \sim .44x$. This relation now easily permits us to improve the table! Our first relation was

$$L(2) = .3 + \tfrac{1}{10}L(1.024).$$

x	Preliminary L(x)	Corrected L(x)	Error in 4th place	x	Preliminary L(x)	Corrected L(x)	Error in 4th place
1	.000	.0000	0	11	1.038	1.0416	2
2	.300	.3011	1	12	1.075	1.0794	2
3	.475	.4772	1	13	1.113	1.1140	1
4	.600	.6022	1	14	1.144	1.1463	2
5	.700	.6989	−1	15	1.175	1.1761	0
6	.775	.7783	1	16	1.200	1.2044	3
7	.844	.8452	1	17	1.225	1.2307	3
8	.900	.9033	2	18	1.250	1.2555	2
9	.950	.9544	2	19	1.275	1.2789	1
10	1.000	1.0000	0	20	1.300	1.3011	1

With $L(1.024) \sim .44 \times .024 = .011$, we have $L(2) = .3011$. Similarly, $4L(3) = 1 + 3L(2) + L(1.0125)$ or $L(3) = .4772$, etc. These corrected values are shown in the above table. Using these, we find

$$\begin{aligned} L(\ .96) &= -.0173 \\ L(\ .98) &= -.0085 \\ L(1.00) &= \ 0 \\ L(1.02) &= \ \ .0090 \\ L(1.04) &= \ \ .0173. \end{aligned}$$

These values suggest that $L(1 + x) = .435x$ for $|x|$ small. If this is used instead of $.44x$, the small errors in the fourth place may be reduced.

The procedure beyond L(20) is clear. Linear interpolation between L(18) and L(20) for L(19) is in error by .0006; very soon we may obtain the new prime logarithms by interpolation. The table should be carried to L(100), checked by differencing, and rounded to 3D.

Several variants on the above approach may be exploited:

(1) It is fun to look for two linear relations between L(2) and L(3) to start the process. For example:

$$\begin{aligned} 6^9 &= 10{,}077{,}696 \sim 10^7 \\ 3^{13} &= 1{,}594{,}323 \sim 2^4 \times 10^5 \end{aligned}$$

yield $L(2) = .30066$, $L(3) = .47712$.

(2) Starting with $L(10) = 1$ and taking successive square roots (e.g., from a table) yields logarithms of the form $(\frac{1}{2})^b$. When these are dense enough, inverse interpolation gives the L function.

(3) Begin with $L(1.01) = .01$ and, by taking powers of 1.01, develop a table of natural logarithms. Show that both systems yield the same multiplicative results.

(4) It is possible to generate relations involving several primes and to solve the resulting sets of linear equations for the logarithms. A good way to search is to look for three consecutive integers which have the desired prime factors. For instance, if we wish to try for 2, 3, and 7, to the base 10 (so that 5 is automatic), we observe 8, 9, 10; 14, 15, 16; and 48, 49, 50 as possible triples. Then the distributive property applied in the form $(n - 1)(n + 1) = n^2 - 1$ gives

$$\begin{aligned} 9^2 &\sim (10)(8) = 80 \\ 15^2 &\sim (14)(16) = 224 \\ 49^2 &\sim (48)(50) = 2400 \end{aligned}$$

The resulting three equations in log 2, log 3, log 7 yield log 2 = .301, log 3 = .476, log 7 = .845 to three decimal places. If more accuracy is desired, one should drop 80 = 81 and look for something more precise. For instance, if we observe 125, 126, 128 as having the desired factors (although not consecutive), we can then say $(126)^3 \sim (125)^2(128)$, and proceed from there.

(5) In a somewhat more sophisticated approach, the approximate equations used above are carried as inequalities; e.g., $4 \log 3 > 3 \log 2 + 1$. This leads to definite inequalities for the logarithms, which become sharper and sharper as we go along.

(6) There are any number of variations of this problem, some of which may occur to the brighter students. For example, one might take advantage of convexity (which could be taken as an additional assumption, or possibly even proved) in interpolation to get inequalities.

After students have made their own partial tables for (say) two significant figures, they can explore the use of tables given in books. One can avoid the notion of characteristics and mantissas by using the table *exactly* as printed (as is commonly done with tables of natural logarithms). Later, when the table has been extended to larger and larger numbers, the students will probably discover for themselves the utility of keeping only the table of mantissas.

Appendix C

THE INTRODUCTION TO FORMAL GEOMETRY

This is an essay in support of one of the two main viewpoints on geometry that appeared at the conference. An exposition of the other is found in Section 6, first proposal (pp. 55-57).

