and then taking $O_{p+1} = X_{p+1}/\|X_{p+1}\|$. Since we are orthonormalizing in a vector space of finite dimension $n^2$, we eventually have $X_{m+1} = 0$ for some $m \leqslant n^2$. The Gram-Schmidt process stops here. Gather up the various coefficients computed in the orthonormalization process and express each $O_k$ as a linear combination of $\phi(A^k), \cdots, \phi(A), \phi(I)$ and thereby obtain a relation

$$\phi(A^m) + \sum_{k=1}^{m} a_k \phi(A^{m-k}) = O.$$

Since the process stops at the first power of $A$ that is linearly dependent on the preceding (lower) powers of $A$, it is clear that

$$m_A(\lambda) = \lambda^m + \sum_{k=1}^{m} a_k \lambda^{m-k}.$$

We close with a few brief observations. Although it is not hard to write a computer program for carrying out the above procedure, it is apparently not easy to compare the efficiency of this approach to that of using the formula $m_A(\lambda) = \det(A - \lambda I)/D_{n-1}(\lambda)$ where $D_{n-1}(\lambda) = \text{GCD}$ of all the $(n-1)$ st order minors of $\det(A - \lambda I)$. Finally if $v$ is any vector in $C^n$ the Gram-Schmidt process applied as above to the vectors $v, Av, A^2v, \ldots$ in $C^n$ leads to the so-called *order* $m_{A,v}(\lambda)$ of $v$ with respect to $A$, i.e., to the unique monic polynomial

$$\lambda^\mu + \sum_{k=1}^{\mu} b_k \lambda^{\mu-k}$$

of least degree $\mu$ such that $m_{A,v}(A)v = 0$. Clearly $\mu < n$ and then by the Cayley-Hamilton theorem $m \leqslant n$.

---

## THE TEACHING OF MATHEMATICS

EDITED BY MARY R. WARDROP AND ROBERT F. WARDROP

*Material for this department should be sent to Professor Robert F. Wardrop, Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859.*

### MATHEMATICS APPRECIATION COURSES: THE REPORT OF A CUPM PANEL*

In 1977 the Committee on the Undergraduate Program in Mathematics (CUPM) established a panel to consider the content of those college and university courses that treat mathematics appreciation for students in the arts and humanities. Such courses are taken by a large number of students, frequently as their last formal contact with mathematics. Yet in most institutions they are given very low priority; they are frequently taught perfunctorily, without a clear set of objectives, by faculty who lack appropriate interest or credentials. Since these courses may play a major role in molding nonscientists' opinions of mathematics and its role in society, CUPM decided that it should call attention to the importance of these courses and offer some suggestions on how they may be organized and taught effectively.

This is the report, approved by CUPM, of the CUPM Panel on Mathematics Appreciation Courses. While the panel has many guidelines and recommendations to offer, it does not feel that a particular selection of topics or teaching strategy should be universally adopted for mathematics appreciation courses. A main goal of such courses is to get students to appreciate the significant role that mathematics plays in society, both past and present. All material presented in such courses should be well motivated and related to the role of mathematics in culture and technology.

*Reprints of this article and the related bibliography contained in the Center Section may be purchased for $1.00 from MAA Publications Department, 1529 Eighteenth Street, NW, Washington, D. C. 20036.

**Acknowledgements**

Many people, including nonmathematicians, responded to the Panel's request for information and advice. An open meeting of the panel at the MAA meeting in Biloxi, Mississippi, in January, 1979, was attended by more than sixty people who had much to contribute. The panel particularly wishes to thank Professors Henry Alder, Dorothy Bernstein, Donald Bushaw, William Lucas, David Penney, William M. Priestley, David Roselle, and Alice Schafer for their helpful comments.

Bettye Anne Case
John Conway
Richard J. Duffin
Jerome A. Goldstein, Chairman
Elaine Koppelman
Kenneth Rebman
Lynn Arthur Steen
James Vineyard

**I. Mathematics Appreciation: A Philosophy.** The inclusion of a mathematics appreciation course in the undergraduate curriculum is common in the nation's colleges and universities. This trend is a direct result of an underlying belief, held by most mathematicians, that every well-educated person should be mathematically literate. Whether or not a mathematics course is required at a particular institution often depends, among other things, upon the extent to which this belief is shared by the general faculty. But the ultimate success of an "appreciation" course in mathematics should not depend upon mandatory enrollment. Rather, the value and importance of such a course should be directly attributable to the care and understanding with which it is conceived and taught.

