CUPM Discussion Papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know?

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Contents

Mathematics and the Mathematical Sciences in 2010: What Should Students Know?

ABSTRACT: This report is a call for participation from the mathematics profession. The Committee on the Undergraduate Program in Mathematics (CUPM) is gathering information toward the preparation of a *Curriculum Guide* to help departments plan the further development and evolution of their undergraduate curriculum. Over the past year and a half, CUPM has held paper sessions, panels, and focus groups on issues to consider when planning curricular change. CUPM has also sponsored workshops where faculty from mathematics and mathematics-intensive disciplines have met to discuss the mathematical needs of other disciplines. In the summer of 2000, CUPM invited a number of mathematicians to write thoughtful and provocative papers describing their views on topics in curricular development. In September 2000, CUPM met with these authors and a few other leaders in undergraduate mathematics education at a small three-day workshop. The goal was to identify major issues and develop tentative recommendations for a *Curriculum Guide*. They are summarized in this report. CUPM now invites you, as a member of the mathematical community, to contribute your comments and suggestions to the discussion.

What have you learned about the mathematics curriculum? What are your views on this curriculum? Faculty expend great effort thinking about, preparing for, and facilitating the learning of mathematics. Every child is now promised a college education. The job market values analytical skills above all others. Students graduating in mathematics take jobs and pursue further education in ever-widening areas. Students who once might have majored in mathematics are now concentrating in allied mathematically intensive disciplines. A new surge of students is entering higher education expecting faculty to prepare them in their discipline and provide the background for future careers. Acute teacher shortages are developing in mathematics and science throughout the United States. Universities are instituting procedures of accountability.

What is the appropriate program in this new environment? In order to answer this question for their own students, many departments keep track of their majors, ask employers what characteristics make for a successful employee, and collect evidence about the results of their placement policies. Many faculty conduct studies concerning education innovations. The Committee on Undergraduate Programs in Mathematics (CUPM) is seeking information from those who have undertaken such work; it is also seeking considered views from the community of mathematicians and experts in other disciplines.

Please, page through this report; think about the possible recommendations that are contained in it; identify the ideas in the articles that ring true for you; analyze those questions that bother you; raise issues the report overlooks; but above all, discuss curricula and expectations with your colleagues and share your views with CUPM (`CUPM-curric@maa.org`).

Part I. The Goal of the Curriculum Initiative

The Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America (MAA) issues curriculum guidelines about once every decade. The present committee is currently engaged in gathering information leading to preparation of a *Curriculum Guide*. With far greater diversity among institutions, preparations of students, and expectations of higher education, it is no longer either possible or desirable to simply list and describe the content of a few courses that would constitute a major. We must ask ourselves, "What should students know?" The very process of answering this question will serve as a useful base on which to build a curriculum that can address the needs of a particular college or university. With this in mind, CUPM will prepare a document differing from those in the past. The focus of the *Guide* will not be "What courses should a department offer or require?" but rather "What do we want our students to know and be able to do?" Planning a program in answer to this question is a process. A good deal of thought, information, and discussion are required. CUPM hopes to provide guidance to departments as they go through this process.

To facilitate a planning process, CUPM can assist departments by posing the right questions, suggesting a structure to find answers, and providing examples and information so that departments can benefit from the experience of colleagues elsewhere. The preliminary recommendations are organized into seven broad areas outlined below.

A. Mathematical Knowledge: General Expectations

What should students know upon graduation? Student knowledge grows in several ways. Closest to our own discipline are recommendations about the mathematical content that we expect students to understand. But as a second critical foundation, we also expect a level of skill development. A new third dimension has been added as technology has grown in importance and as more of our students enter the job market directly. Students now may need experience with powerful mathematical software tools not only to promote learning in mathematics, but also to be prepared for a wide range of software tools they will encounter later.

Should every student be given an opportunity to "put it all together" through a group or individual experience? Employers expect graduates to be able to read, write, and speak about mathematics both for their expert colleagues and for others whose background is not in mathematics. We expect our graduates to enjoy mathematics and to be emissaries for it in later life. What classroom experiences foster the attainment of goals like these?

B. A Special Responsibility to Future Teachers

How often have you met someone casually, explained that you "do mathematics," and heard in reply, "Math, I could never do that"? Our first lines of defense against such illiteracy are the teachers in our schools. The country is encountering critical shortages of mathematics and science teachers. Not only do we want high quality teachers who love mathematics, but we want those teachers to teach as we want our children to learn. While many universities assume as their primary responsibility the preparation of future teachers, other universities also have mathematics students who contemplate becoming teachers. Every department should discuss the role of its program in the preparation of future teachers of mathematics and science.

C. Extracurricular Learning Experiences

Some of the most successful departments have thought about and shaped the extracurricular experience for their students. Because motivation so strongly influences learning, these departments have fostered a large number of loyal students. What parts of the environment outside the classroom should a department address? What actions benefit students?

D. Assessment

We would all like to know that we are succeeding. We also want to know if we are falling short of our expectations. Such knowledge comes from recognizing our goals and gathering information that can tell us what progress we are making toward them. How are our students being assessed? Is this compatible with reaching our expectations of them? Is our program succeeding? Are we recruiting, retaining, and graduating the audience we expected? Are students reporting their satisfaction with their preparation once they leave us?

E. The Role and Responsibility of the Department

Consider

- a major land grant university serving 20,000 students in a region that includes an entire state,
- a state university of 13,000 students chartered to serve everyone in a local region,
- a privately endowed university charged to reach out to the "best and brightest" throughout the world,
- a college of 2,000 students focusing on undergraduate education in the liberal arts drawn from across the country,
- a college of 1,500 undergraduates serving particular needs in a local area, and
- a two-year college of 3,000 students preparing some students to immediately enter the workforce and sending other students on to a variety of four-year institutions to prepare for a wide variety of careers (including teaching).

These institutions serve their students in many different kinds of classroom settings providing a wide range of opportunities. Clearly, the programs at these varied institutions are different, in course offerings, in sequence, and in experiences. A program designed locally and informed by who its students are and what its institution's mission is best serves its students. But doesn't it seem reasonable that students graduating with intensive mathematical training from any of these institutions should possess knowledge and skills that are recognizable as preparation in mathematics?

If the faculty is to assist students in meeting our expectations, then what are the needs of the faculty? How should expectations be set and fulfilled so that faculty members enjoy their careers and grow as teachers and mathematicians?

F. Issues for the Discipline

The *Curriculum Guide* will be the first of its type from CUPM. There is a responsibility to produce something of substantial value to departments and to provide them with the kinds of information they need. Many tell CUPM that models of successful efforts are particularly helpful, like the undergraduate programs in *Models That Work* (MAA Notes 38) and the programs at research universities in *Toward Excellence* (AMS, 1999). What information and guidance should the profession provide to departments?

Between 4 and 5 percent of an age group majors in Mathematics, Science, or Engineering, that is, in one of the traditional mathematics-intensive disciplines. This percentage has been fairly constant since the 1950's through both mathematics reform and "back-to-basics" movements. Majors grow or shrink by reapportioning students in this 5 percent group. In this decade, students who would formerly have majored in mathematics are now choosing to major in other disciplines. Is there a responsibility to reach out to them? Should we be doing a better job of addressing the mathematical needs of advanced students in these other mathematics-intensive disciplines? Should we encourage more of these students to major in mathematics? What can the profession do to assist departments as they address these questions?

Part II. The Information Gathering Process

Prior to releasing this report, CUPM engaged in a series of information gathering projects. The first was to examine past CUPM recommendations. The 1981 recommendations recognized that many students wish to combine the study of mathematics with that of other disciplines in order to broaden their knowledge base and enhance their future career opportunities. The 1981 recommendations were reissued in 1988 as *Reshaping College Mathematics* (MAA Notes 13). The 1991 CUPM recommendations appear at the end of *Heeding the Call for Change* (MAA Notes 22). By this time, a list of courses for the major was no longer easy to state and the mathematics community was engaged in major discussions about calculus. Therefore, the 1991 curriculum document is brief. Since 1991, *Models That Work* (MAA Notes 38) described exemplary programs in mathematics, and *Confronting the Core Curriculum* (MAA Notes 45) addressed preparation in the first two years. The recent book on assessment, *Assessment Practices in Undergraduate Mathematics* (MAA Notes 49) examines a broad range of assessment issues and includes models illustrating productive approaches. All of these documents remain useful for departments planning their programs. They also serve as a base for the current curricular discussions.

Since the Summer of 1999, CUPM has been gathering information directly from the profession in the following ways.

Meeting sessions and panels

- At Mathfest 1999, CUPM sponsored a panel/audience reaction session in front of an involved and packed audience.
- At the January 2000 AMS/MAA/SIAM meetings, CUPM sponsored a panel/audience reaction session again to a packed room, plus two very well attended contributed paper sessions.
- At Mathfest 2000, CUPM sponsored a panel of industry members commenting on the curriculum for a full and active audience.

Focus groups

- In January 2000, CUPM invited mathematicians to participate in focus groups discussing curricular issues, clustering participants by institution type.

Interdisciplinary conferences

- Under the title "Curriculum Foundations Project," the subcommittee Calculus Reform and the First Two Years (CRAFTY) is holding 11 different workshops bringing mathematicians together with faculty from other mathematics-intensive disciplines to discuss the undergraduate curriculum. From these meetings a sequence of reports will be issued by those other disciplines on the mathematical needs of their students. Two panels at the January 2001 AMS/MAA/SIAM meetings will discuss the findings of workshops to date. The remaining workshops will be held in 2001, and the reports from the various disciplines will be combined for distribution later that year. The reports themselves will be made available prior to that time on the web.

Invited papers followed by a workshop discussion (September 2000)

- CUPM solicited papers from a number of mathematicians asking them to address issues central to developing a planning document for departments. Writers, members of CUPM, and a few others met in September 2000 for a workshop informed by and taking off from the issues raised in the papers. The invited papers appear in this report along with a summary of the deliberations at the workshop and tentative recommendations that CUPM would like to discuss with the larger community.

A call for discussion

- This report is being issued in time for the January 2001 AMS/MAA/SIAM meetings. At that meeting, an extended panel session has been scheduled to elicit comment from the mathematical community. A set of focus groups is being arranged to obtain reactions in depth from participating mathematicians.

Involvement of mathematics-intensive disciplines

- Representatives of sister societies in engineering, physics, and economics attended the September 2000 workshop reported on here. AMS has representation on CUPM through membership on CUPM and through a liaison appointed by the President of AMS. The presidents of MAA, SIAM, and AMATYC contributed papers and participated in the workshop. During the next year, liaisons from a broad range of additional disciplines will be asked to contribute to CUPM's deliberations.

CUPM seeks to involve the major stakeholders, with greatest emphasis on the mathematics profession itself. If we can engage you in active discussion, with your colleagues and through your profession, then we will have already done a great deal for curricular awareness and improvement. Further, through the information you will provide us, we will be able to offer the profession the best guidance and advice.

We have had regular articles about our work appear in *Focus*, and there will be more. There is an email address that directly reaches CUPM and is open for your use (CUPM-curric@maa.org). By the time this report appears we hope also to have a web page. CUPM wants to listen to your views.

Part III. Issues and Possible Recommendations Identified at the September 2000 CUPM Workshop

As described in Part II of this report, members of CUPM are engaged in a series of conversations in preparation for developing a guide to help mathematics departments develop, modify, and refine their overall undergraduate programs. This final guide has been tentatively entitled *Mathematics and Other Mathematics-Intensive Majors: A CUPM Curriculum Guide*.

The September 2000 CUPM Workshop referred to in Part II was attended by members of CUPM, the authors of the papers that are included in this report, representatives of professional societies for some mathematics-intensive disciplines, and a small number of other leaders in the mathematics community. During the workshop, the participants met in small groups to begin to identify key issues and questions to be addressed in the *Curriculum Guide*.

What follows is framed as a sequence of possible recommendations accompanied by some discussion of the rationale supporting the recommendations, along with concerns that might argue against them. These recommendations can be viewed as a sequence of questions that raise issues of importance. *They do not constitute a draft of the Curriculum Guide. Moreover, it should be emphasized that neither the full CUPM Workshop nor the CUPM as a committee has endorsed any of these possible recommendations*. In fact, these groups did not even formally decide that these issues are indeed the most important. They were selected because they emerged in our discussions, and they do reflect the thinking of a large number of CUPM members and others who have been involved in the process during the past year.

CUPM believes that now is the appropriate time to share this thinking with the entire mathematics community and to obtain reaction to this preliminary work. We hope and expect that the final *Guide* will be a useful tool to shape departmental discussions and to present to deans and other administrators to explain the needs and expectations of a well-functioning undergraduate mathematics program. Please keep this dual audience in mind as you make recommendations to CUPM for the *Guide*. Recommendations are needed both on the particular areas under consideration here and on anything perceived to be missing.

The first three sets of possible recommendations directly address the question, "What do we want our students to be able to do?" Throughout its work, the CUPM defines as "our students" those who are enrolled in mathematical sciences courses and are majoring in mathematics or other mathematics-intensive disciplines, including physical sciences, engineering, computer science, and many subfields of a large number of disciplines. Expectations having to do with content mastery have not been separated from those having to do with broader thinking skills and supporting competencies, although the more general expectations are divided from more specific ones.

A. Mathematical Knowledge: General Expectations

1. **Possible recommendation: All students should achieve mastery of a rich and diverse set of mathematical ideas.**
   The particular ideas will vary — no single list will be right for every institution or program, let alone every student. But mathematics is about ideas, and the conceptual content of the discipline deserves

attention. Especially as student programs become more diverse, attention should be paid to interconnections and broad themes (linearization, optimization, symmetry, approximation, etc.) that give coherence to a major.

Concerns. A minority argues that we should attempt to identify a core of ideas with which every major should be familiar.

2. **Possible recommendation: All students should be able to think analytically and critically and to formulate problems, solve them, and interpret their solutions.**

   As the list of topics that students are to master is developed, the technology skills are stated, and the ability to communicate is emphasized, it is possible that the central goal of teaching students to think analytically and critically may be obscured. Faculty should emphasize these general thinking skills and support students as they learn to think mathematically. For example, even something as simple as asking students whether they have seen a similar problem before is an opportunity to reflect on the use of analogy as a tool.

Concerns. Is it appropriate for CUPM to be making such a recommendation? Mathematical thinking is central to our discipline and perhaps doesn't need to be explicitly stated as an expectation.