Our proposed program for the first six grades consists of arithmetic together with a variety of intuitive "pre-mathematics." We propose that this be followed by a more formal introduction to algebra. This would not be a fully postulational treatment; it would be deductive piecemeal, in the small but not in the large. It would, however, undertake to organize and solidify the student's knowledge of the number system and of the use of variables in algebra.

The scheme of the geometry course would be somewhat different: it would be the student's first experience with a formal mathematical structure. For several reasons, we believe that geometry is a good choice for this purpose.

The reality of the subject is palpable. In algebra there is a serious danger that the student will confuse names with things and think that algebra deals with the former rather than the latter. Geometry, on the other hand, appears at this maturity level to be the study of physical space. (For this reason, nothing analogous to the number-numeral distinction has been used, or needed, in geometry courses.) Postulational geometry is thus not only the prototype of the postulational method in the rest of mathematics but also the prototype of the use of mathematical models in natural science; thus Euclid was the precursor not only of Bourbaki but also of Newton. In geometry, as almost nowhere else in mathematics, the student is continually confronted with concepts whose intuitive meaning is clear; he is then given descriptions of these concepts, sometimes in postulates but more often in definitions; he is invited to compare the former with the latter, to see whether the descriptions really fit the things that they are supposed to describe; he then investigates *things* at great length, using the *descriptions* as tools in the investigation. This kind of study represents an important opportunity. For this reason we believe that geometry has an educational significance above and beyond the importance of its substantive content.

There are, however, a number of ways in which this opportunity can be missed or thrown away.

1. *By excessive looseness and inaccuracy.* It is possible to write postulates and definitions so vaguely and inaccurately that in fact and practice they are unused and unusable. The reader and the student infer ideas intuitively from the context, regarding the supposed formal structure merely as a gesture. Even if the objective were merely to learn the facts of geometry, we believe that such treatments should be regarded as inferior; they obscure the coherence of the subject and prolong unduly the student's reliance on authority. And if the objective includes the learning of mathematical and scientific method, such treatments are very inferior indeed.

2. *By fragmentation.* The meaning of geometry as a deductive subject depends (like the artistic effect of colossal statuary) partly on its sheer size; it depends on the fact that a large and complicated mathematical object, intrinsically worthy of interest and respect, is investigated at length and in depth by the application of the deductive method to a mathematical description of it. If geometry is taught only intermittently, in various "integrated" courses, without ever having a course to itself, the meaning of this picture as a whole is likely to be lost. (Here we are not objecting to a possible scheme under which a student might study a two-hour algebra course and a three-hour geometry course, or vice versa, out of two different books concurrently. Our idea is merely that the geometry should form a readily recognizable unit.)

3. *By excessive delicacy and austerity.* If the postulates are so weak that the transition to significant statements is a lengthy chore, and the logic is so formalized that only the most trivial proofs can be written in a small finite number of words, then the deductive method is not likely to look either attractive or powerful, and much of the mathematical substance is likely to be crowded out. The choice of a logical level somewhere between raw intuition and total formalism does not look easy; probably it should depend on long experimentation. It seems obvious, however, that the prevailing level can be raised without loss of time or of teachability. For example, Euclid and later writers have used a very vague omnibus postulate which says that "the whole is equal to the sum of its parts." The meaning of this in Euclid is too complicated to discuss here. In contemporary textbooks it is interpreted, in three different contexts, to mean that, under certain conditions, length, area, and angular measure are additive. It is surely no burden on the student to say straightforwardly the three things that we really mean. In this case, the

price of precision is zero. In many other cases, we believe that its price is small.

4. *By the use of "parachute postulates."* One of the most striking devices of modern mathematics is the use of flank-attacks on conceptual and technical problems. A familiar example of this is the use of the definition

$$\ln x = \int_1^x \frac{dt}{t}$$

as a point of entry to the study of exponentials and logarithms. Using this definition, we attack all difficulties from the rear, as it were, and get easy proofs of theorems which would otherwise be very hard. (Indeed, in other treatments the proofs are so tedious that some of them are usually omitted not merely by college freshmen, but by everybody else.) The flank-attack, however, is extremely artificial to a person who does not already understand the subject in outline. For example, the definitions $\exp = \ln^{-1}$ and $a^b = \exp(b \ln a)$ give us an almost bewildering reason for believing that $a^b a^c = a^{b+c}$. We also have a simple but sophisticated theorem which asserts that for $x > 0$, our new definition of x^n agrees with the old one.