If as mathematicians we accept the notion that an educated person should know something about mathematics, then we must also accept the responsibility for conscientiously providing appropriate training. Students in the mathematical, physical, life, and some social sciences, and usually those in business, study mathematics as an inherent part of the undergraduate curriculum. It is not to these students, but rather to majors in the arts, in the humanities, and in certain social sciences that we must direct the mathematics appreciation course. At the outset we must take into account the background and interests of the prospective students. In many cases they have chosen their majors precisely because of a weak or unpleasant mathematical background; a college course that reinforces this negative experience with mathematics certainly cannot be called an appreciation course.

At most institutions the great majority of students in a mathematics appreciation course will have studied less than four years of high school mathematics; moreover, many of these students will have had poor experience in mathematics, or will have had very weak courses. However, high school mathematics study is predominantly concerned with developing skills, and while such skills are of unquestioned importance, they are not necessarily prerequisite to (nor should teaching them be a part of) a mathematics appreciation course.

Among all fundamental academic disciplines, mathematics is perhaps unique in the degree to which it is not understood (or is misunderstood) by students and even faculty from other areas of study. By taking an introductory course in chemistry, history, or psychology, a student is expected to gain an understanding of the general techniques, accomplishments, and goals of the discipline, and will learn to appreciate the work of the contemporary professional practitioner of the subject, sometimes even to the extent of reading the current journals. But an undergraduate major in mathematics is unlikely to have comparable insight into mathematics. Thus the challenge of a mathematics appreciation course is enormous.

The ultimate goal of such courses is defined by our umbrella title—to instill in the student an appreciation of mathematics. For this to occur, students must come to understand the historical and contemporary role of mathematics, and to place the discipline properly in the context of other human intellectual achievements.

From the beginning of recorded history, mathematics has proved to be an indispensable aid to the empirical sciences; the great successes (and failures) of mathematical reasoning in the furtherance of human knowledge are tales begging to be told. Even the direct impact of mathematics on developments in virtually all disciplines is often not realized by the mathematical layman.

But of course, to mathematicians, the subject is more than a tool of applied science, more than a universal language useful for communication and research in other disciplines. Mathematicians see mathematics as an intellectually exciting discipline, one that holds great aesthetic appeal for its practitioners. This idea of mathematics as art is often difficult for nonmathematicians to appreciate, yet is fundamental to understanding the development and role of the subject.

Finally, to appreciate mathematics fully, one must recognize it as a vital, on-going discipline, one that is practiced by a worldwide community of dedicated, sometimes passionate, and frequently brilliant scholars. It is a surprise to many that mathematics is a living, changing, developing subject. A true appreciation of mathematics requires some knowledge of contemporary developments.

The entire mathematical community should be concerned with what view educated, informed people have of mathematics. Thus, courses in mathematics appreciation, while presumably benefiting primarily the students, may also have a long-term positive effect on the discipline itself. Obvious benefits will accrue if leaders in education, industry, business, and government have a better understanding of the nature, role, and importance of contemporary mathematics.

It is a sad commentary on the attitudes of mathematicians that courses in mathematics appreciation frequently command pejorative (albeit informal) labels such as "Math for Poets." Even the supposedly neutral title of "Math for Liberal Arts Students" may convey the connotation of condescension. We must recall that liberal arts education, for a large percentage of the college educated population, is a rigorous, disciplined encounter with the best elements of man's history and culture. The major clientele of the mathematics appreciation courses are liberal arts students, and it is from their ranks that many of society's leaders will emerge.

The panel believes that it is better to describe courses of this type in terms of their objectives rather than their audience. Since the term "mathematics appreciation" brings to mind similar courses in other special fields (e.g., "music appreciation") that generally carry positive connotations with regard to their role in general undergraduate education, and since it conveys concisely what such courses intend to accomplish, standing as a brief reminder of this intention to both teachers and students, the majority of the Panel prefers this title.