3. **Possible recommendation: All students should achieve an understanding of the nature of proof.**

   Proof is what makes mathematics special. Students should understand and appreciate the core of mathematical culture: the value and validity of careful reasoning, precise definition, and close argument. The development of this understanding can begin with activities such as finding a counterexample to a false statement, reading and critiquing a short proof, or completing a proof given its first step or two. Indeed, all mathematics courses should contribute to the development of critical thinking/reasoning skills in order to teach students to think, rather than just seeing mathematics in procedural terms. Those majoring in mathematics should move beyond the developmental activities described above and develop an ability to write some complete proofs.

Concerns. Might the inclusion of these encounters with proof alienate many students? No one has argued that courses, particularly those designed also to meet the needs of non-mathematics majors, should be structured in a theorem-proof mode. However, a recommendation to include proof as a central component of the mathematics curriculum could lead to empty formalism that is meaningless to students or could distract attention from the power of well-chosen examples to motivate ideas and illuminate their interrelations. Also, issues of proof can become "code" for the divide between those who believe in a very traditional mathematics major aimed at students who might ultimately earn PhDs, and those who are concerned with what students who major in mathematics can do with only a bachelor's degree.

4. **Possible recommendation: All students should have experience applying knowledge from one branch of mathematics to another and from mathematics to other disciplines.**

   Different working groups discussed the needs of students who were intending to pursue graduate work in mathematics, enter the work force, or prepare for teaching. Each working group recommended that the students they were considering should have practice in translating problems from other disciplines into mathematics and then expressing the solution in language understandable to those with relatively little mathematical training. Courses in statistics, computational science, computer graphics, and operations research are particularly valuable for students entering the work force. All students should appreciate the role mathematics plays in applications, especially in the development of new technologies; for example, for computing, medical imaging, and communication.

Concerns. Some urge a stronger recommendation: that all students should have a deep understanding of at least one other discipline that uses mathematics.

5. **Possible recommendation: All students should experience mathematics as an engaging field with contemporary open questions as opposed to an elegant body of knowledge that is complete and static.**

   The primary motivating examples should play a central role in all abstract courses, and there should be significant exposure to the deep interconnections and interplay among diverse mathematical topics and current applications. This understanding should be developed throughout the curriculum and then reinforced through an in-depth study of some specific area, perhaps through a two-course sequence. The subject matter of such a two-course sequence is less important than its ability to develop student confidence and facility with reading mathematics, analyzing mathematical arguments, creating their own mathematical arguments, exploring ideas, and coming up with their own questions and conjectures. Direct contact with open questions is key at this stage.

Concerns. There were no concerns expressed with this recommendation.

6. **Possible recommendation: All students majoring in mathematics and mathematics-intensive fields should be able to use a variety of technological tools: e.g., algebraic and visualization software, statistical packages, a high-level programming language.**

   Technology has become a major tool in all scientific and engineering investigations. Research mathematicians use powerful computer tools to investigate examples and sort through a myriad of situations. Scientists use software to model theories and to cope with large-scale data gathering and analysis. Engineers often rely more on computational models than on prototypes. Through all of this runs software enabling mathematicians, scientists and engineers to organize and communicate their ideas. Industry has incorporated advanced tools to maintain a competitive position in the marketplace. Experiences using technology and computer languages are valuable tools for understanding and exploring mathematics, and competence in the use of these tools is needed for employment.

Concerns. Potential dangers in using various technologies include: diversion of time and resources. Steep technology learning curves can create new obstacles. Some felt these potential dangers outweigh the possible benefits and questioned whether technology indeed enhances learning. Also, the use of technology can exaggerate various equity issues relating to background and financial resources of students. Moreover, differences in experience with technology can also raise equity issues relating to gender.

7. **Possible recommendation. All students should be able to communicate mathematics both orally and in writing.**

   Students should have many points in their undergraduate mathematics program where they are required to write and give oral presentations about mathematics. These reports and presentations should be accompanied by critical feedback so that students are able to improve these skills. There are many formal and informal situations in which these abilities can be developed. For example, if students answer questions posed by other students during class, this adds to their speaking and listening skills.

Concerns. Providing experiences for all mathematics majors to develop these abilities requires a substantial investment of faculty time and effort. Does the importance of these abilities justify the commitment of resources? (Most argued yes.)

B. Mathematical Knowledge: Specific Expectations

1. **Possible recommendation: All mathematics majors should have a command of ideas and techniques ranging across single and multivariate calculus, discrete mathematics, linear algebra, statistics, and differential equations.**

   These topics stand at the gateway to powerful and deep mathematics. In many mathematical areas, ideas from these areas merge and become powerful tools to understanding deeper mathematics. They provide the foundation for more advanced study of mathematics and the breadth necessary to apply mathematics flexibly. However, this possible recommendation does not mean that these topics must, of necessity, be the first college-level mathematics courses taken by potential mathematics majors.

Concerns. Some participants thought that providing alternate routes to the major would attract different students and energize the major. They argued that the dramatic decline in the number of math majors makes it important to recognize computer science (CS) students as a large source of potential double majors. The new CS curricular recommendations will dramatically downplay the role of calculus, which will lead many CS students to postpone taking it, starting their college programs with discrete mathematics instead. Others were concerned with the sheer length of this list of subjects (see concerns below).

2. **Possible recommendation: All mathematics majors should have experience in algebra, analysis, geometry, probability, and mathematical modeling, with a more substantial experience in at least one of these areas.**

   These experiences should be provided in a coherent program. However, a variety of different courses or combinations of courses could provide each of the experiences. For example, students could have an algebra experience in a variety of different settings other than the traditional abstract algebra course. In addition, it is envisaged that two or more of these experiences could occur within one course.

Concerns. Would the major be weakened by not having all students take an upper-level core? Although there is no consensus on what a core should be, there was a minority view that the CUPM recommendations should seek to develop such a core. From a different perspective, there is a minority view that the recommendations should not call for experiences in any specific content area. Given the diversity of institutions and the wide range of abilities and aspirations of their students, this line of thought urges that CUPM provide a framework that is very flexible, recommending only that students be exposed to a variety of perspectives: algebraic, geometric, formal, intuitive, applications-oriented, analytical, deductive, experimental, etc. Some successful programs do not require majors to take either algebra or analysis, and many do not currently require any geometry (see below). Finally, if the list of topics included in these possible recommendations is too long that risks making the mathematics major less attractive at a time of declining interest in the major.

3. **Possible recommendation: All students should develop skills in three-dimensional visualization and geometry significantly beyond what is currently expected in most undergraduate programs**.

   The three-dimensional visualization skills of undergraduates are not good and can be improved through instruction. Geometry, long relegated to a secondary status in the undergraduate curriculum, needs to be rejuvenated and given a more central role in the education of all students. It provides an excellent framework for courses that integrate diverse branches of mathematics, and workshop participants believe that it is naturally appealing and accessible to undergraduates.

Concerns. Why highlight geometry to this degree, given that many areas of mathematics are not emphasized? However, many argued that the teaching of geometry has been so de-emphasized over the past 40 years that this recommendation is warranted.

4. **Possible recommendation: All mathematics majors should have an experience working on an intensive project that requires them to analyze and create mathematical arguments and then to produce a substantial written and oral report.**

   This experience should enable students to integrate the mathematics they have learned, to hone their skills at reading and analyzing mathematics, and to show that they can establish their own insights and communicate them effectively. For students preparing for graduate school in mathematics this could be a research experience—an opportunity to taste the satisfactions and frustrations of the search for mathematical knowledge. For students planning to enter the work force, this might be a team project that seeks to solve an industrial problem. A modeling course offered at an advanced level would be another good mechanism.

Concerns. Providing this experience for all mathematics majors requires a substantial investment of faculty time.

C. Specific Needs of Future Teachers

Many mathematics majors are preparing to be teachers of mathematics. Most of the general recommendations for majors are appropriate for prospective teachers. But there are some special issues. Members of CUPM are paying close attention to the report of the Conference Board of Mathematical Sciences on the preparation of future teachers. CUPM is particularly concerned about the large numbers of individuals who did not major in mathematics but are now teaching mathematics at the middle and secondary levels. This situation is especially serious at the middle school level. All are agreed that effective teachers require a solid preparation in mathematics, but there is growing discussion of whether a "traditional" major is the best means. What is appropriate mathematics preparation for a prospective teacher merits more attention and examination.

1. **Possible Recommendation. Students preparing to teach high school mathematics need sufficient breadth of study to give them a coherent picture of the discipline; they also need sufficient depth to make connections and deal effectively with student questions.**

   Topics of study should include geometry, statistics, calculus, mathematical modeling, and physics.

Concerns. Some questioned the inclusion of physics in the list. Others felt that discrete mathematics and/or probability should be added.

2. **Possible Recommendation. Students should directly make the connections between what they are learning in their college mathematics courses and what they will be teaching in their high school classroom.**

   These types of connections could be more easily provided if some of the mathematics courses taken by these students were designed specifically for prospective teachers.

Concerns. There was not agreement on the desirability of having special courses designed for future teachers. In small institutions, special courses would likely be impractical.

3. **Possible Recommendation: Future teachers should develop strong teaching skills prior to completing their program.**

   This would require opportunities for teaching experience prior to the formal student-teaching experience at the program's end. (Someone suggested the abbreviation TEU: Teaching Experiences for Undergraduates.) In addition, future teachers should develop the need and desire for continued life-long mathematical learning as part of an ongoing professional development program.

Concerns. Providing these types of experiences for all mathematics majors who are preparing to teach requires a substantial investment of faculty time and effort.

Sets A–C of possible recommendations directly address the question, "What do we want our students to be able to do?" The following sets of recommendations involve the infrastructure and support needed by students and faculty in order for students to develop these abilities.

D. Extracurricular Learning Environment

In the view of the workshop participants, it is probably not possible to develop the called-for skills entirely within the formal classes required of a major. Much needs to be accomplished outside of the classroom.

1. **Possible recommendation: Departments should provide space for informal student contact.**

   An impressive example of a successful student space is the mathematics lounge provided for mathematics majors at one institution. The lounge contains the work area for the secretary who supports the undergraduate program, computers, study tables, pictures of students, and blackboards. Majors are encouraged to eat lunch in the room and to form study groups centered at a table or a blackboard. Having such a space can make possible many of the recommendations described above.

Concerns. Is this recommendation realistic for all mathematics departments?

2. **Possible recommendation: Students should be expected to communicate mathematics in a variety of settings outside of class**.

   While most departments would probably not offer all of the following opportunities, all departments should offer, and expect students majoring in the department to participate in, many of the following: Undergraduate Mathematics Colloquia; Mathematics Clubs or student MAA Chapters; undergraduate research projects; regional/national meetings; employment within the department in a number of different capacities; Mathematical Modeling and Putnam competitions; informal contact with faculty in social situations, lunches, etc. A successful example is the Undergraduate Mathematics Colloquium offered at one institution. The Colloquium meets every other week and features internal as well as external speakers, applications of mathematics, new mathematics, and other enrichment activities. On some occasions recent graduates are invited to return and talk about their job-related experiences and the extent to which they were prepared for employment. As a graduation requirement, all students must attend at least 12 colloquia throughout their college experience and they must write reports about four of the colloquia presentations.

Concerns. Is this recommendation realistic for all mathematics departments?

3. **Possible recommendation: Departments should recognize student achievement in scholarship and for service through a variety of means.**

   Possibilities include: Putnam competition; Modeling competition; Calculus Award; Department Service Award; Outstanding Senior; Sophomore Prize; department scholarships; prizes named after retiring faculty.

Concerns. None were expressed. It would not be difficult for departments to take these steps.

4. **Possible recommendation: Mentoring programs should be set up for all potential majors.**

   Women and members of typically underrepresented groups need to be encouraged to major in mathematics, but all students benefit from mentoring. One particularly effective approach is to invite recent graduates to talk about their job experiences and preparedness. Upper-class students can also be effective mentors of beginning students.

Concerns. None were expressed. It would not be difficult for departments to take these steps, particularly if students were recruited to serve as mentors.

E. Assessment

1. **Possible recommendation: Assessment should be integral to all of our work, not just something we do at the end of a project.**

   Asking questions as we work gives us information and helps shape our thinking. What we learn prompts new questions and more learning. Evaluation is only one purpose of assessment. At least as important is feedback for the purpose of improvement**.** Assessment can also provide valuable documentation of efforts and positive outcomes. Assessment should always be used in some way; it should be designed to be useful.

2. **Possible recommendation. When we formulate an expectation, we should also describe possible indicators from which we can learn the degree to which the expectation has been met.**

   Expectations must be explicitly stated beforehand**.** Continuing and effective improvement requires continual assessment. Departments must have reliable means of measuring how well they are meeting their own goals, and they must have mechanisms in place for using this information to inform and improve their programs. "Closing the loop on assessment" is not just jargon. It is essential.

Concerns. Many mathematicians are unfamiliar with formal assessment procedures. Models of effective assessment in different institutions would be enormously helpful to departments and programs. CUPM is considering describing such models as a portion of its *Curriculum Guide*.

F. The Role and Responsibility of the Department

Strong infrastructure support at the local level is necessary for programs that assure that graduates obtain the various skills, competencies, and viewpoints advocated. Participants in the CUPM Conference believe that effective programs must be developed at the local level and that the development of high quality programs requires strong leadership and support of the faculty.

1. **Possible recommendation: Each undergraduate program must be developed locally. Each department needs to gather information on its own students, its own departmental and institutional resources, and its constraints.**

   The first step in moving each department toward its goals is to articulate these goals and the department's vision for itself. This vision should be focused on what it wants its students to be able to do. Departments need to have available time-series data on such things as teaching loads, course enrollments, completion rates, and various student outcomes in order to frame realistic goals and to move toward them.

Concerns. There were no concerns expressed.

2. **Possible recommendation: A high priority must be placed on coordination and cooperation on curricular design with our partner disciplines.**

   We need to build collaborative relationships with partner disciplines for curriculum planning, on student placement in courses, and on faculty development. We have much to gain from these collaborations.

Concerns. This not be interpreted as a diminution of the department's own needs, relegating its status to that of a mere service department.

3. **Possible recommendation: Departments should be encouraged to investigate and foster possible joint majors with partner disciplines.**

   These joint majors may go beyond the "mathematical sciences" majors recommended in the last CUPM curriculum report and be more evenly balanced between courses in the mathematical sciences and in the partner discipline. Each department would need to understand the strengths in its institution, both within and outside of the mathematics department, to determine which opportunities are most promising.

Concerns. Some participants seek assurance that such programs will not, in some way, weaken the mathematics major.

4. **Possible recommendation: Each department needs to provide appropriate infrastructure to support all faculty (full-time, part-time, GTAs) in using technology.**

   Institutions are used to providing support to faculty in science and computer science programs, but the need to provide a wide range of support for offering appropriate technology in mathematics is not fully recognized.