Devices of this kind should by all means be taught. We do not believe, however, that they should be used in the student's first introduction to the postulational and deductive method. When the student is trying for the first time to learn and use mathematical descriptions of an external reality, the descriptions that are offered to him cannot be justified merely by the fact that after a lengthy development they turn out to be appropriate, adequate, and accurate. Their naturalness should be plain at the outset; the student should feel that he is engaged in a careful analysis of things that he sees. For this reason, we believe that the logical simplicity of a vectorial approach to the geometry of space does not justify its use in a *first* treatment.

What postulates should be used is another question. Suitable modifications of the postulates of Euclid, Hilbert, or Birkhoff are surely workable. Recently, in England and Germany, experimental courses have been developed in which the reflections have been used as the basic apparatus. All of these ideas are worthy of investigation. We propose, however, as a fundamental criterion, that in the first course, geometry be formulated so that the student will feel that he is using natural descriptions of objects of experience.

Appendix D

EXPLORATION

It is proposed that every opportunity be taken to let the students explore mathematical situations on their own. This can be done in every phase of the education, from the homework exercises all the way to the development of major mathematical concepts and theorems. The advantages of such an approach might include:

1. There is a real worry that early mathematical sophistication will condition the students against intuitive arguments. The exploring of mathematical situations by the students as an essential part of the algebraic development will help them develop their intuition alongside their sophistication.

2. There is a real worry that the willingness to do pre-mathematics for the future will cease when doing actual mathematics for the present. Doing pre-mathematical exploration at all times will help to overcome this tendency.

3. Both the practitioner of applied mathematics and the creator of pure mathematics spend much of their time and effort on "here's a situation, explore it," not only on "here's a problem, solve it" or "here's a theorem, prove it." It is good to admit this to the students, and to let them work on mathematics in this manner themselves.

4. It is important for the student to get the feeling that definitions and lines of attack are matters of *choice*. You first explore the situation, and then pick a particular point of view for its convenience and for its power.

We now give a collection of examples of opportunities for such exploration:

1. Explore the different ways in which infinite sequences of numbers can behave — the different patterns of convergence and divergence. A good place to conjecture results.

2. Observe the Pythagorean triples (3, 4, 5), (5, 12, 13), (8, 15, 17), (7, 24, 25), which are the only ones usually seen in school. Are there more? How would you generate them? How is this problem related to algebraic identities? How is it related to circles in the plane?

3. Make a table of logarithms to the base 10 by using the facts that $125 \sim 128$, $80 \sim 81$, $7^4 = 2401 \sim 2400$, etc. Two-place accuracy is easy. How do errors accumulate? How would you improve the accuracy?

4. You know what you mean by the area of a rectangle. What might you like to mean by area of other figures?

5. Do you want to define 1 to be a prime number? (You do have a choice.) Is a notion of "proper factor" useful?

6. What is unusual about quadrilaterals inscribed in a circle as opposed to arbitrary quadrilaterals?

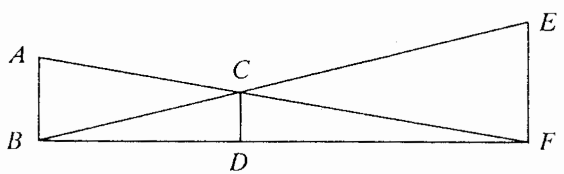
7. Given the system

$$x + y + z = a$$
$$x^2 + y^2 + z^2 = b$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c$$

What can you find easily?

8. How would it be convenient to define inverse trigonometric functions? square roots? fractional exponents?

9. The ladders in the alley problem:



What relations are there among AB, CD, EF, AF, BE, BF, BD, DF? What determines what?

10. How can you characterize a finite set without specifying the *number* of members?

Appendix E

ELEMENTARY MODERN MATHEMATICS FROM THE ADVANCED STANDPOINT

In order that any syllabus devised for the high school should be teachable, there must be those able and available to teach it. Thus we recommend that the syllabus should form the first five semesters of a 6-semester college mathematics course to be taken by potential high school teachers among others. Of course, its content should be familiar to all who have been through a college mathematics course. The 6th semester would be taken up with more advanced material and material omitted from the course as proposed on the grounds of shortage of time (e.g., multiple integrals, Stokes's theorem, elementary group theory, linear programming).