## II. Things to Stress in a Mathematics Appreciation Course.

1. *The relationship between mathematics and our cultural heritage*. Students enrolled in mathematics appreciation courses are generally more interested in, as well as more knowledgeable about, the arts and humanities than the sciences; it is natural, therefore, to capitalize on these strengths by appropriate illustrations of the relations between mathematics and music, art, literature, history, and society.

2. *The role of mathematics in history and the role of history in mathematics*. Although the influence of mathematics is often remote, mathematical discoveries have shaped our world in fundamental ways, altering the course of history as well as the way we live and work. Examples of these influences abound, and should form a major part of any mathematics appreciation course. Historical developments and the evolution of mathematical concepts should be properly emphasized.

3. *The nature of contemporary mathematics*. The mathematics known by most humanities students is ancient mathematics—the geometry of ancient Greece, and the algebra of the early renaissance; not surprisingly, such students have the impression that mathematics is dead. Showing them that it is in fact a vigorous, growing discipline with considerable influence in contemporary society is an important aspect of any course in mathematics appreciation.

4. *The recent emergence of several mathematical sciences*. While the mathematics appreciation course should not be devoted solely to one "modern" area such as statistics, computer science, or operations research, it surely provides an opportunity to use these fields as illustrations of the panoply of contemporary mathematical science.

5. *The necessity of doing mathematics to learn mathematics.* While some parts of the mathematics appreciation course can and should be about mathematics, it is essential that some parts actually engage the students in doing mathematics. Only in this way can they gain a realistic sense of the process and nature of mathematics. Of course it is vitally important that the instructor have appropriate respect for the students' interests and abilities, and that exercises be selected so as to maintain rather than destroy their enthusiasm.

6. *The role of mathematics as a tool for problem solving.* As the language of science and industry, mathematical models are the tool *par excellence* for solving problems. Students in mathematics appreciation courses should be exposed to contemporary mathematical modelling, to gain some appreciation both of its power and its limitations.

7. *The verbalization and reasoning necessary to understand symbolism.* While symbols provide the mathematician and scientist with great power, they obscure the meaning of mathematics from the uninitiated. A great service the teacher of a mathematics appreciation course can provide is to enable students to overcome their fear of symbols, to learn to think through arguments apart from the traditional symbols in which they are expressed.

8. *The existence of a large body of interesting writing about mathematics.* Students in mathematics appreciation courses generally feel comfortable with assignments such as term papers, book reports, and library research because they have become accustomed to these in their humanities courses. There is much good mathematics that can be learned in this way, and assignments can be arranged that utilize these familiar learning tools.

**III. Things to Avoid in a Mathematics Appreciation Course.**

1. Do not leave the assignment of an instructor in the mathematics appreciation course until the last minute and do not assign it on the sole basis of availability. The course requires more planning and preparation than almost any other mathematics course if it is to be successful.

2. Do not simply allow the students to sit back and listen. It is important that they be involved actively. But this need not take the form of daily homework. In fact, drill type assignments should be avoided. The involvement could take the form of projects, papers, book reports, "discovering" mathematics in class, participating in class discussions.

3. Do not overemphasize the history of mathematics. While the history of mathematics could and should be used to enliven the topics covered, a student who knows (and cares) nothing about a mathematical topic is not likely to be interested in its history.

4. Do not stress remedial topics. While many of the students in a mathematics appreciation course may need remedial work, any such material that is covered must be presented as part of a topic that fits into the scope of the course as a whole.

5. Do not make a fetish of rigor; in particular do not prove things that are self-evident to the students. For example, a rigorous presentation of the real numbers in which one proves the uniqueness of zero is entirely inappropriate in courses of this type.

6. Do not cover topics you do not yourself find interesting and important. It is hard to fool these students, and if the teacher does not care, they will not see why they should.

7. Do not be condescending. While the students in such courses may not be mathematically inclined, this does not mean that they are unintelligent. Many who take mathematics appreciation courses are outstanding, creative students, who have simply concentrated in the nonquantitative areas of the curriculum. The attitude of the teacher can help either to open or to close their minds to the material.