Concerns. None were expressed.

5. **Possible recommendation: The faculty as a whole and individual faculty members need to focus on the needs of students.**

   We need to understand the extent and nature of their mathematical background and their background in other areas. This information is important for meeting the needs of students in other mathematics-intensive disciplines and in developing courses and programs for students majoring in mathematics.

Concerns. None were expressed.

G. Issues for the discipline

1. **Possible recommendation: The decline in the number of mathematics majors at a time of great demand must be addressed locally and nationally**.

   We need to consider whether the recommendations proposed here will result in the needed number of mathematics majors nationwide. The MAA can serve as a clearinghouse for innovative ideas and programs from institutions around the country that increase the number of students majoring in the mathematical sciences.

2. **Possible recommendation: The declining number of majors should lead us to consider more proactive ways to encourage women and minorities to enter the profession**.

   Under these circumstances of unmet mathematical demand, it is necessary that we fully utilize our talent pool.

3. **Possible recommendation: CUPM and the broader mathematics community can help most by providing models of successful departments.**

   Models include how they came to embark on their efforts, what information they gathered to prompt and/or guide their work, what and how they changed, and what the consequences were. Models are needed rather than prescriptions since programs offered at each institution need to reflect the make-up of the student body, the strengths and interests of the faculty, and the nature of the institution.

*Concerns***.** No concerns were expressed about these three recommendations. However, CUPM will need the help of the readers of this document to obtain such information.

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Mathematics and Mathematical Sciences in 2010: What Should a Graduate Know? Some Predictions for the Next Decade

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What is different about the way we answer the question in the title now, as compared to ten or twenty years ago? At each of those times, it seemed that mathematicians were more confident about their answers. After all, mathematics had been around for a long time and the content of courses had not undergone much change. But now, over the past decade in particular, the pace of change has changed so greatly that it is hard to imagine how to predict what will be expected ten years hence. The paragraphs below represent an attempt to make some predictions as to what our students will know when they complete undergraduate concentrations in mathematics and the mathematical sciences. Any such predictions fall short of what actually will occur.

Current Expectations

For many years we have agreed on an entry-level sequence for college students interested in subjects that rely on mathematics, namely that everyone should have the equivalent of three semesters of calculus and a semester of linear algebra as basic prerequisites for any mathematical sciences degree. This seems to be accepted by just about everyone. Beyond that, there is a good deal of variation. It is quite possible for a student to complete a mathematics major without taking any probability or statistics, or any geometry or topology, or any ordinary or partial differential equations, although most concentrators will take one or more courses in one or more of these areas.

The usual expectation is that a mathematics concentrator will have at least one course in “Groups, Rings and Fields.” There will always be some quantity requirements as well, usually spelled out in terms of a number of individual courses. Some departments will describe two different concentrations, one of a general sort and the other aimed at providing preparation for graduate school, the second requiring more specific courses, sometimes in sequences.

Future Placement

That describes the present. What about the future? Over the next ten years, there will be a widening of the gap between well-prepared and less well-prepared students coming to colleges and universities. It is already true in many places that the majority of students who want to graduate with a degree in mathematics have taken calculus in secondary school. Those who have taken an Advanced Placement test can reliably be placed in an appropriate college course that will not require them to repeat material they have mastered. For most students, this amounts to a semester of placement, meaning that by the end of the first year they will complete the equivalent of three semesters of calculus. Within ten years, the students in this category will be presenting reliable evidence indicating that they have completed the equivalent of a full year of college calculus.

For more and more of these students, the background in pre-calculus and calculus will be acquired through distance learning programs administered by private agencies as well as by extension divisions of colleges and univerisities. In some cases these will be group efforts, either for a number of students at a particular secondary school who have gone beyond the mathematics that the faculty can offer, or in a "virtual class" made up of individual students in different locations, under the direction of a course coordinator working for the distance learning agency. Colleges and universities will have to accept the challenge of dealing with these new ways of obtaining and certifying achievement in courses traditionally taken in college.

In addition to courses in the one-variable calculus sequence, students will be presenting evidence of course work in other subjects, most commonly multivariable calculus, differential equations, and linear algebra. There will also be more students who have experience with statistics and discrete math courses. There will be occasional courses in dynamical systems or number theory or modern algebra topics. It is difficult to see how students with such varying backgrounds could be accommodated in current undergraduate programs, and it will be necessary for departments in the mathematical sciences to find ways of building on the experiences of beginning students with varied preparations. This can become an easier process as traditional boundaries between courses become more flexible, and this is connected with the continued development of supertexts.

Emergence of Supertexts

With the continued development of hypertext technology, and with the accelerated construction of Internet-based courses, the demarcation lines between courses will be less clear and courses will become more flexible. A course will be seen as a collection of modules, with the order of topics and the level of treatment selected by the instructor. That happens to a certain extent today, as instructors attempt to pick and choose among the offerings in a textbook, sometimes proposing to alter the order of presentation. The difference in the hypertext situation is that each topic in a course will be accompanied by background material for students with gaps in their preparation, as well as advanced discussion of more subtle points, as well as applications to a variety of different disciplines. A student can choose material appropriate to his or her background, level of understanding, and particular interests. There can also be lateral tie-ins with other courses in mathematics and related disciplines, so students can get a preview of higher-level mathematics courses and even of research problems in mathematics and related disciplines. A hypertext can also provide historical and philosophical background at different levels, more so than can be presented by occasional sidebars and footnotes or endnotes.

Since I am currently preparing for my courses in multivariable calculus and differential geometry, let me mention a few ideas from those subjects.

Example: A teacher of multivariable calculus will be able to introduce a collection of definitions of continuity, recalling the standard definition for a real-valued function of a single real variable. A function f is continuous at a point x_0 if, for an ε there is a δ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. It is rather remarkable that this definition still holds true if x is considered to be a vector in m-space and $f(x)$ is a vector in n-space, where the absolute value sign is interpreted as the length of a vector. The same definition will work if x and $f(x)$ are interpreted as complex numbers.

Furthermore many of the proofs from one variable go over without change in this more general context. For example, if f is a continuous function and the collection of values $f(x)$ is a subset of the domain of a continuous function g, then the composition $g \circ f$ is also continuous.

Proof: Given $\varepsilon > 0$, there is a δ such that $|g(y) - g(y_0)| < \varepsilon$ if $|y - y_0| < \delta$, and there is a δ_i such that $|f(x) - f(x_0)| < \delta$ if $|x - x_0| < \delta_i$. Therefore given ε, there is a δ_i such that $|g(f(x)) - g(f(x_0))| < \varepsilon$ if $|x - x_0| < \delta_i$. A student who understands that in the simplest case will be able to understand it in all dimensions, once the absolute value of the difference is interpreted as a length.

A slightly more subtle example is given by the Intermediate Value Property. A real-valued function f of a real variable has the IVP if when $f(a) < C < f(b)$ for an interval $[a,b]$ in the domain, there must be a c in that interval such that $f(c) = C$. We can obtain an analogue of this function for maps of n-space to n-space. If D is a disc of radius r about x_0 in the domain of a contiunous function f from n-space to n-space, and if there is a ray starting at a point C intersecting the image of the sphere $\{x$ such that $|x - x_0| = r\}$ an odd number of times, then for some c in D, we have $f(c) = C$. In order for a student to understand this theorem, he or she must appreciate the idea that the boundary of a ball of dimension n is a sphere of dimension $n - 1$. The details of such arguments are usually treated in courses in differential topology, and it is a good idea to let students know what lies ahead.

Generalization should be a natural reaction for the mathematics a major of 2010. If something does not generalize, then that should be a surprise. What about Lipschitz continuity? We say a real-valued function of one real variable is Lipschitz continuous if there is a k such that $|f(x) - f(y)| < k|x - y|$ for all x and y in the domain. The geometric interpretation is that the graph of the function is contained in any wedge with angle dependent on k about a point of the domain. Then $|f(x) - f(x_0)| < k|x - x_0|$ so we automatically get continuity by choosing $\delta = (1/k)\varepsilon$. Once again, this works in all dimensions. If the domain is two-dimensional, then instead of a double wedge, we get a double cone as the comparison surface. Similarly, Hoelder continuity with a coefficient α means that the graph is contained not in a double cone but in a double "α cone," of the form $|f(x) - f(x_0)| < k|x - x_0|^{\alpha}$. Thus a function can be Hoelder continuous without being Lipschitz continuous. The same test examples that work for functions of a single variable, will work in the general case. There is a subtle difference between stating extremely general definitions and then applying them to the simplest situation, as opposed to starting with the simplest cases and stating the definition in such a way that it generalizes automatically.

Breaking Course Barriers

These examples are meant to suggest that in the future there will be fewer distinctions between subjects than there are now, and the text materials will make it much easier for students to go from one level to another. Isolated texts in individual subjects will be replaced by "supertexts" covering a wide range of material, most probably written not by a single author or a pair of authors, but rather by "authoring teams" comprised of experts with complementary specialties. In addition to content specialists, there will be professionals in pedagogy, as well as designers of hypertext materials, interfaces for the Internet, and assessment and evaluation tools. These efforts will try to provide text material for the total content of a wide range of courses, with a consistent notation throughout. At any point of entry in such a supertext, a reader can inquire about what background is necessary to follow any particular argument, and the appropriate pathways will be displayed on some sort of universal table of contents. In many current-day texts, there are flow charts included at the end of a preface, and they are almost universally ignored. For a supertext, interaction graphs will be the primary navigation device, for teacher and for students.

The closest we have come to a supertext in the past is a series of books at the beginning undergraduate level, for example, by Earl Swokowski. An example at a somewhat different level is the remarkable collection of texts by Serge Lang. It will be less likely in the future that a supercollection will be produced by a single individual. For one thing, there will be continual maintenance, so that editions and printings will become a thing of the past. It might be expected or required that students will use the same changeable text as their teacher, much in the same way that students currently use the first two-thirds of a large calculus book as their introductory course, and then have to buy the new edition of the last third the following year. Any supertext project will have to have a staff that maintains the integrity of the text. It may be that "maintainers" will make substantial revisions only at fixed periods. There can easily be a constantly updated errata file, with flags at the appropriate spots in the text.

As we have illustrated above, one of the advantages of such approaches will be a consistent treatment of one-variable and many-variable calculus. A natural follow-up will be a course in functions of a complex variable, which every student of mathematics or mathematical sciences can be expected to understand, rather than the somewhat fragmented situation that now exists, where complex analysis is quite different than its real counterpart.

Another natural course to follow multivariable calculus and linear algebra is differential geometry, a course that can intensify understanding of curves and surfaces and lead up to a big result, the Gauss-Bonnet Theorem. It will be very nice to know that students share the same background when they take this course, and that there will be a natural place to which to direct students who lack some specific background.

At present, advanced texts have to keep designating problems "for those familiar with complex analysis" or "for those who know differential equations." Such linking among various courses will be quite natural and easy to achieve in a supertext environment.

Profile Evaluations

In the age of the supertext, the expectations will be somewhat different from what they are now. Students in 2010 who want to go on to graduate school will be able to certify their levels of achievement in a way that will make preliminary examinations unnecessary. A version of Graduate Record Exams will be available for all standard undergraduate topics, with subscores that indicate precisely what each student can or cannot handle at the requested competence level. Students can come equipped with a complete profile, indicating what they can do quickly and what they can do with sufficient reflection. The single-grade transcript will be considered a laughable vestige of an unenlightened past. A global profile will take the place of a Grade Point Average. It is possible that this global profiling can be a matter of choice, much in the same way that students today can designate some of their courses as pass/fail or satisfactory/no credit. Anyone can rely on letters of recommendation in the old style, available from a universal dossier depository. Selection of entering graduate classes will be done largely by filtering processes from the data base that takes the place of the Graduate Record Examinations. Competency tests will be standardized, with record including sound clips as well as sample writing assignments under varying time constraints.

Teaching assistants will be chosen from among students who have already served as undergraduate assistants running virtual recitation sections. There will be simulated synchronous discussions of examples and homeworks, with no necessity for students to congregate in a single room any more than they would be expected to assemble at specific times in laboratories. Empathy for different styles of teaching and learning will be guaranteed when students have the opportunity not only to hear contributions of classmates proffered in class sessions, but also to read what they submit to structured online discussions. Within a course, everyone gets to read what everyone else has submitted, and to some extent, they are required to do so in order to be full participants. Varying levels of participation are possible as well. For classes of sufficient size, there can be learning subcommunities (corresponding to the obsolete recitation sections). These can, of course, be monitored by the course leader to guarantee consistency and quality control, much in the same way that there is a day of reckoning in multisection courses at the time of the common final examination. Exams will take place on a universal honor system, with built-in checks to identify cheating. Students will decide when to take a test, and sign up for a secure booth where they are isolated from other individuals and other sources of information. What they have access to is controlled.

Mathematics and Computer Science Concentrations in 2010

At the present time, we still distinguish between math majors and computer science (CS) majors, but by 2010, that distinction might well be obsolete, as antediluvian as an economics major that did not even require a semester of calculus. All of our best majors at Brown in recent years have had a great deal of experience with computers. Most of them have taken several courses in the Computer Science Department, and those who have not have had no difficulty learning a number of computer languages and complex programs. Some of these students complete a computer science major along with a mathematics major. The best of these students will end up with a Bachelor of Science combined major in mathematics and computer science. Other students will take a number of CS courses but not pursue a degree in that subject. Naturally a CS degree presupposes a breadth of understanding in a number of different areas in that subject, whereas a mathematics student might be primarily interested in a particular subfield, such as robotics or computer graphics, and go up to the graduate level in that area while avoiding courses in hardware or operating systems. It may be desirable to preserve such options, but for all practical purposes, the 2010 mathematics student who does not know a good deal of computer graphics will be rare indeed.

Some of my best undergraduates, students who have taken more than one of my courses and who have acted as my assistants and my collaborators, have gone on to careers in computer science, most often in computer graphics. Several of them have obtained PhDs and are teaching in colleges and universities. They use a great deal of mathematics in their work, and they encourage their students to take many courses in mathematics. That is good for our subject. On the other hand, I have a number of students who have gone on for a degree in mathematics, even some of them who have avoided courses in computer science. But they will not encourage their students to avoid computer science. In any case, I believe that the caricature of a pure mathematician with a disdain for computers is a thing of the past. I do not predict that we mathematicians will have to teach computer science, although some younger colleagues will do so effortlessly, especially in very small departments.