We have aimed at achieving the right level for high school mathematics; the material itself should be suitable for those with the interests and aptitudes of high school students, and it is of supreme importance that the attitude inculcated should be right. Indeed, we place less emphasis on actual content than on level and point of view. Nor do we insist in all cases on the precise order of presentation laid down in our draft although there are certain principles (algebra before analysis as preparation for the calculus, for example) which we would not sacrifice. But we believe this curriculum may be regarded as a l.u.b. in future discussions. We can make no valid prediction about the time required in the high school for the topics discussed in the syllabus; but we are quite confident that they can be taught and absorbed in 5 semesters of college. (We would enunciate one principle to guide the choice of omissions from the syllabus: we should not sacrifice practice to conserve theory, since it is no use understanding what you would be talking about but having nothing to say.)

Our advocacy of the suitability in college teaching of the proposed curriculum should not be interpreted as a retreat from our claim that the level, point of view, and nature of the material are, in fact, appropriate to a high school syllabus. But the realities of the educational situation require that in the first instance we must train the teachers to teach it — and that we have in any case to wait many years before students will reach the 7th grade equipped, as we would wish them to be equipped, for launching on the course. However, we would immodestly claim that in fact the course is superior in many ways to that currently given in many colleges. We point out that the part of the algebra course which would be completed in the first semester by presenting the formal differentiation of polynomials, behavior at infinity, orders of magnitude, etc. would provide a good preparation for elementary physics.

The course would almost certainly require the writing of special textbooks. However, rather than wait for the production of such textbooks — and in the meanwhile introduce no reform into college courses — it would clearly be preferable to introduce certain reforms immediately and utilize such high-quality texts as do in fact exist. In this connection we might mention the texts by Moise (geometry), Mostow, et al (algebra) and Courant (calculus). We believe it would be difficult, if not impossible, to devise textbooks suitable for the transmission of this material to both college and high school students. It might be added that the provision of special textbooks may incline colleges to adopt the syllabus, at least experimentally; and it helps to avoid misunderstanding.

We offer some comments on the impressions we have gained from our deliberations. We have been aware of the danger of putting ourselves — or, rather, our contemporaries — in the position of the students for whom we have been trying to cater. There is a tendency, in judging the difficulty of certain mathematical concepts and pieces of mathematical reasoning, to suppose that the student approaches these things equipped as our contemporaries were equipped at the same educational grade. Nowhere, perhaps, is this potential fallacy more glaring than in estimating the relative difficulties of algebra, geometry, and calculus.

There is a feeling among some that the algebraic content of the course is too stiff, that, for example, the notion of a polynomial is too sophisticated to be described to a 7th grader. The fact is that this notion is one of the few which can be described honestly and precisely at that level, and it is certainly mathematically useful at the stage at which one is interested in factorization; we also believe it will be quite sufficiently "real" to the 7th grader. We acknowledge that the notion of plane Euclidean geometry is also useful to the student at that stage and has enormous intuitive content — but converting it from an intuitive synthetic-geometrical concept into a precise analytical concept, replete with coordinatization procedures, presents formidable difficulties. Not least of these is the quite general difficulty of passing from the intuitive to the precise; for, in the nature of things, no mathematical demonstration can validate the passage. Yet there may be some who maintain that it would be better to suppress polynomials rather because there is no vague notion, familiar to the student at this stage, to be replaced by a precise one; it seems to be held (elsewhere than in this subgroup) that advantage should be taken of the familiarity of the function concept in order to confuse the issue. In fact, the greatest difficulty we have had in clarifying our own *mathematical* thinking about the topics appropriate to a high school course have centered round the problem of tying up intuitive with precise concepts. Apart from geometry, the examples which naturally come to mind are logarithms, the trigonometric functions, and concepts of length and area associated with curves. Certainly we do not deny that these topics inevitably appear first in imprecise form and it would be sterilizing to await the mathematical sophistication needed to render them quite precise. But the very fact of their arrival in the course in immature form poses problems of teaching and mathematical understanding from which algebra is very largely free. Thus we are unrepentant in giving the prominence we have to algebra, though we readily admit, as stated earlier, that it may be preferable to change the order of presentation in certain instances and to interleave the algebra with some geometry and elementary probability theory in the 7th and 8th grades.

A second— and related — point concerns the amount of unlearning that should figure in a well-designed course. The process of unlearning is frequently painful to both student and teacher; in the absence of really excellent rapport between student and teacher it can destroy the student's confidence. Thus it is seen to be particularly important to present the student with an intuitive and imprecise approximation only when he is not yet ready for the real thing.