8. Do not cover topics which you cannot relate in some way to ideas familiar to the students.

Clock arithmetic and symbolic logic, for example, are of little value to mathematics appreciation courses unless you can find applications the students can appreciate and understand.

9. Do not make the course too easy. The material should not be way over the heads of the students, but it should not be trivial either.

10. Do not accept anyone else's blueprint for a mathematics appreciation course. If you can communicate, in your own way, why you believe that mathematics is beautiful and important, the course will fulfill its purpose.

**IV. Approaches to Course Organization.** There are nearly as many ways to teach a course in mathematics appreciation as there are teachers of these courses. While some strategies will work superbly in some contexts, none can be recommended for all; the teacher's enthusiasm for what is being done as well as the appropriateness of the strategy for the students in the course are generally more important than the actual strategy adopted. Nevertheless, to encourage flexibility, we list below some approaches to teaching mathematics appreciation that have been effective in certain contexts.

1. A sampler approach, featuring a variety of more or less independent topics. The advantage of this method is that it covers many areas without requiring a sustained continuity of interest; students who fall behind or simply fail to comprehend one topic always know that they have a chance for a fresh start in a few days. The disadvantage is that of all survey courses: not enough time spent on any one thing to ensure long-term learning.

2. A single-thread approach, built around a common theme, for example, $2 \times 2$ matrices, or algorithms, or patterns of symmetry. Doing this takes careful planning, and runs the risk of alienating some of the class who find the thread incomprehensible. But it guarantees a solid example of the intellectual coherence that is so much a part of contemporary mathematics, that ideas arising in one context find applications in others, and that a common abstract structure underlies them all.

3. A socratic approach, in which the instructor works carefully to let the students develop their own reasoning. This works well in small classes with a highly motivated instructor. While the content of such courses is hard to guarantee in advance, the achievement for students who are able to think for themselves, perceiving patterns where others simply see chaos, is a worthy objective for a course in mathematics appreciation.

**V. Examples of Topics.** The topics available for courses in mathematics appreciation are as diverse as mathematical science itself. Standard textbooks offer a rather traditional assortment of topics: probability, graph theory, finite difference equations, computers, matrices, statistics, exponential growth, set theory, and logic seem to dominate. But there are numerous other themes that can be used for large or small components of courses. Here are a few of the many possible examples.

1. Understanding how to use the buttons on a pocket calculator. It used to be that the number $e$ was a complete mystery to those who had not studied calculus, and that "sin" had for humanities students more the connotation of theology than of mathematics. But no more. Virtually everyone has, or has seen, inexpensive hand calculators with buttons that perform operations involving exponential, trigonometric, and basic statistical functions. Teaching a class what these buttons do is an exciting new way to explore some traditional parts of classical mathematics.

2. Tracing the modern descendants of classical mathematical ideas can illustrate the power of mathematics to influence the real world, as well as its remoteness from it. For example, classical Greek geometry involving conic sections led to models for planetary motion, and ultimately to the

possibility of space flight. And probability, which had its origins in seventeenth-century discussions about gambling, now dominates actuarial and fiscal policy, influencing government and corporate budgets, thus affecting the level of interest, of unemployment, and the health of the entire economy.

3. Connecting mathematics with Nobel Prizes. Nobel prizes are not given in mathematics (and the apocryphal reasons for this are quite amusing). But the work that led to Nobel prizes (e.g., of Libby, of Arrow, of Lederberg, and others) often has an intrinsically mathematical basis. The study of this scientific work provides an opportunity to show how mathematics is important in the most profound discoveries of modern science.

4. Applying exponential growth models. The applications of traditional topics from elementary mathematics can often be explored more fully than has usually been the case. Exponential growth and decay models provide a striking example. Simple noncalculus approaches to models of growth provide a basis for discussion not only of interest and inflation, but also of such things as radiocarbon dating, cooling and heating of houses, population dynamics, strategies for controlling epidemics, and even detection of art forgeries.