I guess I am reacting to the virtual panic some of our colleagues express when they realize that computer science is continuing to get more majors than math and that some of their potentially brightest mathematics students are being lured away. I just don't see that that is going to be such an either/or choice ten years in the future.

Interdisciplinary Mathematics in 2010

The same can be said about applications and interdisciplinary work. More and more mathematicians and their students will become involved in projects with professionals in other areas, not just the traditional disciplines of engineering, physics, and statistics but also in chemistry, biology, geology, economics, demography, art and design. There will be more collaborations involving algebraists and geometers, not just hard analysts. It will happen even with mathematicians who are not expert in the expected areas of statistics or numerical analysis or differential equations. Applications, especially those connected with modeling, will enlist the participation of mathematicians from a wide variety of subfields.

But what of those who simply do not care about applications, who prefer to work in a field so abstruse that only the best expositors can begin to give an inkling of the key problems in their field, even to a group of professional mathematicians? There will always be a place for them, although it will be a smaller place.

During the second week of August, I attended the AMS conference on Mathematical Challenges of the Twenty-First Century. What will the 2010 students have to know in order to participate in the research on these challenging problems? By and large, they will have to learn a lot of traditional mathematics, and at the same time begin to learn how to interact with biologists, as well as physicists and data analysts. They will have to keep abreast of new ideas as they become available, and hopefully they will have to learn how to come up with new ideas of their own. The main difference is the degree of interaction to be expected with researchers outside of mathematics. So the 2010 mathematics students will have to learn how to be good communicators, to learn the vocabularies and techniques of people outside mathematics to a deeper level than their teachers had.

Time and Distance Learning Displacement

We are rapidly approaching a society where people will follow more or less the same television programs but not at the same times. Most of the programs are canned to some degree, and even those that are designed to be spontaneous can be stored on tape and seen at the viewer's convenience (even to the point of seeing a version with the commercials excised, a great threat to the pricing structure of television advertising). Why should teachers expect different treatment? Some students will want to attend the live audience presentation. Others will be satisfied to listen later on, fast forwarding past the parts that are familiar and slowing or repeating the new parts. This is even better than a mere replay of an earlier videotaped lecture, since each student can "ask a question" at any time. The professor can tape answers to FAQ's and allow all students to hear an explanation, just as they would if it had been asked in class. Students might expect to be rewarded for posing questions that are particularly helpful in elucidating some point made in class. That can happen even if there is an outline provided after each lecture, not the full transcript. Of course that places the instructor in a somewhat vulnerable position, but the payoff is that a ragged or confusing lecture can be tidied up or corrected online before the next class and before too much damage has been done. Interactive technology is going to change everything.

Here I insert a short parable, written earlier this summer for presentation at a panel for NExT participants:

THE GIFT OF GIEUS

Professor Gieus was unhappy about his Tuesday class. As he walked back to his office, he reviewed what had happened—a new insight had come in the middle of his otherwise standard lecture and he had strayed from his prepared notes. The bright students had perked up a while, then settled back since it was clear that the new idea had been lost in his fumbling attempts to express it. The slow students were clearly irritated by the distraction. At the end of the hour and twenty minutes, he had rushed through the last half of his intended topics, just sketching the last one as students began gathering their books and papers to leave for their next classes.

"Did anyone understand anything at all?" he asked himself, wondering how he would be able to salvage the lecture on Thursday. As he came to his office, he saw a young woman waiting whom he did not recognize, "probably a prospective freshmen," he thought and he asked her to come in.

"I come bearing a gift," she said, "a gift that will let you know by tomorrow night exactly what your students understood from your lecture today, after they have a had a chance to think about it. The only catch is that you will have to live with that knowledge." "What kind of gift could that possibly be?" "Welcome to the Internet," she explained.

By Wednesday evening, Professor Gieus had received electronic messages from nearly all of his students. Most of them started by commenting on the reading he had half-heartedly assigned when it was clear he was not going to be able to finish his prepared presentation on Tuesday. Almost everyone said they understood the additional examples in the book that paralleled the ones he had given in class, but several were unsure of some of the steps in the argument in the second section, even after they went back and studied the illustrations. Could there be a brief discussion on this point on Thursday?

Then came several new messages. "When the files were opened Wednesday at midnight and we could see what the other students had written," one of them wrote, "I saw how to put together a couple of comments and to come up with a new example that helped me see the problem I had described earlier. Does anybody else have an idea about this?" Professor Gieus was astounded — the student who submitted that message was a B student in his mind. She did reasonable work on homeworks and tests, but nothing special. She never raised her hand in class or answered any of his semi-rhetorical questions. When he had tried a small group exercise in class one time, she sat there not participating as others traded fast comments. And here she had figured out by reading other students' difficulties exactly the right thing to consider to clear everything up. She had seen how to formulate his class insight better than he had himself.

A couple of the A students jumped in to add something to the discussion. One asked a new question Professor Gieus had never seen before. He found himself giving a response on the Internet, then deciding to transmit it to the entire class. He went on to summarize the lecture he had attempted to give and to repeat a few of the paragraphs at the end that he had rushed through. He couldn't wait for Thursday's class, to take up right where they had left off two days earlier, knowing exactly where the students were and with the students sharing that knowledge with each other.

He looked at his watch and realized it was 2 a.m. He had been at his computer for over three hours.

"Oh wad some power the giftie gie us to see oursels as ithers see us" (from "To a Louse"). When my father used to quote Robert Burns, I thought he was saying "the gift of Gieus." Hence my title character, and the little story, a kind of Midas tale about wishing for something and then having to face the consequences. The Internet does provide new forms of immediate feedback and adaptive response. It really does offer something new to teaching and learning.

Class Summaries

For each class, or at least for each week, there can be a summary: "What did we do in class?" Along with this, perhaps equally important, there be a section "What could we have done?" This might easily be quite long sometimes, including optional material as well as previews of what will occur later in the course. If we are successful in describing the syllabus in a true multi-dimensional array, then as part of the summary, we can "check off" the sites we have visited, even though we might well want to visit them again.

Note that this kind of summary is quite different from "posting lecture notes," a process that seems to indicate that the professor already knew all the important points that were going to be covered in class, and in what order. If the lecturer is going to entertain any questions at all, they will necessarily affect the sequence of topics covered and the amount of time to be spent on each. Of course it is desirable for the instructor to have a clear idea of the topics to be covered either in the upcoming class or in the next two or three classes. Often the order of some of these topics can be changed, depending on the interests of class members. Spontaneity has a lot to do with the ability to respond to questions or to reactions from the audience. A successful spontaneous teacher is not necessarily less prepared than a colleague who stays with a prewritten set of notes, undeterred by student response. On the contrary, it is harder to be prepared to lead the class in a discussion of several different possible topics. The details can be cleaned up in the afternotes.

Afternotes

Doesn't that require a good deal more effort than simply writing notes ahead of time and making them available either before or immediately after the lecture, even on paper? Yes, but the quality difference is great. This style of teaching is (probably) not for everyone, but the rewards for those who choose to work this way are substantial. Perhaps we can train our students to be spontaneous in this way by suggesting that they prepare topics, even a number of related topics, rather than a set lecture. With the chance to write up afternotes, the stress of lecture preparation is shifted. No longer will it be necessary to squelch good questions in order to be able to "cover the material." I used to try to cover for my inadequate lectures by producing handwritten notes. I'm not sure that anyone ever read them. Somehow I expect that the new kinds of afternotes will be more likely to be read.

Another benefit of afternotes can help those lecturers who prepare everything ahead of time but have to accelerate to an impossible speed to finish by the end of the period. Such a chance to put those final transparencies on line might provide a good alternative to flashing them at the conclusion of a formal lecture, after the moderator has signalled time.

Articulation and Placement

The supertext phenomenon will also affect the secondary schools, in particular in the subjects leading up to calculus. Student teachers who have used such materials in college courses will be well prepared to carry over the same pedagogy to school classrooms. For each course, there will be teacher supplements, which can serve as the basis of in-service training. To some extent, this will be true even for elementary school courses.

Articulation will be rather automatic ten years from now. Students entering the college and university system will be able to validate their knowledge and experience in the same way that students present dossiers for graduate study. There will be little difference between students who are self-taught, or taught through distance learning, or educated in secondary schools or two-year institutions. Standard examinations, with multiple scales indicating exactly how well individuals have learned how to handle certain kinds of problems, will make it possible to place students with much more confidence and accuracy than before.

There will, of course, be a recalibration of the Advanced Placement Tests. It is already true that it is too easy to get a 5 on the BC exam, as reported by some students who find themselves much less well prepared than their classmates. The SAT scores have been unreliable at the high end for a number of years. Better high-end measurements will enable colleges and universities to do a better job of placing students and fitting their backgrounds to a variety of continuations at the next levels.

Final Thoughts on Teaching in 2010

What will the role of teachers be in 2010? Even though the supertexts will individualize learning to a much greater degree, there will still be opportunity for group interaction. Precisely because people learn at different rates and with different understanding levels, it will still be valuable for them to learn together. There will still be classes. There will still be some teachers who depend completely on the text materials, and others who will feel free to deviate from those materials. The students will have a much better chance to move from one level of learning to another, and to present evidence that they have indeed accomplished something. This seems to indicate that evaluation will be much more on a performance basis than a competitive one, with the emphasis on profile certification rather than single-letter designation of accomplishment. Teaching and learning are multi-dimensional activities and they deserve to be treated as such. There will still be much for teachers to do in 2010.

Conclusion

Will any of these projections take place in the next ten years? Will there be other trends within education, or outside demands, that will change things in totally different directions? One thing is certain, and that is that we will not have to wait too long to find out.

The Mathematics-Intensive Undergraduate Major

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Introduction

I'm writing this piece on the undergraduate curriculum from the perspective of a long-time teacher of undergraduate mathematics at a large, rather isolated state university. Our university receives relatively hardworking, moderately well-prepared, if somewhat mathematically unsophisticated, high school graduates, and our student population is hardly a diverse one. So, at the outset, I invite the reader to a certain degree of skepticism, grounded in other realities.

Secondly, I plan to comment only on curriculum related to the teaching of geometry and, to a lesser extent, algebra, and to curriculum for traditional mathematics majors and for mathematics teaching majors, that is, students preparing for a career as a secondary school mathematics teacher.

Much of what there is to say flows from what I perceive to be two hard facts of life. First of all, even many mathematically "well-prepared" students arriving at my university are underachievers when it comes to spatial intuition, so that almost all mathematics becomes harder than it should be. So right at the beginning, geometry in the undergraduate curriculum is forced to include some vehicle for a remedial course in the kind of skills some develop by building with blocks and Lincoln Logs, sewing, building models, etc. In a very broad sense, the role of spatial intuition in mathematics is to prearrange and presort mathematical ideas, half-truths, inklings, and experiences on some sort of multi-dimensional mental table or in some sort of mental greenhouse. More narrowly, this skill is the ability to intuit properties of objects in three dimensions from their shadows or from a set of specifications, a pattern or sketch, or some set of specifications or other numerical medium like an equation. For example the fraction 5/7 has a serious spatial component that, over the years, seeps into our unconscious reaction to seeing the symbol in any form. If we do not experience that "seeping in" we do not do well with $x/(x+2)$ or with $\lim_{x\to\infty} x/(x+2)$ or with l'Hopital's rule, etc.

Secondly, at least at my institution, there is no common intellectual agenda between traditional mathematics majors and mathematics teaching majors after one-variable calculus. While there are exceptions on both sides, many teaching majors acquire less advanced mathematics than other traditional mathematics majors. Of the last 30 graduating teaching majors in mathematics at my university, 3 had the advanced coursework in mathematics required of the traditional major, while the other 27 averaged less than 0.5 advanced course per student (traditional majors are required to complete 6 advanced semester courses). Nowhere is that disparity more in evidence than in analysis and abstract algebra. While it is true that the preparation in these areas for the two groups needs to have different emphases, in fact these two groups often live in different intellectual worlds with few viable bridges between them. In practice this translates into the fact that one finds only future teachers in upper division mathematics courses like Foundations of Geometry and Foundations of Algebra and almost no teachers in upper division mathematics courses like Introduction to Number Theory and Curves and Surfaces in Euclidean Space. And, at least at my institution, the common requisite of Foundations of Analysis (which used to be called “baby Rudin” by some) is experienced by most preservice secondary math teachers as the “gulag” of their undergraduate math career.

So my suggestions in this short paper will be driven by the above realities which, for me, fall into the “sad but true” category.

Geometry in the first two undergraduate years

Gatekeeping in the mathematics major is of course much easier to achieve when it comes to computational skills than it is for spatial skills. Entering students with deficiencies in algebra are screened into “College Algebra” courses before proceeding further but there is no real “College Geometry” alternative for those with geometric deficiencies. Trigonometry is still used as a geometric gatekeeper at some institutions. But somewhat better alternatives like the old Analytic Geometry courses seem to have disappeared entirely, or to have had their subject matter relegated to a late topic in the freshman calculus course—so late in fact that spatial skill and intuition evoked there is “too little, too late” to influence fundamental perception and understanding of the calculus itself. Some device for spatial gatekeeping would be useful, in that it would significantly increase the tools available for the teaching and learning of one-variable calculus, the efficiency with which it can be taught, and the depth with which it is learned.

Let me give an example of what I mean. The notion of (continuous) function is fundamental to calculus but is not the natural way that quantitative relationships arise. Quantitative relationships come to us as *relationships* between quantities or positions where there is usually no naturally preferred causal or “independent” variable or quantity. Many if not most understand such a relationship first spatially in a context in which no variable is preferred. For example, the trajectory of a thrown ball or the shape of a rainbow is determined and conceptualized in our experience without the choice of a variable (horizontal displacement, time, etc.). The choice of the “independent variable” or “coordinate” is an artificial mental construct, albeit an important tool to aid our understanding of what is essentially and finally a “coordinate-free” or “synthetic” relationship between quantities or positions. This reality reflects itself, for example, in the fact that most of the key derivations in differential calculus are actually “implicit differentiations,” which I feel are difficult to comprehend or to teach without the above fundamental realization, which, I maintain, is fundamentally geometric.