Third, we feel it is essential to show the greatest of pedagogical skill and insight in helping the student to decide the extent to which he should require proof of validity before feeling entitled to use a technique. It is very important not to inhibit the student's enthusiasm and facility for solving problems by preoccupying him excessively with scruples about rigor. The essential point here seems to be to develop intellectual honesty, so that the student knows what is being assumed and what has been proved, where the concept is quite precise and where it is imperfect. This is, of course, an end in itself; but it also has the immensely desirable effect of enabling the student to steer a middle course between the extremes of glib indifference to mathematical principles and paralyzing obsession with mathematical rigor. Discovery and proof are both vital in mathematics.

A fourth point concerns the relation of applications to mathematics. It has seemed throughout the preparation of this syllabus that the need to develop proficiency in certain mathematical techniques by reason of their applicability has led inevitably to the discussion of important mathematical concepts (e.g., function, polynomial, vector space, limit); and, conversely, that the introduction of good mathematical topics, judged by the criteria of coherence and power, has led easily and smoothly to significant applications. Thus we find no justification, in our own thinking, for the view that, at the high school level, at any rate, the needs of the potential professional mathematician are different from those of the potential professional user of mathematics and even more different from those of the intelligent citizen of the 21st century. Of course, speeds and styles of presentation will differ and so too will the facility of absorption, even between individuals belonging to the same broad category. But there seems to be a compelling and inescapable quality about good mathematics — and this must, in the last analysis, constitute the justification of the curriculum.

Appendix F

OPPORTUNITIES FOR PROOF-MAKING IN THE ELEMENTARY SCHOOL

This material is presented in the belief that more logical inference must be made with the mathematical concepts of grades K to 6 while the student is still in grades 4 to 6. Proofs, generally of deductive nature, of interesting theorems must be evolved from the axiomatic material given the student in great abundance (although these axioms have not always been formulated explicitly). The proofs must follow a recognizable logic, although the need for a detailed logical style or the naming of logical steps is not implied here. Reasons for presenting proofs in grades 4 to 6 include (1) economizing on the number of concepts (axioms) required, (2) the illustration of the power of the concepts already introduced in proving more elaborate and surprising statements, (3) the unifying of the material of K to 6, and (4) a sufficient development of the logical discipline to prepare the students for the large scale of formal geometry and algebra to be introduced in 7 and 8. The last point does not interfere with the psychological stimulation a student entering junior high school should experience from an overall new approach. There is still a vast difference between proving an occasional theorem and having a course of organized theorems based largely on each other.

In order to eliminate the need for describing a seven-year course in detail we will propose a few theorems that can be inserted into the present SMSG course for grades 4 to 6. It will be apparent in other frameworks where similar theorems can be proved. Many more theorems will suggest themselves in any given course and we should think that about twenty theorems should be distributed through 4 to 6. Those presented below are samples consistent with SMSG material.

The form of proof thought suitable will be outlined. It is expected that the pedagogical approach to each proof will be lengthy; that it will involve elements of self-discovery and directed discovery somewhat as in the Madison Project approach. That project is a rich source of proved theorems. The emphasis here is perhaps on theorems of a somewhat more general kind.

(1) Theorems on sets.

It is agreed that emphasizing a formal approach to theorems such as:

$$\begin{aligned} A \cap B &= B \cap A \\ A \cup B &= B \cup A \\ A \cap B &\subset A \end{aligned} \tag{1}$$

would have the disadvantage of laboring at the obvious. It is proposed that two or three such theorems be briefly used to introduce the required language. When the student verbalizes the last theorem above as "what is in both A and B is of course in A," the teacher supplies the language (not even stated to be a proof):

$$y \in A \cap B \Rightarrow \text{(by definition) } y \in A \text{ and } y \in B.$$
$$\therefore\ y \in A \text{ as required.}$$

This approach can then be applied to the theorems below, which have sufficient complexity to make their proof interesting to the student.

$$\begin{aligned} (A - B) \cap (A - C) &= A - (B \cup C) \\ (A \cap B) \cap C &= A \cap (B \cap C) \\ (A \cup B) \cup C &= A \cup (B \cup C) \\ (A \cup B) \cap C &\neq A \cup (B \cap C) \text{ unless } A \subset C \end{aligned} \tag{2}$$

The proof of the first of these would start

$$y \in (A - B) \cap (A - C) \Rightarrow y \in (A - B) \text{ and } y \in (A - C)$$
etc.

If Venn diagrams are considered helpful, they may be used alongside the above type of proof.