5. Relating traditional mathematics to new applications. A discussion of beginning probability theory can quickly lead to a treatment of the Hardy-Weinberg law of genetics and a calculation of the probability of winning state lotteries. An introductory treatment of statistics can quickly lead to a discussion of political polls, the design and interpretation of surveys and of related decision-making problems. Modern applications of elementary network theory include recent work in computational complexity and almost unbreakable codes.

6. Introducing problems involving decision-making. There are many situations described by elementary mathematics in which one must choose "rationally" among possible options. One can discuss quantifying risk and uncertainty, fair division schemes, applications of network flows, pursuit and navigation problems, game theory and numerous other topics. Political science is full of unexpected but usually interesting topics, including Arrow's theorem and its offshoot theories of voting, the recently discovered problems associated with apportionment of legislatures, and strategies of fair voting in multiple candidate elections.

7. Exploring the powers and limitations of mathematical models. Each of the modern social sciences abounds with applications of elementary mathematics. All of the examples mentioned above, and many more, involve the use of mathematical models. Sometimes these models are quite accurate and sometimes they are not. But even in the latter case the model can help clarify one's thinking about the underlying problem. An example of this use of mathematical modelling is the prisoner's dilemma argument of game theory and its possible connection with U.S.-U.S.S.R. relations.

## References

Numerous references for mathematics appreciation courses are given in Section IX of this report. We list here only a few specific suggestions for the numbered topics mentioned above.

1. See the handbooks for various calculators.

2. Much of this is in standard textbooks. Morris Kline's *Mathematics in Western Culture* and George Pólya's *Mathematical Models in Science* are helpful sources.

3. Libby's work is briefly discussed in several elementary texts on ordinary differential equations. e.g., in *Differential Equations with Applications and Historical Notes* by George F. Simmons. For some work of Arrow see Edward Bender, *An Introduction to Mathematical Modeling*, Wiley, New York, 1978. Lederberg published an article in this MONTHLY entitled "Hamilton circuits of convex trivalent polyhedra (up to 18 vertices)" in vol. 74, pp. 522–527.

4. See any modern text on ordinary differential equations. A particularly good one is Martin Braun, *Differential Equations and Their Applications*, Springer-Verlag, New York, 1975.

5. The Hardy-Weinberg law appears in several texts on finite mathematics, e.g., *Applied Finite Mathematics* by Anton and Kolman. *How to Lie with Statistics* by Darrel Huff and Irving Geis, Norton, 1954, and other texts contain situational mathematics which can be discussed according to the interests of the audience. For the two topics mentioned last, see *Scientific American*, Jan. 1978, p. 96, and Aug. 1977.

6. See Bender (as in 3); the articles by William Lucas in vol. 2 of the forthcoming *Modules in Applied Mathematics* (Springer-Verlag, New York); M. Balinski and H. P. Young, *Proc. Nat. Acad. Sci. U.S.A.*, 77 (January 1980) 1–4; H. Hamburger, *J. Math. Sociology*, 3 (1973) 27–48; David Gale, UMAP Module 317, 1978; W. Stromquist, this MONTHLY 87 (1980) 640–644; and George Minty's article in M. D. Thompson, ed., *Discrete Mathematics and its Applications* (Indiana University, Bloomington, 1977).

**VI. Two-Year Colleges.** Many courses that ought to follow the "mathematics appreciation" philosophy are taught in two-year colleges. Innovative approaches and curriculum development by some two-year faculty are reflected by their texts and articles in this area. Although the preceding sections of this report are applicable to mathematics appreciation courses in all colleges, this separate section appears because of the special problems created in two-year colleges by generally heavy teaching loads, by staffing in some cases by faculty whose mathematical experiences are not sufficient to make them comfortable with the broad range of topics demanded by these courses, and by the regrettable frequency of administrative procedures which allow students needing remediation to enroll in these courses. The following suggestions may help to overcome these impediments to two-year college implementation of the goals of mathematics appreciation courses.

1. When there is a choice among faculty members for assignment to the mathematics appreciation course, only those having a broad range of mathematical experiences and expressing interest in the course should teach it. Mathematics program administrators should provide extra guidance to faculty teaching this course for the first time. In the two-year college there will usually be one text used by all teachers, often supplemented by a reading list and/or other texts; a description of special uses of these materials, as well as sample course outlines, supplementary and classwork materials, and tests, will be helpful. Entrance and/or exit requirements may be matters of policy and should be explained. Lists of applicable resource material owned by the school should be provided to new teachers, along with knowledge of the school's film rental policies.