Some might say that spatial gatekeeping at the university level might have more to do with the secondary math curriculum than the undergraduate curriculum. In a “better mathematical world” I might agree. My “impossible dream” is that we might slow down the imprudent “rush to calculus” so prevalent in the secondary math curriculum for talented and/or hard-working secondary students. One might actually encourage more time spent and depth plumbed in geometry in high school as an alternative to AP calculus which often has to be relearned (or not) with more understanding (or not) at the university level anyway. At the high school level, traditional axiomatic Euclidean geometry has often been replaced by “informal” or intuitive geometry or by the integration of geometric concepts in other math courses. Partially this may be explained by past excesses—the religion of “two column proofs” about geometrically obvious facts such as “all right angles are congruent.” But the fact is that geometry needs its formal rituals as well as the cultivation of intuition and visualization. The subject takes life only when the two operate in tandem. For most students this process takes far longer to develop than is currently allowed in most middle and high schools. As with algebra or analysis, geometry too needs its simple, sometimes mindless rituals that precede true understanding. We seem to be ready to throw them out because of past excess and perversions, without having any good accessible alternative. We would not think of throwing out drill on systems of linear equations, factoring and the quadratic formula in algebra. Indeed I think the problem in algebra is the opposite one—we leave out all the richness of the numerics out of which these algebraic rituals arose.

At the college or university level, the vehicles for spatial gatekeeping and remediation are not so clear. The return of Analytic Geometry as the spiritual “spatial” partner of College Algebra might be one possibility. But there may be many others, especially these days in which technology offers so many possibilities. Could one even think of a software-based graphic design course not far from what might be offered in an Art

Department, as a vehicle for remedial spatial (geometric) education? Such a course could be the beginning of a track in computational geometry in later undergraduate years.

Geometry in later undergraduate years for the traditional math major

Again, at my institution, geometry often takes a back seat to analysis, applied math, probability and statistics, and even abstract algebra. Formal geometry appears in the curriculum only as a one-semester course in introductory topology and a one-semester course usually titled “Curves and Surfaces in 3-dimensional Euclidean space.” (I might add that, when I arrived here 25 years ago, no courses in either elementary differential geometry or elementary number theory had been offered in the previous 10 years—I suspect that there are quite a few places around the country where the same is true even today.) This course, comprising the notion of curvature and torsion of one-dimensional loci and the three “fundamental forms,” parallelism and geodesics on surfaces, is, when well taught, a fundamental link between Euclidean geometry and modern differential geometry, physics, differential equations and graphics. It is also replete with classical computational jewels, like the closed form for the equation of geodesics on tori.

So there is plenty of room to suggest greater emphasis for geometry and (to a lesser extent) topology in the upper-division curriculum and plenty of justification coming from the directions of current research and applications in the very fields (analysis, applied math, probability and statistics, and even abstract algebra) which are the so-called competitors of geometry for time and attention. Titles of recent texts such as Fraenkel’s *The Geometry of Physics—An Introduction* and, at a considerably more elementary level, Shifrin’s *Abstract Algebra, a Geometric Approach* suggest a bit of these trends.

Here again, my tendency would be to opt for deeper rather than more, and for connections to other fields rather than narrow internal development. How can a curriculum reconcile these seemingly contradictory goals? I personally believe that the three courses

Curves and Surfaces in 3-dimensional Euclidean space,

One-variable Complex Analysis,

Elementary Number Theory,

when properly conceived and presented, are the three most beautiful and central subjects in the entire pure mathematics curriculum, undergraduate or graduate. So I would base a good undergraduate formation in *geometry* on the first two of these three. The next addition would be a good introduction to the fundamental group, not as one of several tools in algebraic topology, but as the most profound yet accessible tool linking geometry, analysis, linear and abstract algebra, combinatorics, and differential equations. A complementary modern necessity is a course in computational methods.

Again on the issue of deeper versus more, let me pose a question: How many undergraduate math majors, upon completion of their degree requirements, could give a proof of the Pythagorean theorem (or even the quadratic formula)? Or how many could say anything intelligent about the relationship between the Pythagorean theorem and the quadratic formula? It has been my experience that well-trained undergraduate majors can compute (so can *Mathematica* or *Maple*) but precious few can articulate what it is that they are computing or what the significance of the computation is.

Mathematics vs. the mathematics teaching majors

If the question “How many undergraduate math majors, upon completion of their degree requirements, could give a proof of the Pythagorean theorem (or even the quadratic formula)?” has a positive answer for a particular group of undergraduates, it is (and should be) the group of undergraduate math education majors. This is one of the good changes brought about by the evolution of the mathematics preparation for teaching majors at many institutions. But it appears to me that in practice this insight is near the apogee of the intellectual trajectory traversed by most teaching majors during their undergraduate mathematics experience. More sophisticated understanding of the role of measure or metric is entirely missing. For example, the role of complex analysis in the solution of the heat equation with simple boundary conditions is not even on their radar screen, given that many of them never get beyond the most rudimentary familiarity with the complex numbers, let alone complex analysis. Such a background leaves the preservice teacher with a woefully impoverished understanding of geometry. Too often the end result is a teacher with few mathematical resources, and so at the mercy of a particular book or district-dictated curriculum.

This problem is not restricted to geometry and is a tough one. I know of only one university that maintains that the mathematical level attained by the secondary math teaching majors is equivalent to that of their other math majors. And there, I must say, I would have to see it to believe it. The fundamental question for math departments that is raised by this disparity is whether or not to raise the mathematical bar for teaching majors and run the risk of driving away the clientele. Another question is what it means to raise the bar "nearer" to that for the traditional math major. If indeed it is true that traditional math majors can't prove the Pythagorean theorem, then their preparation is perhaps not so great either.

Can these two groups interact to the mathematical benefit of each?

One direction that seems to have some promise at the upper division level is making REUs, summer internships, and undergraduate research seminars, colloquia and theses as an integral part of the undergraduate curriculum. These activities emphasize the role of active *communication* of mathematics in learning and understanding of mathematics. Peer tutoring is another such enterprise. Might there be a way to formalize this? Communication skills in mathematics are more highly valued these days, not only for future teachers but also more widely in the academic and industrial world. But a strong if still somewhat under-appreciated case can be made for communication as a learning tool. The old adage "I never understood that until I taught it" is more than an adage or an accident of inadequate preparation. The act of communicating, and the mental organization, the critical listening, the heightened awareness, even raw fear that it engenders, takes understanding to a new level.

Another promising direction is that of raising expectations, as long as the means are provided for the hardworking to fulfill them and someone important is truly interested in the outcome. One of the points of the famous movie *Stand and Deliver* was the gradual transference of that role of "someone important" from the teacher to classmates. Could we not envision mechanisms to engender some of that same transference in our own classrooms? Unfortunately such mechanisms are always labor-intensive and so difficult to sustain over time. In a large commuter school such as my own, it seems quite hard to effect such an experience, and it is probably impossible to achieve it on a large scale. But could there not be at least one math communications requirement introduced, one in which the target of the communication, and the evaluators of its effectiveness are other students or coworkers?

Such a requirement would also bring traditional majors and teaching majors back together late in their undergraduate career on relatively equal footing. Presumably teaching majors by that point would be better communicators and have a deeper, move flexible hold on more elementary mathematics and the theorems and proofs that support it. Traditional majors would presumably have more advanced math under their belts and could be challenged to communicate some of that knowledge to teachers.

Geometry in later undergraduate years for the math teaching major

To me the fundamental course in geometry for future secondary teachers is a serious course in two-dimensional geometry concentrating on non-trivial results in Euclidean geometry, but also with the three or four fundamental results of spherical and hyperbolic geometry, in particular, the "excess angle theorem." This course should also include a serious segment on groups of symmetry and, more generally, the group of rigid motions of one-, two-, and three-dimensional Euclidean space. Finally the notion of "inducing on dimension" should be explored by such exercises as inductively counting faces and vertices on n-simplices and n-cubes. Beyond this, and the required foundations of algebra and foundations of analysis courses, it would be wonderful if a future teacher of secondary mathematics could be equipped with a mature understanding of the "big three" courses I mention above:

Curves and Surfaces in 3-dimensional Euclidean space,
Elementary Number Theory,
One-variable Complex Analysis,

but, at least at our institution, this is a bit more than the market will bear in the way of preservice requirements. However one promising aspect of the current national dialogue on teacher education is the growing realization that career-long mathematical study and development are a necessary component of teaching excellence. Hopefully the day will come before long when the kind of mature understanding I mention above can be gradually achieved through in-service released time for study, sabbaticals, etc., and through guided peer presentations and seminars.

Here again the communication requirements and experiences I mentioned for math majors enter. Such activities are a paradigm for continued learning. Not only would they comprise a new instrument for bringing teachers together with traditional math majors, they would also serve as models for future in-service mathematical experiences; models with more mathematical depth than what currently passes for in-service teacher enhancement.

Algebra in later undergraduate years for the math teaching major

If "Curves and Surfaces in 3-dimensional Euclidean space" is the geometry course most lacking in the preservice preparation of secondary teaching, "Elementary Number Theory" is, to my way of thinking, the algebra course too many graduate without taking. Far more than a full course in groups, rings and fields, Elementary Number theory has soul. Loading up secondary teachers with a "capstone" course that is primarily a set of tools for future use does not seem to be the right answer. Experiencing some of the most beautiful mathematics of all time, mathematics that lies at the heart of algebra, makes far more sense. But here again it is probably a more reasonable goal to make the mathematics of this course and of a course in groups, rings and fields (which makes more sense in conjunction with number theory than alone) central to in-service study and mathematical development over the professional life of a secondary teacher.

Conclusions

The main points that I wanted to make in this very short article are the following:

1. Traditional mathematics majors need more ability to communicate what they know. In fact the value of exercises in communication (peer tutoring, undergraduate seminars and colloquia, undergraduate research experiences) extends to learning itself and fosters deeper knowledge and the desire for continued learning.
2. Secondary teachers need a bridge to lifelong mathematical learning. Given the other mathematical and methodological requirements, it is probably unrealistic to require of them the kind of deep appreciation and understanding of higher algebra and geometry that ideally should be at the center of their professional mathematical lives. Perhaps the fundamental challenge to the current preservice curriculum is to instill the intellectual and professional "felt need" to continue the pursuit of this appreciation and understanding.

This second point is to me the most important and most challenging issue facing us with respect to the care and feeding of secondary teachers of mathematics. The crucial ingredient necessary for improvement in the teaching of mathematics in elementary, middle or secondary school is lifelong professional development, both mathematically and pedagogically. Just as, at the university level, we become gradually dead and ineffective if we do not continue to combine our teaching with learning and research, broadly conceived, so teachers become dead and ineffective if there is no mechanism for life-long mathematical learning and for deepening pedagogical insight. "Use it or lose it" is just as true in this instance as it is in most other areas of human activity.

And too—a point that we university mathematicians rarely realize, let alone acknowledge—this change in the culture and professionalism of school math teachers is also in our own self-interest in purely selfish terms. We fret over declining enrollments in pure math courses yet ignore a very large and sustained source of potential students. Teachers of math, even those whose mathematical preparation or abilities are modest, can over their lives achieve high levels of mathematical understanding through sustained interest, support and hard work. We will be the teachers of these lifelong students, once it becomes clear, as sooner or later it will, that this is a *sine qua non* of improvement, and once the political consequences of that realization play out in a combination of improved working conditions and higher standards of professionalism for teachers.

Mathematics and the Mathematical Sciences in 2010: What Should Graduates Know?

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Kinds and uses of technology in the curriculum and their impact on the undergraduate program

There are four aspects of technology that I would like to discuss:
1. Computational software
2. Internet interconnectivity
3. Presentation software
4 Tutorial software

1. Computational software

Students in this century will become accustomed to using a variety of software tools in learning and doing mathematics. The standard CAS packages such as *Maple*, *Mathematica*, and *Derive* have been integrated into many courses that are part of the first two years of collegiate mathematics—courses that mathematics majors are required to take. The integration of the technology is often seen as an enhancement or enrichment of the basic material. The graphical, numerical, and symbolic computational capabilities of the technology is not seen as a fundamental part of the course.

It is possible to develop a course based on the notion that the concepts should be introduced with the use of a CAS and that those concepts that cannot be fully explored (at the introductory calculus level) with the technology can be enhanced and enriched with paper-and-pencil activities. Such a course might begin with an exploration of change from a difference equation or recursion formula point of view. Then, when exploring situations where such methods fail, the concept of the derivative can be introduced and explored. Once the derivative is defined, then a transition can be made from difference equations to differential equations. Since most differential equations that occur in the real world do not have closed-form solutions but can be investigated and solved using numerical (discrete) methods, this approach provides students with a working knowledge of modeling with both discrete and continuous approaches and shows the relationship between them. Such an approach begins to satisfy the needs of the computer science and engineering disciplines that beginning to ask for more discrete mathematics in the introductory mathematics service courses. At the same time, such courses are exciting and enticing for students and can act as a recruitment method for mathematics majors.

In dealing with integration in the first two-years of collegiate mathematics, we can see that it is perhaps not as central an idea as it is in the current freshman and sophomore courses in mathematics might have us believe. The Fundamental Theorem of Calculus is, of course, still important and should be proved, but its importance may now lie in the role that it plays in proving the existence of solutions to certain differential equations. The standard real world problems such as the area, volume, and surface area of regular geometric objects that involve Riemann sums would be greatly reduced (as it once was in Granville, Smith, and Longley's calculus text of the first half of the 20th century). The theoretical existence of solutions is very important if one is to look for a solution using discrete numerical methods.

Developing a course based on technology might satisfy the needs of some of our client disciplines. It might also reduce the time needed for students to master the limit, derivative, and integration concepts because less time would be spent on computing such limits, derivatives, antiderivatives, and definite integrals of a variety of functions, since the CAS would provide the results faster and more accurately. Still, there are some basic skills with limits, derivatives, and integrals that one should know by heart—$\sin(x)/x$ as x approaches 0, the limit of the derivatives of polynomials, the chain rule and product rule for derivatives, and substitution and integration by parts come to mind.

Using CAS software in some experimental classes has cut by one-third the time needed to cover the first two semesters of calculus. This might allow the coverage of differential equations (in a qualitative manner as well as symbolically and numerically as mentioned above) to occur in the first year. In the second year, students could cover the basics of linear algebra (again using CAS software tools), multivariate calculus, and probability and statistics at the level required by engineering schools. Much of this has been done at the United States Military Academy as part of their seven into four project.

Some will say that students will not develop the symbol manipulation capabilities often developed in the first two years of collegiate mathematics in the past. This is certainly true, but many of our client disciplines do not need that symbol manipulation capability and some need the computer software capability much more. In addition, much of the beauty and power of the wonderful symbol manipulation activities that occur in our current calculus courses are lost on freshmen and sophomores. Perhaps techniques of integration should be left as a capstone course for majors who would appreciate and may need these activities and skills to move on to higher mathematics.