(2) The C-A-D axioms of operations on integers are used very well in SMSG to show the correctness of the algorithms of addition, multiplication, etc. of multi-digit numbers. This is the only important proof material in SMSG grades 4 to 6, and it should form an important part of the deductive training in these grades. We propose that the SMSG approach be extended to include a systematic statement of the proofs, one equation leading to the next through axioms and previous equations. One should also prove the validity of the usual manipulations with fractions.

(3) Number factoring theorems.

This is surely an important prelude to the factoring of polynomials in grade 7. The series of theorems below ends with a rather important theorem.

a. If n is a factor of p and q, it is a factor of $p + q$.
Given $p = na$ and $q = nb$, a and b integers.
$p + q = na + nb = n(a + b)$, n is a factor of $p + q$.
Corollary: If n is a factor of p but not of q, then it is not a divisor of $p + q$. Proof by contradiction of above theorem.

b. If ab is a factor of q, then both a and b are factors of q.

$$q = (ab)c = a(bc) = b(ac)$$

c. There is no largest prime.
If P_n is largest, consider $Q = P_1P_2 \ldots P_n + 1$. As 1 has no factors, by the corollary of theorem 1 above the factors of Q do not include P_1, P_2, . . ., P_n. Therefore if Q is not prime, it has a prime factor greater than P_n. If it is prime, then Q itself is a prime greater than P_n.

(4) A nontrivial use of induction following a theorem already seen "geometrically."

Theorem: $\sum_{n=1}^{m} (2n - 1) = m^2$ for every m.

This is first shown "geometrically" by adding squares around two sides and a corner of a square. Then we prove it as a nontrivial application of mathematical induction.

$$m = 1 \qquad 1 = 1^2$$

If true for m, then

$$\begin{aligned} \sum_{n=1}^{m+1} (2n - 1) &= \sum_{n=1}^{m} (2n - 1) + [2(m + 1) - 1] \\ &= m^2 + 2m + 1 \\ &= (m + 1)^2 \end{aligned}$$

We may then look at the sum of other finite series in the same way.

Appendix G

THE USE OF UNITS

It appears to us that the teaching of the use of units of measurement involves some serious problems. Most of these problems are due to the incongruity between the simplicity of the formal operations and the conceptual obscurity of the underlying ideas. Thus it is easy to teach a student to cancel out "sec." in the expression

$$(10 \text{ ft./sec.}^2)(5 \text{ sec.}),$$

but it is not so easy to describe this proceeding in a rational way.

The best response to this problem that we can think of is to discuss the use of units piecemeal, as the occasion requires. Very early, the student should learn about units of distance, area, weight, and so on; he should know that in each case many choices are possible, and that therefore we must indicate which choice we have made; and he should know how to convert from one unit to another, e.g., feet to yards to meters to miles, pounds to grams, and so on. Manipulative processes, e.g.,

$$5 \text{ ft.} + 10 \text{ ft.} = 15 \text{ ft.}$$

should be discussed in the context of applications. For example, the above equation should be interpreted to mean that if segments of length 5 and 10 (measured in feet) are laid end to end, then the length of their union (measured in feet) is 15. This, we think, is what the equation means in the contexts in which it is used. It is not an equation in a new system of "impure numbers." The use of units can, indeed, be thought of as a kind of abstract algebra, but it is not commonly so thought of by the people who use it; and even if young students could be taught to think of it in this way, they would still need to bridge the gap between their algebra and the extra-algebraic interpretations which make it useful.

The next problem is the relation between units of length and area. Sq. ft. is the unit of area for which it is true that a square one foot along an edge has area 1. Thus the statement that

$$(a \text{ ft.})(b \text{ ft.}) = ab \text{ sq. ft.}$$

is a statement of fact about rectangles; its validity does not depend on a purely algebraic claim that the left-hand member in the above equation is an associative and commutative fourfold product. It is possible, in these terms, to explain conversion of sq. ft. to sq. yd.; as before, the substance of the statements is regarded as geometric. If the student observes that "ft." and "yd." are behaving as if they were variables, so much the better. But the teacher should not attempt to insist that this formalism has any force or validity of its own, because this aspect of the matter is incapable of intelligible explanation at an elementary level.

The same principle applies to the use of units of distance, time, and speed. In each such case, it is the fact which justifies the formalism, rather than the formalism which establishes the fact.

We believe that this sort of treatment is adequate for the uses of units which come up naturally in mathematics courses. Further ramifications of it will be needed in physics; but we believe that they should be taught in connection with the discussions which at the same time require them and elucidate them. We believe that this scheme has two main advantages:

1. It tends to connect the language of mathematics with that of physics, and suggests that these fields live in a spirit of mutual coexistence.