2. Special attention should be paid to the needs of this course by the library, audio-visual, and computer facilities. The mathematics program administrator should be sure these courses are adequately supported.

3. Since mathematics appreciation courses, properly taught, take an enormous amount of preparation time, any load relief possible would be appropriate. In a suitable lecture room, the course can be effectively taught to "double" sections of 60–90 students if doing so would leave the teacher several hours more preparation time each week. (Such a load might be counted as two or three sections, corresponding to the grading load.)

4. Sharing materials and ideas and perhaps team-teaching would be reasonable for mathematics appreciation courses. One teacher at a school might be most qualified for teaching, say, a computer unit, and might "rotate" across several sections. Many more "hand-out" materials seem to be necessary for mathematics appreciation courses than for traditional courses; these might be used by several teachers in a given term, or re-used in succeeding terms. Faculty teaching mathematics appreciation courses seem to enjoy sharing materials and methods.

5. In-course remediation should be avoided. If students are enrolled who cannot handle elementary operations at the level needed for the work of the course, a "math lab" facility might

be used to design and administer individual remediation programs. It cannot be over-emphasized that a mathematics appreciation course cannot fulfill its goal if it degenerates into the teaching of arithmetic computations or pre-algebra skills, or if it is limited to a topic such as "consumer mathematics."

6. A large proportion of students enter two-year colleges with little realistic expectation concerning majors. Many of these students have had poor experiences with mathematics and, if there is a general education mathematics requirement which may be satisfied by either a mathematics appreciation course or a pre-calculus/calculus course, they will often elect the mathematics appreciation course. Well into a successful term, the student may begin to think realistically about mathematics requirements of various university majors. Since most majors outside the humanities will necessitate at least some mathematics at a technical level rarely achieved in the typical two-year college mathematics appreciation course, an important service of this course can be to channel these students back into regular sequence mathematics courses. Without violating the spirit of a mathematics appreciation course, it is possible to include a topically organized unit requiring the review and use of elementary algebra and graphing techniques; this may give the student a successful experience in doing mathematics that serve as encouragement to return to regular sequence mathematics courses. (A linear programming unit, for example, requires the students to review or acquire facility with graphing and algebra techniques. Many of the topics suitable for a mathematics appreciation course can be handled in this way.) Students with the experience will frequently place higher in the sequence courses than they would have upon original enrollment, and will go on as solid, though late-blooming, students.

**VII. Films.** Since students in mathematics appreciation courses frequently have little experience in sustaining interest in regular mathematics lectures, it is usually appropriate in these courses to provide a variety of class activities. Films are a useful but under-utilized medium for mathematics instruction generally. They are especially useful for the mathematics appreciation course.

We list in the Center Section a selection of films about mathematical subjects that are suitable for lay audiences. (Distributor addresses are listed at the end.) Further information on these and other films is available in the booklet *Annotated Bibliography of Films and Videotapes for College Mathematics* by David Schneider (M.A.A., 1980).

**VIII. Classroom Aids.** Certain topics treated in mathematics appreciation courses are particularly amenable to demonstration with physical or geometric devices. Useful exhibits can often be seen at NCTM meetings. A list of major suppliers of mathematics classroom devices is given in the Center Section.

**IX. References.** Since many of the topics that arise in mathematics appreciation courses occur nowhere else in the mathematics curriculum, it is quite important that instructors be aware of the expository literature of mathematics that treats its relations to science and society. Student term papers in courses on mathematics appreciation typically tax the instructor's knowledge of the literature more than any other course in the mathematics curriculum.

To aid instructors of mathematics appreciation courses, we list in the Center Section major references that would be suitable for background reading, and as sources for special projects. This list does not include textbooks, partly because we do not wish to endorse some books over others, and partly because texts go in and out of print much more rapidly than the reference classics.

---

**ANSWER TO "PHOTO" ON PAGE 11**

Julia Robinson, current president of the American Mathematical Society. Photo by George Bergman.