I recommend that the mathematics enterprise look carefully at restructuring and redeveloping the activities that we ask students to do in the first two years of collegiate mathematics so as to take full advantage of the current software that will almost certainly be at the fingertips of all those who use mathematical sciences in their careers. And, of course, we must be mindful of the changes in outlook, understanding, and skills that such new courses might cause in our mathematical science majors. These new activities and topics could ensure that our majors have the mathematical maturity we have always desired and at the same time provide marketable work force skills for our majors.

2. Internet Interconnectivity

Many instructors who are interested in the applications of mathematics and feel that their students should have exposure to such applications also feel that they do not know enough about such applications to present them to students. The Internet provides a method for working around this problem. A one-unit calculus laboratory (for lack of a better word) course could be developed where students could chose to work on an application in their major area or in a discipline of interest to them with students from other institutions under the tutelage of an instructor from one of the participating institutions. In this way, students from different disciplines could see the applications of calculus in their area of interest, and mathematicians could concentrate on knowing applications from just one field. Mathematicians have a great deal of understanding of all disciplines as a group, but no one mathematics department is likely to have instructors versed in more than four or five. Using the Internet to tap these resources (both academic and industrial) would broaden our departments in many beneficial ways. Our students (majors included) would learn about collaborative work at a distance and the instructors would learn about different disciplines and about perhaps different mathematics.

I recommend the development of laboratory courses and the infrastructure at the national level to support those courses.

3. Presentation software

Presentation software such as *PowerPoint* allows faculty members to present elaborate and effective mathematical lectures. Such software allows for the quick and colorful development of slides that can give insight into mathematical theory and applications. Faculty should be careful in using such software to make sure that the student understands what is being so artfully displayed and has some way to carry the information flashed on the screen home to study and absorb.

I recommend that faculty who wish to use such software be given the institutional support to do so effectively in class—a ceiling mounted data projector, permanently installed computer facilities (not on a cart), Internet connection in class, and training on the instructional use of such software. Students in advanced classes should be encouraged to make presentations on projects using such tools.

4. Tutorial Software

Tutorial software, like that available from Academic Systems and from the ALEKS Corporation can help us cope with the ever-present problems we are having with remediating students. These software/book packages provide an opportunity for the mathematics community to ensure levels of student proficiency with basic mathematics skills. This allows the faculty to concentrate on improving the quality of the collegiate mathematics instruction. ALEKS, for example, can allow students to assess their mathematical achievement from home (perhaps before school begins) and then work through tutorial activities on those parts of the required mathematics that they have not yet mastered. Thus, students who have only small deficiencies could correct them before the school year and save themselves and their colleges much time and repetitious activity. Students who under the current paper-and-pencil placement tests might be doomed to repeat at least one course could bring their skills up to par before school begins and could then advance through their mathematics requirements on schedule.

Also, in courses such as statistics that are often taken later in students' careers after they have been away from mathematics for some time, students could assess themselves against a mathematical skills standard set by the professor, participate in appropriate remedial activities for a week prior to the course, and then begin the course without the usual several days of review.

I recommend that such software/book packages be investigated as a way to reduce some forms of remediation.

Paths leading to a major

There are many paths to a major. One that is particularly attractive to community college students is a path that leads to high school or community college teaching. When we see able mathematics students at the community college level, we must hasten to ask them to major in mathematics and to ask them if they are interested in teaching. In many cases, we are the first teachers of mathematics they have seen who actually know some mathematics and for this reason are seen as individuals to emulate. This is not to say for an instant that high school mathematics teachers cannot teach or know mathematics. It does, however, speak to the large number of economically disadvantaged students who, through no fault of their own, have had awful mathematics instruction from teachers who are good people and are fully certified in science, history, or English, but do not have (and know that they do not have) a sound background in mathematics but, nonetheless, are forced to teach mathematics.

I recommend that courses be developed at the community college level to help such students understand the value of mathematics. These courses might be of a capstone nature or perhaps courses in mathematical modeling. These courses should be seen as ways to entice students into the exciting world of mathematics as an avocation and as a career.

Transition issues from two- and four-year institutions affecting math intensive majors

Two- year college students always face articulation problems when transferring to four-year colleges. The two-year college is on a semester system, while the four-year college is on a quarters system or the reverse. All faculty at whatever level find that the preparation of students in their courses is lacking in some fashion. Four-year college faculty sometimes ascribe the difficulties of their students to their previous education at some other institution (like a two-year college). This is normal, but still presents a problem for the two-year college students and faculty to overcome. As it turns out, at least in California, students from two-year colleges usually do as well or better than students who have taken all their courses at the four-year college or university.

I, therefore, recommend that data on the GPAs of majors in the mathematical sciences at four-year institutions be compiled and reported to the faculty each year. These reports should be sent to two-year colleges whose students are majors. These reports should be used as a basis for continued articulation between the two- and four-year colleges.

The package of mathematics courses that a two-year college mathematics major takes should include calculus and linear algebra. These courses should give the student an understanding of the importance of theory in the application of mathematics, an understanding of the need for proof in establishing theory, and an understanding of the usefulness of symbolic, graphical, and numerical software in developing theory and in solving applied problems. These courses should provide students with a foundation for further study at the advanced undergraduate level. The requirements for a major at a two-year college include more than these courses, but these courses should be the basis of the students' two-year background.

A one-unit course on the historical development of mathematical theory might be introduced to make sure students have seen interconnections of mathematics theory with the real world. Students who attend community colleges often take many years to complete their course work and may not have the opportunity to see the connectedness of the many courses that they have taken. Students should also be encouraged to join the Mathematics Club and to participate in both the Putnam Examination and the Modeling Contest.

So that a student can understand the learning outcomes that the mathematics department expects of those transferring to four-year colleges in mathematics, I recommend that a majors advising process be put in place at community colleges and that this advising process include the use of the GPA articulation data that comes from the local four-year colleges.

Two-year college mathematics faculty often have greater access to computing facilities than their four-year college counterparts. In such cases, the four-year college departments may deny their transfer students the use of graphing calculators, symbolic manipulation graphing calculators, and computers that they have used at their two-year college. Therefore, it is important for two-year college faculty to know that, while technology should be integrated into the development of mathematical understanding, students should have experience in doing extended pen-and-paper computations. The transition from two-year to four-year institutions can be eased by making sure that students have already completed their calculus and linear algebra courses (courses that can be greatly enriched and deepened with the use of technology, but where not all faculty allow its use).

The Mathematical Education of Prospective Teachers of Secondary School Mathematics: Old Assumptions, New Challenges[1]

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We address two questions: What mathematics do prospective secondary school mathematics teachers need to know? In what context should they come to know it? Consideration of both matters has implications for the revision of the undergraduate program in mathematics.

What mathematics do prospective secondary school mathematics teachers need to know?

Teachers must know the mathematics they teach. Deciding exactly what this means, and then determining what more mathematics they need, are not simple matters. Typically, two perspectives have influenced the design of programs for the preparation of secondary teachers, and both are relevant to mathematics departments:

1. Prospective high school teachers should study essentially whatever *mathematics majors* study—because this will best equip them with a coherent picture of the discipline of mathematics and the directions in which it is heading, which should influence the school curriculum.
2. Prospective high school teachers should study *mathematics education*—methods of teaching mathematics, pedagogical knowledge in mathematics, the 9–12 mathematics curriculum, etc.

We argue in this paper that there is substantial knowledge that is necessary for effective teaching but which is neglected by this two-pronged approach. Furthermore, much of this knowledge is mathematical in character, and, as such, should be a responsibility of mathematics departments. Because this knowledge is particular for the teaching of mathematics, it lies, in a sense, between mathematics education and traditional undergraduate mathematics content. Keep in mind, however, that there is much outside of mathematics and mathematics education that all secondary school teachers need to know, about students, about learning, about teaching, about curriculum, and about the contexts of schooling.

History of recommendations[2]

The dominant approach to the mathematical preparation of secondary school teachers in the United States in recent years is to require that they complete an undergraduate major (or a near-equivalent) in mathematics.

[1] The authors wish to thank Deborah Loewenberg Ball, Dick Stanley, Tom Rishel, Merle Heidemann, and Dawn Berk for their comments and assistance in preparing this paper.

[2] For more detail on the following history, see Gibb, Karnes, & Wren, 1970.

Interestingly enough, a quick review of the recommendations in this century about the mathematical preparation of teachers reveals that this trend toward a near-major has generally grown stronger with each set of recommendations. For instance, the 1911 report of the American subcommittee of the International Commission on the Teaching of Mathematics recommended preparation in several areas of pure mathematics, applied mathematics (e.g., mechanics, astronomy, physics), surveying, a "strong course on the teaching of secondary mathematics," other education, and "a course of an encyclopedic nature dealing critically with the field of elementary mathematics from the higher standpoint" (International Commission on the Teaching of Mathematics, 1911, pp. 13–14). There is no explicit call for a major in mathematics. Likewise, the 1935 recommendations of the Mathematical Association of America's Commission on the Training and Utilization of Advanced Students of Mathematics calls for "minimum training in mathematics that goes as far as 6 hours of calculus, Euclidean geometry, theory of equations, and a history of mathematics course" (Commission on the Training and Utilization of Advanced Students in Mathematics, 1935). The courses that might have been more typical of a major at that time (advanced calculus, mechanics, projective geometry, additional algebra) are described as "desirable additional training."

In reports from various groups in the late '50s and early '60s, the expectations for secondary teachers began to sound like a major, with calls for 24 semester hours of mathematics courses (National Council of Teachers of Mathematics [NCTM], 1959), and 30 semester hours, including abstract algebra (American Association for the Advancement of Science, 1959). It was the Committee on the Undergraduate Program in Mathematics (CUPM) that first recommended that "prospective teachers of high school mathematics beyond the elements of algebra and geometry should complete a major in mathematics" (CUPM, 1961). Ten years later, this sentiment was still strongly held: "We regard it as a matter of great importance that a program for teachers should be identical to the one offered to other mathematics majors, except for a few courses peculiarly appropriate to prospective high school teachers" (CUPM, 1971, p. 170).

The 1983 CUPM recommendations do not explicitly call for a mathematics major, but instead list 13 courses, including a 3-course calculus sequence, as the minimal preparation, with a call for additional work for teachers of calculus (CUPM, 1983). It is worth noting that 13 courses is *more* than a major in some institutions. In 1991, the MAA's Committee on the Mathematical Education of Teachers (COMET) assumed responsibility for the preparation of teachers: "These recommendations assume that those preparing to teach mathematics at the 9–12 level will complete the equivalent of a major in mathematics, but one quite different from that currently in place at most institutions" (Leitzel, 1991, p. 27). The recommendations list standards in seven content areas (e.g., geometry, continuous change, and mathematical structures) rather than specific courses.

Since the first CUPM recommendations, most major sets of national committee recommendations offered by the mathematics community and most recommendations from the education community have recommended the equivalent of a major in mathematics as the fundamental preparation for the secondary teacher. Sometimes the recommendation is general and assumes that whatever is considered appropriate as a major is appropriate for future mathematics teachers. For instance, the new recommendations of the National Council of Accreditation of Teacher Education (NCATE), to go into effect next year, expect that candidates for teaching should "know the content of their field (a major or the substantial equivalent of a major)" (NCATE, 2000). The most current recommendations being developed for the mathematical education of teachers (the Conference Board of the Mathematical Sciences [CBMS] Mathematical Education of Teachers project), though reflecting some more current general issues about the undergraduate major, still make the same basic argument, as is evident in the following excerpt:

> The following outline of mathematics and supporting courses is one way to provide core knowledge for future high school teachers while satisfying many requirements in a standard mathematics major.
>
> Year One: Calculus, Introduction to Statistics, Supporting Science
>
> Year Two: Calculus, Linear Algebra, and Introduction to Computer Science
>
> Year Three: Abstract Algebra, Geometry, Discrete Mathematics, and Statistics
>
> Year Four: Introduction to Real Analysis, Capstone, and Mathematics Education Courses
>
> (CBMS, in preparation)

There is no question that teachers need to know mathematics in order to teach well in secondary schools—the logic in this seems unassailable. Yet at the same time, research studies do not demonstrate a convincing relationship between teachers' knowledge of mathematics (often measured by the number of college mathematics courses taken) and their students' mathematical performance (see Begle, 1979; Monk, 1994). Perhaps teachers fail to learn the content of these courses, or they do learn it but find that it doesn't connect in any recognizable way with their classroom practice.

There are at least two problems with requiring the same preparation for mathematics teaching as for graduate school in mathematics. First, high school teachers are preparing for a professional practice that is completely different from that of conducting mathematical research. The mathematical demands they will face are different. But we are not arguing for less mathematical preparation for teachers. In fact, we would argue that with a typical major in mathematics, teachers may have *too little mathematical preparation of the kind they will need*. Second, by keeping content separate from pedagogy, prospective teachers may fail to acquire what Shulman (1987, p. 8) called *pedagogical content knowledge*—"an understanding of how particular topics, problems, or issues are organized, presented, and adapted to the diverse interests and abilities of learners, and presented for instruction".

The mathematics of the secondary school curriculum

Today's secondary school context is radically different from that of 20 years ago. First of all, secondary schools today take seriously the commitment to educate all students to be prepared for a rapidly changing world—and thus *all* students need to be prepared for the possibility of higher education or a highly technical workplace. This has meant increasing trends away from vocational or general tracks, and toward a foundation of significant mathematics for all students (see the recommendations of the NCTM 2000 *Principles and Standards for School Mathematics*). The range of options today in high school curricular materials reflects this shift. While the algebra 1, geometry, algebra 2, precalculus, calculus sequence that many of us experienced is still alive and well, today's instructional materials also include substantial emphasis on data and statistics, on discrete mathematics, on dynamic geometry, and on early treatment of functions and modeling. Some series are fully integrated, with titles like "math 1, 2, 3, 4," or are completely organized around applications of mathematics and so-called contextual problems. The once unchallenged high school end-goal of advanced placement calculus has given way to other equally strong possibilities, such as advanced placement statistics, or sophisticated technical courses focusing on CAD-CAM technologies, finance, or applications of mathematics to the world of work. These trends are at odds with what has been the traditional mathematics major, with its historic emphasis on abstract algebra and analysis as end-goals. If one takes seriously the notion that being prepared to handle the mathematics of the secondary school is something crucial for teachers, then it seems that these shifts in the nature of secondary school mathematics education need to be taken quite seriously by those who prepare teachers.