2. It helps to avoid creating the impression that some kinds of mathematics are hopelessly mystical.

It should be understood that in recommending that units be used in this way, we do not mean to suggest that the language and notation of units be used on every occasion where they might be. It is quite possible, in a purely geometric theory of length and area, to treat lengths and areas as pure numbers. (Indeed, this is the universal practice in advanced treatises on measure theory.) Some elementary texts treat measurement in both ways; and this strikes us as a reasonable proceeding. Measurement by pure numbers must, in fact, come up eventually, if we are to calculate areas by integration: in the formula,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

the left-hand member is the area of a region, and the right-hand member is the length of a segment.

Appendix H

REMARKS ON SIGNIFICANT FIGURES

In the elementary mathematics curriculum as now planned, numbers obtained from measurements are to be manipulated by the students to aid in relating the mathematics to the real world and to science, to aid in the intuitive study of geometry, and to assist with the ideas of models. Questions are bound to arise about the proper treatment of these numbers obtained by measuring. In handling the "scientific notation" a body of information about the treatment of "significant figures" has grown up some of which is useful but not all of which is satisfactory. The present remarks are designed to discourage too formal a treatment of the idea of "significant figures" and to give some positive suggestions about calculations with numbers obtained from measuring. In a revised form the remarks might be suitable for a teacher's manual.

The notion of number of significant figures tries to do two jobs at once — to report the result of a calculation and to report on its accuracy all in the same number. This double burden creates an extra strain that arithmetic and the decimal system cannot cope with quite satisfactorily.

Suppose significant figures have been defined so that the student knows 87000 has at least two significant figures and that if it has more than two, someone should have mentioned it. He knows that 0.0246 has three significant figures.

A major source of trouble in dealing with significant figures is the folklore about hurrying to round back to the number of significant figures in the original measurements. This is unwise. It is wise to keep everyone informed about the significant figures in the original numbers and it can be wise to keep more than the original number of significant figures both in the intermediate calculation and even in the final answer.

Example. *Averages.* If we have 10 independent measurements in a physics experiment, each given to two significant figures, like 8.3, 8.4, 8.1, 8.4, 8.4, 8.2, 8.6, 8.3, 8.5, 8.2, and we have reason to average these measurements to get a good single descriptive value, or to estimate a constant, it is sensible to report an extra place in the final answer. The average obtained has a good chance of being closer to the desired number than the rounded number. Probability theory shows that averages can be more reliable than the figures that go to make them up. Why add to the unavoidable error in the measurement by deliberately rounding? On the other hand, excessive accuracy of the report of the final number is useless so that carrying 5 extra places, while not strictly wrong, suggests that the reporter has lost his sense of proportion.

Products

We generally think of a number stated to a given number of significant figures as possibly varying each way 5 digits in the next place, so that a measurement of 80 to one significant figure is believed to be, or even known to be, between 75 and 85. Under these circumstances, if we have to multiply 2 one-significant-figure numbers, say 20×80, we might well think that the true product is between 15×75 and 25×85. That is, we have a number represented by 1600 that we think is somewhere between 1125 and 2125, and the swing here is substantial. In spite of this, to round the number 1600 off to its one-significant-figure representation of 2000, merely to preserve the consistency of winding up with no more significant figures than we had in the multiplicative components, adds error. In this problem we have little reason to think the number 2000 is preferable to 1600 and some reason to prefer the latter. These remarks are intended to explain why attempts to make standard rules about the handling of significant figures usually run into difficulty.

Rounding early in more extended calculations

Let a, b, c, and d be numbers given to a certain number of significant figures each. We wish to make a calculation of the form $ab - cd$. For convenience of exposition let us illustrate with one-significant-digit numbers. Let $a = 8$, $b = 7$, $c = 9$, $d = 6$. Then $ab = 8 \cdot 7 = 56$ and $cd = 9 \cdot 6 = 54$. It is important not to round at this intermediate stage of the calculation. If you do you get, in this example,

$$ab \sim 60, \qquad cd \sim 50$$

and then

$$ab - cd \sim 60 - 50 = 10,$$

instead of the more direct $56 - 54 = 2$.