For secondary mathematics teachers, it is ironic that, except for occasional concepts that might be called upon in calculus, the entire four years of an undergraduate mathematics major address content that is, on the surface, unrelated to the topics of the high school curriculum. The only place where prospective secondary teachers are very likely to learn about such secondary school topics as the Law of Cosines, the Rational Root Theorem, the proof of Side-Angle-Side congruence, the Zero Product Principle or tests of divisibility is in the secondary school, as students themselves. More substantially, the kinds of integration of mathematical ideas and connections that are necessary in teaching a coherent secondary program, are unlikely to be obvious to students on the basis of their undergraduate program. Consider the following example of a student teacher episode. This student teacher had been a strong undergraduate mathematics major at a small state university; she had taken courses in abstract algebra, geometry at a junior level, and advanced calculus. She was conducting a lesson in algebra 2 class and was presenting the absolute value function. She showed the students the notation, $f(x) = |x|$, and drew the graph. A student said something like the following: "That graph reminds me of angles in geometry. Can we use the absolute value function as a way to write a formula for any angle?” The teacher was completely taken aback by the question. The question would have required the teacher to make a number of mathematical judgments on the spot, and also to connect ideas across content areas in unexpected ways.

As professors working with prospective high school teachers, are we confident that our students will be able to answer the following typical high school students’ questions, in ways that are both mathematically sound and also accessible and compelling to a 15 year-old?

- Why does a negative times a negative equal a positive?
- Why do I switch the direction of the less than symbol when I multiply both sides by a negative number?
- In every triangle that I tried in Sketchpad, the angles add up to 180. I don’t need to do a proof, do I?
- I am not convinced that .99999… = 1.
- How do I know parallel lines never intersect?
- How do I know that the asymptote never hits the line? I mean, it crosses the line near 0.
- I think that 100 *is* divisible by 3—the answer is 33 and one third.
- Why is it OK to use 22/7 for the value of pi, sometimes?
- I think that the number 1 has three different square roots: 1, -1, and .99999999. I am sure that .99999999 is a square root of 1 because when I multiply it by itself on my calculator I get 1.00000000.

Mathematics for Teaching

Suppose we could construct a curriculum for secondary school teachers that, in terms of mathematical content, was in tune with the current secondary curriculum and its directions of change. Moreover, suppose that it genuinely offered students a chance to see both where the concepts of the high school curriculum are embedded in a larger picture of mathematics and also to see elementary mathematics from an advanced standpoint or to develop “profound understanding of fundamental mathematics” (Ma, 1999). This task probably would require designing some courses especially for teachers, thus breaking with the tradition that what’s good for the math major is good for the prospective teacher. Both majors require substantial study of serious mathematics, but there may be reasons why the body of mathematical content is different in some ways.

We suspect there is another body of knowledge that high school teachers also need, which is mostly mathematical in character, and which is probably more within the purview of the mathematics department than it is the school of education. Mathematics education researchers and mathematics educators are struggling with how to describe and talk about this knowledge. Ball and Bass have studied this notion in the elementary grades and call it “pedagogically useful mathematical understanding” (Ball & Bass, 2000). Zalman Usiskin (personal communication) considers “teachers’ mathematics” to be a branch of applied mathematics, in the sense that (1) it emanates from the classroom in much the same way that operations research arises from problems in business, and (2) it includes specialized mathematical knowledge that may not be known by mathematicians in other applied or pure areas. In addition to having profound understanding of the content that is taught in the secondary grades, mathematics teachers at this level need to be able to draw on and use other knowledge that is mathematical in character, such as:

- finding the logic in someone else’s argument or the meaning in someone else’s representation;
- deciding which of several mathematical ideas has the most promise, and what to emphasize;
- making and explaining connections among mathematical ideas;
- situating a mathematical idea in a broader mathematical context;
- choosing representations that are mathematically profitable; and
- maintaining essential features of a mathematical idea while simplifying other aspects to help students understand the idea.

These kinds of mathematical activities, we would argue, are essential in teaching. They arise while teachers are planning lessons, designing tasks for class and for assessments, interacting directly with students around content, answering their questions, and correcting their work. If such activities are essential, then prospective teachers need to learn the necessary mathematical skills—ideally, we contend, through mathematical material that is close to the material of high school classrooms, under the guidance of mathematicians in university mathematics departments.

There are at least three approaches for creating connections between undergraduate and high school mathematics:

A mathematical approach. Prospective teachers should study high school mathematics from an advanced standpoint.[3] The approach is to find and exploit topics in the high school curriculum that can be extended and elaborated in ways that are sophisticated mathematically. (See Box 1 for a list of principles that guide this approach.) An alternative is to find topics in the typical undergraduate curriculum and look for ways to connect them with key areas of the high school curriculum—this is the idea behind capstone courses or "shadow courses" that prospective teachers take alongside such courses as abstract algebra. Although these approaches may represent an improvement over some mathematics courses that have no connection to high school mathematics, the approach runs the risk of leaving prospective teachers without sufficient pedagogical content knowledge, which lies at the intersection of content and pedagogy.

An integrative approach. Integrate the goals of the mathematics content and pedagogy courses so that teachers might be better able to see connections and later use them (see, e.g., Cooney et al., 1996).

An emergent approach. Analyze the practice of teaching and determine what mathematical knowledge teachers draw upon in their practice. Then use real mathematical "problems" of mathematics teaching practice as sites for learning mathematics, taking advantage of the mathematical opportunities that emerge while working on the problems.

We believe that all of these approaches are worth pursuing. The emergent approach, however, is the most unusual and requires the most explanation. To understand this approach, it is useful to characterize the traditional approach as a failure at helping teachers transfer their mathematical knowledge into practice. Rather than constructing a solution apart from teaching, the approach begins in the context of teaching practice. We try to identify the interpreting, problem-solving, and decision-making activities in which a teacher actually engages, so that we may infer what mathematics is actually used. The next step would be to design a curriculum around such activities, in much the same way one might create a mathematics curriculum for engineers or social scientists by looking at the mathematical problems they have to solve.

Mathematics education researchers and professional developers (see, e.g., Ball & Cohen, 1999; Schifter, Bastable, & Russell, 1999; Shulman, 1992; Stein, Smith, Henningson, & Silver, 2000; Barnett, Goldenstein, & Jackson, 1994) have begun to explore this approach through the use of videos of classrooms, student work, written cases, and student curriculum materials. The notion of using the actual work of teaching as a starting point for thinking about the mathematical preparation of teachers was explored further at the Teacher Preparation Mathematics Content Workshop hosted in 1999 by the Mathematical Sciences Education Board (National Research Council, 2000). The ideas are just beginning to take shape in the mathematics education community. Future development will require some concerted work in conceptualizing the emergent approach more fully and designing experiences in which prospective teachers might profitably acquire this mathematical knowledge.

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[3] The University of California at Berkeley together with the University of Chicago School Mathematics Project (UCSMP) are major participants in a grant, awarded by the Stuart Foundation of San Francisco, to create and test a new type of college mathematics course described as "High School Mathematics from an Advanced Standpoint." Principal writers for the materials are Dick Stanley of the University of California at Berkeley, UCSMP Director Zalman Usiskin of the University of Chicago, Anthony Peressini of the University of Illinois, and Elena Marchisotto of California State University, Northridge. Initial pilot testing is underway.

[4]We wish to acknowledge the original work and thinking of Deborah Ball and Hyman Bass in advancing this particular line of argument.

Box 1. Principles for extended problem analysis (Stanley & Callahan, in progress)

The mathematical content involved in extended analyses of problems can be expressed as a set of mathematical "principles." They include:

1. Selecting parameters to represent key quantities in a problem situation. Typically the parameters replace numerical values of some of the quantities that are used in the initial statement of the problem. There are many subthemes here, such as:
   a. considering which quantities to parametrize;
   b. being alert for ways to generalize the results being found, and at the same time looking for important special cases;
   c. replacing a variable x that has a particular range $0 < x < L$ with a proportional variable p with a range $0 < p < L$.
2. Coaxing expressions into their most useful forms. Again, there are many subthemes:
   a. collapsing separate occurrences of the independent variable;
   b. making use of ratios in particular and dimensionless factors in general.
3. Representing relationships in a situation in several different kinds of ways to get different insights. Examples are diagrams, graphs, tables, formulas.
4. Looking for connections to different kinds of mathematics. For example:
   a. looking for geometrical interpretations of analytic results, and conversely;
   b. looking to connect discrete mathematics with continuous mathematics.
5. Anticipating, asking, and answering many of the sorts of questions that may occur to a reader who is trying to understand the ideas. Standard treatments often bypass such questions since they are not part of the most efficient and elegant presentation.

It is worth pointing out that this approach, while affording some new opportunities, is not without potential pitfalls. One potential pitfall is that discussion among current or future teachers might remain mired in school mathematics and fail to move toward higher mathematics. Another is that most of the discussion might be spent on more general teaching and learning issues. Our experience in presenting these ideas to a variety of audiences, however, is that with a well-chosen example the discussion can be deep and substantive, often leading to mathematical territory that is unexplored or implicit in the traditional curriculum.

Conclusions

Some of the current challenges in secondary school mathematics education today, coupled with new insights from research in mathematics education, suggest that it may be time to move away from the little questioned assumption that has historically guided mathematics education in the past several decades—that a major in mathematics, or something that deviates from it only marginally, is the best mathematical preparation for prospective teachers of secondary school. Teachers need to understand mathematics deeply. They must understand its applications and how ideas are integrated across subject matters. And they must be able to see mathematical possibilities in students' statements or written work. Could new majors be designed specifically for prospective secondary school mathematics teachers, that bring together the three kinds of mathematical knowledge described here, in ways that would serve our secondary teachers well? Given that as many as half of the mathematics majors, at least in some research universities, intend to become high school teachers, such development is warranted. The recommendations of the Mathematical Education of Teachers draft (CBMS, in preparation) are timely in their recognition of the need to diversify the offerings in the undergraduate curriculum for the prospective secondary teacher. Increasingly, the set of recommended offerings diverges from what at least has been the mathematics major. We need to prepare teachers to solve the kinds of mathematical problems that actually arise in teaching. This kind of thinking has not prevailed for secondary school teachers in mathematics.

In What Context Should Prospective Teachers Come to Know Mathematics?

The responsibility for the mathematical content preparation of secondary school mathematics teachers has been, historically, the responsibility of departments of mathematics—and this should continue to be the case, in our view. Some rather serious issues need to be confronted, however, if departments are to provide effective preparation. Within the faculty, who is responsible for the mathematical content knowledge of secondary teachers? What expertise do faculty members need, and how do they acquire it? Do they need to be working in schools, conferring with secondary teachers, and staying current in their knowledge of mathematics education? And, if their primary background is in mathematics education, how do they remain current in mathematics?

Departmental environment

Little research has been conducted about the student learning environment for mathematics majors who intend to teach secondary school mathematics, although there is much anecdotal information available. A frequent complaint among these students is that their experience in mathematics courses, in particular the nature of the mathematics instruction, is inconsistent with what they are learning in their education courses about the best ways to help students learn. In education courses about pedagogy, where they may be learning about the importance of actively engaging their future students, finding ways to make the subject matter meaningful and to connect it to other concepts, building on what students know, and using embedded assessments that call for explanation, they are developing certain knowledge and images about what effective mathematics teaching is like. In many upper-level mathematics courses, however, the instruction does not include these elements. So, despite the fact that these upper-level majors often are good mathematics students and have succeeded within the system, they sometimes are very conflicted about what should happen in their own mathematics teaching, because of the variation and dissonance they experience.

The use of technology in the undergraduate experience for prospective teachers is also an issue. NCTM and state standards recommend the use of technology in secondary schools to support mathematics learning. Thus, prospective teachers need experience as learners in using technology in advanced mathematical settings. Some institutions offer “reform” calculus or technology-rich calculus courses to students in the life sciences, or in some engineering and science tracks, but not to students majoring in mathematics, and therefore not to prospective secondary school teachers. In this case the teachers, as mathematics majors, take the more theoretical, less applied calculus option and don’t see the applications, connections, or experience the role of technology.

There are also aspects about the student learning environment (outside the classroom) that are problematic for strong students who have expressed interests in secondary school mathematics teaching. In conversations with their faculty advisors and mentors, such students report that sometimes mathematician advisors discourage them from teaching, with arguments about seeking more lucrative and prestigious options. Yet across the country, and especially in some urban areas, substantial numbers of middle school and secondary school children are being taught mathematics by teachers without even the equivalent of a minor in the field. It follows that without good preparation of students in mathematics in high school, the pressure to offer remedial courses at the undergraduate level and the lack of a supply of strong students into mathematics will continue to be problems facing higher education. Undergraduate mathematics faculty should be eager to encourage good and interested students into mathematics teaching.

A second difficult aspect of the student environment is related to advising. Students intending to be teachers need to meet the requirements of their major, as well as a set of course requirements and clinical experience requirements in professional education. Teacher education students typically have very full and demanding programs. Taking courses in the correct sequence at the correct stage of their undergraduate career is crucial to staying on track. With increasing numbers of teacher education programs being fifth-year programs, or five-year programs, or combined bachelors and masters degree programs, issues about when to apply to the program and deadlines for registering for student teaching, for internships, and for various state and national exams, become crucial to students' ability to complete their programs. Mathematics department advisors need to be aware that advising mathematics education students is complicated, and should work closely with college of education advisors to be sure they have the most up-to-date guidance about the overall program expectations.

Accreditation

The preparation of teachers is professional preparation, and as such brings with it some features that are often unfamiliar to mathematics departments. Teachers in the nation's public schools must hold licenses; therefore, the programs that prepare them generally must be accredited, either through the state or through national organizations or both. Generally colleges of education bear the responsibility for maintaining accreditation. This involves preparing periodic self-study reports, addressing the standards and expectations of the accrediting agencies, organizing site visits for outside reviewers, and responding to reviewer concerns. The activities of subject matter departments fall within the purview of these accreditation agencies. Therefore, faculty in mathematics departments are called upon to help prepare self-study reports and to meet with accreditation teams. This means that someone in the department needs to be aware on a continuing basis of the accreditation issues and of changes and new trends that emerge.

The Need for Collaboration

Lack of mutual respect and cooperation between faculty in colleges of arts and sciences and faculty in education is a long-standing obstacle to the effective education of teachers.[5] Unfortunately, it is quite common for undergraduate students to hear faculty in mathematics criticize faculty in education for lacking high standards, for not understanding mathematics, or for teaching material that has no substance. And, conversely, students hear their education professors complain about poor teaching in the mathematics department or lack of attention by mathematics faculty to current issues such as the role of technology. A variety of programs, conferences, and initiatives that are intended to bring together administrators and faculty in colleges of arts and science with those in education have been initiated over the years, although there is little evidence that such programs have effect. At the level of specific mathematics departments, some things can help: hiring faculty members whose professional scholarship is in mathematics education; arranging joint or adjunct appointments for mathematics faculty in education; including faculty from education in programmatic review or development efforts; holding regular meetings of those who advise prospective secondary school teachers in mathematics and those who do so in education; arranging to host visiting teachers-in-residence from local high schools; and facilitating joint efforts on specialized projects in research, curriculum, or teacher education. Deans and chairs can enable such things to happen, but they will need to make special efforts to monitor and discourage the very negative conversations that sometimes happen along these lines.