Of course, with these measurements, errors of $\frac{1}{2}$ in the worst directions can be given an upper limit to the result of

$$8.5(7.5) - 8.5(5.5) = 17.0$$

and a lower limit of

$$7.5(6.5) - 9.5(6.5) = -13.0$$

But that is not a good reason for reporting 10 instead of 2. These examples all suggest that the student can profitably explore the bounds on the answers given by manipulation of measurement numbers. He will get arithmetic practice in a new context, and he will get another earthy experience with inequalities and bounds. These are all to the good. On the other hand, it should not be done to death, first because a little of such work is interesting but a lot can become tedious. Second, we are partly training a mistaken attitude, and in the early grades at least we are not in a position to correct it satisfactorily.

The exact bounds do not carry all the information in the system. As mentioned above, we may merely believe or choose to act as if the true measurement number were within 5 units each way from the one we obtained. In many measurement processes the rounding is not that good, as a little experience with slide rule calculations done side-by-side with exact calculations will show. Number preferences and misreadings distort the meaning of significant figures. Thus the bounds may underestimate the swings the answer can have.

On the other hand, when the numbers really are afflicted only with the kind of errors which arise from unbiased measurement an appropriate theory of the distribution of errors exists. This theory lays the basis for keeping more places in the answer than in the original measurements in such instances as the average mentioned earlier. And in general that theory suggests that we should not be so pessimistic as the extreme bounds would suggest. Since this theory is complicated and uses more probability theory than we plan to give in grades K through 6, we cannot hope to supply it to the student. Furthermore it is only used occasionally even in advanced work.

To sum up, the bounds can be misleading in either direction, depending on the appropriate model for measurement. Much more glaring examples of the effects of rounding can be given — many in which the intermediate rounding throws the final answer completely outside the possible range that a careful error analysis would give.

Discontinuity of significant figures

This related example is intended to illustrate a different point. Let us calculate $10/x$ where x is a one-significant digit number, say 8, and the 10 is an exact number:

$$10/8 = 1.25 \quad \text{(1 for the eager rounder).}$$

But

$$10/7.5 \sim 1.33, \ 10/8.5 \sim 1.18.$$

By rounding to one significant figure we get out of the range.

The trouble is that the notion of a given number of significant figures does not carry with it the full value of the information contained in the number, whereas the notion of per-cent error does better, while still ignoring any probability theory that might underlie the problem. Thus the one-significant-digit number 1 offers a 50 per cent error measured from its current position, while the number 9 offers only a bit more than 5 per cent. The two-significant digit number 10 also offers about 5 per cent error. This sharp discontinuity in the number of significant digits, while the per-cent error is continuous, is one source of dissatisfaction with the notion of significant digits because it is the per-cent error that we ordinarily wish to control.

One does not wish to go hog-wild in keeping significant figures, however. If we multiply two 3-significant-digit numbers 384 × 529, we get 203,136. If this is the end of the calculation, the final 36 is meaningless. And, of course, we are uncertain even about the 3 in the thousands position. The student should know that these last few digits are unreliable and that we do not want to keep endless meaningless digits. This side of the calculation is the one that the usual approaches to significant digits are correctly meant to cure. Their overemphasis has bred loss of information, but here is where the real value of rounding rules resides. In such a short calculation we would not be justified in keeping more than 203,100. And the reporter should say that there is uncertainty in the third significant figure.

The student should also know that when he multiplies a 2-significant-digit number by a 5-significant-digit number, the result is essentially 2-figure accuracy. Related remarks can be made about division, and the other arithmetic operations.

To sum up, the notion of significant figures has served the community fairly well, and the rules about them when used with discretion can be useful, though in the hands of a novice, worrisome. A basic trouble is that many people feel that the "algebra" of significant figures can be made rigorous if someone will take the trouble to do it, or, taking the blame upon themselves, is rigorous, but they had not studied it sufficiently carefully. Instead, the system has basic weaknesses that are rigidly fixed into the framework. We might distinguish between rules of thumb and rules of crumb. Both are useful. If a man makes a small barrel hoop by multiplying the diameter of the barrel by 3 and adding on a generous thumb length, he is using a rule of thumb, and mathematicians can tell him how to fix the rule if he gets into trouble with a big barrel. Rules of crumb are handy rules that often help one do better than nothing, but that cannot be sufficiently improved or extended, because they are not based on any theory, right or wrong. Thus it is worthwhile to continue to think in significant digits, but to regard that enterprise as a very useful rule of crumb.

As a final recommendation for calculations with measurement numbers through secondary school: carry them to as many places as you would if they were exact, and then consider at the end what sort of reporting is sensible. Often you will need a statement in English as well as a number. The statement may say something about the possible percentage error of the result or about the first figure that seems uncertain.

INDEX