Mathematics Education

The body of research about mathematics teaching and learning is substantial and growing. The most recent major synthesis (Grouws, 1992) includes a chapter on advanced mathematical thinking and a chapter on

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[5] Historically, it was in the university-based arenas that the liberal arts vs. professional education tension developed most strongly: "The collegiate [or university level] institutions entertained grave fears regarding the ability of the normal schools to properly equip teachers for the high schools, while the normal school was certain that the failure to provide practice teaching and the need to adapt subject matter to the mind of the high school student rendered collegiate education inadequate" (Pangburn, 1932, p. 52).

teacher education—both areas that bear upon aspects of the undergraduate major that contribute to teacher education.

The research on undergraduate mathematics education is rich in its documentation of student difficulties, in evidence about interventions that can support deep student understanding, and in portraying how technology can be used in the undergraduate arena to help support student learning (see, e.g., Dubinsky, Schoenfeld, & Kaput, 1994; Kaput, Schoenfeld, & Dubinsky, 1996; Schoenfeld, Kaput, & Dubinsky, 1998). Those who are concerned with the improvement of undergraduate education in general, and teacher education in particular, might find this literature useful. In fact, in most programs, prospective teachers read research about mathematics teaching and learning in their teacher education programs. Perhaps such reading lists would be good background material for the mathematics faculty who are providing their content background.

Generally, education research has little direct impact on practice, either at K–12 or in higher education. The impact of research tends to be indirect. In addition to the findings, both the theoretical perspectives and the methodologies of research can be useful. For instance, a standard research methodology used in gaining insights about a student's understanding of particular concepts is the clinical interview.[6] We have adapted this methodology and used it in mathematics courses for teachers, asking prospective teachers to interview a high school student on some difficult topic.

Relationship with the Major

An important consideration for mathematics departments is the reality that prospective secondary school teachers comprise a growing and significant fraction of the set of math majors in many departments. Although national data are not readily available, from reports at some institutions it seems plausible to guess that more than half of the mathematics majors nationally may intend to teach in secondary school. If this is the case, then mathematics departments must seriously consider what it takes to prepare a teacher mathematically. Will university departments be able to recognize that "teachers' mathematics" exists, is conceptually difficult, and should be offered through departments of mathematics? This is a non-trivial problem that deserves substantial intellectual and institutional resources.

Conclusion

At the core of departmental work relative to teacher education should be the very nature of the course and experiences for learning content that are provided to students. National conversations along these lines are moving very quickly to develop a concept of "mathematics for teaching," as described in previous sections. Exploration of what this would mean at the secondary school level is less well developed than at the elementary level, but if researchers begin to take up this line of work, there will ultimately be implications for, and challenges to, the traditional practice of having prospective teachers take only the mathematics courses taken by prospective graduate students. This is a big challenge, because the ideas are still nascent and the research is just taking shape. Little is known about what this involves in practice, let alone in teaching students how to do it. The issues may vary considerably by content area, and methods for helping prospective teachers learn about this are highly underdetermined. We hope that this paper will serve to broaden the community of mathematicians, mathematics educators, and mathematics education researchers who are willing to contribute to this important line of work.

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[6] In the interview the subject is presented with a mathematical problem and is asked to solve it and tell the interviewer what she/he is thinking along the way. The interviewer does some probing and prompting, but does not tutor the student or move the student toward a solution. The sole purpose is for the interviewer to elicit the student's thinking, so that a "theory" can be built about the student's understanding of the concept at hand. Usually clinical interviews are tape recorded, transcribed, coded and become part of a larger data set that can yield information about understanding of a particular concept across a range of students.

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Two Critical Issues for the Math Curriculum

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Rather than attempt a general discussion of the curriculum, I would like to address two issues that I regard as especially pressing. The first is a broad issue affecting the whole mathematics curriculum, and not only at the college level: the role of reasoning and proof. The second is much more specialized: what to do about geometry.

Reasoning and Proof

In this era of powerful applications and diverse students, a prime concern of mathematics departments must be to protect the core of mathematical culture: the value and validity of careful reasoning, of precise definition, and close argument. This is an activity that is most highly cultivated in mathematics, but vital for the whole of society. Here are some actions that will promote this goal.

1. Providing for training in sustained logical thought in the mathematics major

Promoting the ability to reason carefully and flexibly should be a goal of every mathematics program. This is a skill which will stand graduates in good stead, whatever profession they follow.

All of the core upper division mathematics courses (basic abstract algebra and further algebra, real analysis, complex analysis, harmonic analysis, topology/geometry) should be thoroughly based on proof. Care should be given to acclimating potential mathematics majors to a regime of careful argument, precise definition and consistent justification. This may be done by means of one or more "bridge courses", in which an introduction to proof and definition is built into the course structure. Such courses have been implemented at many colleges and universities. They should be designed to fit as well as possible into the major, and to present important mathematical knowledge as well as to develop reasoning capabilities. Below I propose a two-course sequence in geometry. The first course of this sequence can serve as a bridge course.

The role of reasoning in transitional courses, such as linear algebra and multivariable calculus, needs to be carefully examined. Overemphasis on logical development in such courses may make them forbidding to a large portion of their potential clientele. (It is here especially that we suffer from poor logical preparation in high school.) Underemphasis on logical structure will leave mathematics majors and other takers of more advanced mathematics without an adequate grasp of the conceptual structure of these preliminary subjects. The situation is perhaps most serious in linear algebra, which is a pivotal course in the mathematics curriculum. Linear algebra is a basic subject both for pure mathematics majors and for students in a broad swathe of applied areas. In an ideal world with lots of time, it would be given in an introductory fashion and followed up by a course in which it was done right, much as calculus is reviewed and rigorized in the real analysis course. Such a follow-up course typically does not exist. Even if it did, such a scheme would create articulation problems with multivariable (here meaning, more than three dimensions) calculus, which draws heavily on linear, and even multilinear, algebra. When possible, different levels of linear algebra should be offered, one more numerically based and one more conceptually/logically based. Students should be carefully advised as to which to take, depending on their future intentions.

2. Working for mathematical and logical integrity of service courses

This heading includes the construction of courses which do justice to the mathematical core of the subject, while also satisfying the applications clientele that the course is sufficiently relevant to their needs. These courses should be constructed in consultation with faculty from the relevant fields of application. The consultations should seek to find and satisfy the main needs of the applications field. Service offerings should be as focused as is feasible in the context of a given department. For example, if there is sufficient demand and resources, calculus tailored to individual application fields (e.g., calculus for biologists) is preferable to a generic calculus course. The main advantage is not that the mathematics covered would be substantially different, but that it can be tied to contexts which are particularly meaningful for the students, and deeper examples can be presented. Currently, many wonderful applications of calculus to physics cannot be presented in a generic calculus course, because most students have insufficient physics background. At the same time that they promote the effective adaptation of service courses to the needs of application areas, consultation sessions can be occasions for advocacy of the conceptual underpinnings of the subject.

Rigor and reasoning should always be presented in a way that can be meaningful to the audience. When deep ideas and reasoning are brought up in service courses, they should be treated in a “great ideas” fashion. One example is the theoretical core of the calculus (Existence of Extrema, Rolle’s Theorem, Mean Value Theorem, Fundamental Theorem of Calculus). As well as discussing the necessity of the hypotheses, one can paint in the historical context, noting how the development of coordinate geometry set the stage for the momentous discovery of the Fundamental Theorem, uniting two basic problems which seem utterly disparate. A more focussed example is the definition of continuity. Present it as the intellectual triumph that it was. Another example is the algebraicization in linear algebra of the deeply geometric notion of dimension. The significance of the results, the role of the hypotheses, the use of deep theorems as axioms, and such considerations should be given proper attention. Difficult or tricky aspects of the demonstrations should be discussed. It is legitimate to ask questions about such logical development on exams, without either asking for proofs, or for applications of the theorems.

CUPM can perform a valuable service by undertaking studies of the major service courses, with the goal of clarifying what are the most essential theoretical aspects of the course, and recommending appropriate levels of treatment for the important concepts. Recommendations for alternate treatments of a given subject, depending on the type of clientele, would be desirable. Linear algebra is the subject probably in most need of such attention.

3. Establishing careful reasoning and problem solving as a feature of the liberal arts curriculum

As much as its technical accomplishments, the habits of thought used to discover and develop them are a vital part of the legacy of mathematics for the world at large. Define as sharply as possible what you are talking about, be as explicit as you can about your assumptions, specify conditions on the validity of your reasoning, and argue as carefully and precisely as you can. Distinguish between a statement and its converse, but know that arguing the contrapositive will establish the original. Understand the difference between impossibility, possibility and necessity (existential and universal quantifiers). Make necessary distinctions, analyze by cases, distinguish essential differences from superficial ones, be sure to cover all the cases. Break down complicated problems into pieces. Make simplifying assumptions or try simple cases or simpler analogous problems. These are practices that can be valuable in the lives of many who will never use technical mathematics. Particularly since the logical training once provided by the high school course in Euclidean geometry has been becoming less and less a feature of modern American schooling, there is ample motivation for a course at the college level of basic mental hygiene in reasoning. I don’t have in mind here a formal course in logic, though logical issues would be a prominent theme in the course.

I am thinking of a "Daily Problems" course, with analogous aims to the "Daily Themes" courses that used to be a feature of freshman English. Such a course would be full of problems, of a greatly variegated nature, some clearly mathematical, some less so, more focussed on logic or common sense. The goal would not be to cover a specified syllabus of mathematical topics, nor to provide a sampling of fun or interesting mathematical ideas, but to challenge students to think and reason, to confront them with a variety of problem situations, some from the "real world" and some purely formal, some requiring some technical knowledge to deal with, some immediately accessible to all. This would be a chance for a faculty member to roll out his/her list of favorite puzzles, and to learn new ones. Problems could be grouped to illustrate common types, and also to demonstrate how diverse kinds of problems may invoke common principles. In other words, the course could serve both as a "how to" for certain common kinds of problems, and as a means to pursue broader intellectual goals. I would hope that such a course could be a distinctive contribution of mathematics departments to the liberal arts curriculum. It could also be especially valuable for prospective high school teachers, and could be part of a program to insert such a course into the high school curriculum. The most stunning success such a course could have would be, that it would have to be abandoned because most entering freshman had already had the experience it offers in high school.

4. Pay close attention to the training of teachers, and advocate to strengthen it

For high school teachers, an enthusiasm for, and an appreciation of the need of, careful and sustained thinking is as important, and probably harder to attain, than a knowledge of the details of subject matter of high school mathematics. Indeed, a high school teacher who does not understand and appreciate the logical structure of the subjects of high school mathematics, is missing a serious, though officially unrecognized, part of his or her qualifications. There should be courses specifically designed to address this matter. The problems course advocated just above, and the geometry sequence detailed below can be part of the solution.

5. Advocating (through the MAA) for the renewed attention to careful reasoning in high school

The progressive demise of the traditional geometry course which paid serious attention to issues of logic and reasoning, is leaving the idea and practice of careful thinking without a place in the high school curriculum. Loss of the opportunity for high school students to have an introduction to systematic reasoning would constitute a major institutional failure, but it seems to be happening. This is unfortunate for society in general, but particularly unfortunate for mathematics departments, as even potential mathematics majors enter college without a basic acquaintance with mathematical proof, and unpracticed in logical argument from given premises, or in careful attention to definition.

College mathematics departments, collectively through the MAA, should advocate for increased emphasis on careful reasoning in high school. All high school mathematics courses should be seen as loci of reasoning and proof. Teacher preparation programs, and likewise inservice teacher development, should emphasize the need for and the opportunities for, logical thinking and deduction in algebra, geometry (both synthetic and analytic), and pre-calculus and calculus. Simply pointing out that certain derivations are in fact deductive arguments, and making clear the lines of reasoning, could be helpful. Likewise, the short sequence of theorems (described above, in section 2) that lie at the heart of calculus, should receive careful attention. (These should not be a shock to students used to seeing and dealing with logical argument from earlier courses. In presentations of such interludes of rigor, it is not enough to present simply the logical development. Students will benefit also from considering why such developments are desirable, what they accomplish, and the role and need for the assumptions in the various theorems.) However, besides increased attention to careful thought and logical deduction in all high school mathematics, there should be a course or courses in which this is a central focus. Besides the traditional geometry course, other possible loci for this activity could be a problem-solving course or a basic computer science course. At least one such course should be required for college entrance.

In these efforts to strengthen the high school curriculum, the college and university mathematics community should work with and through the extensive movement for reform of K-12 mathematics education. This movement should be a natural instrument for strengthening the values of careful thought in high school. The voice of mathematicians needs to be heard in this movement, to balance the insistent calls for applicability, which tend to emphasize the artifacts of mathematics (such as calculus in the 1960s and the normal distribution and graphing calculators now) over the core processes.

Geometry

Geometry today is clearly the invalid of the college mathematics curriculum. In many institutions there is no regular offering in geometry, and in others, the main offering is a course specifically aimed at high school teachers, giving some mixture of axiomatics, less familiar results from Euclidean plane geometry, surveys of basic ideas and results of the non-Euclidean (hyperbolic and elliptic) geometries and/or plane projective geometry, and perhaps basic ideas of isometries.

It should not be so. Geometry was for centuries the heart of mathematics. Euclid's Elements stood for over 2000 years at the summit of pure thought. Geometry played a pivotal role in the 19th century revolution that established the essentially abstract nature of mathematics.

Geometry is still a vital part of mathematics. As represented in Lie theory, algebraic geometry, dynamical systems, Riemannian geometry and other research areas, it is a central part of the research enterprise. It continues as a major link between mathematics and physics. Computerized in CAD-CAM, it serves as a vehicle for major industrial applications of mathematics.

Many research mathematicians active today remember their high school course in geometry as a formative experience, the time when they discovered the power and elegance of mathematical proof. That course in increasingly endangered. With greater numbers of students taking geometry, proof and reasoning receive less and less attention. This has implications for more than just the geometric understanding of students reaching college: it means that there is increasingly no place in high school where careful thinking is explicitly attended to. Not just geometric intuition, but basic capacities for logical thought, and especially, mathematical proof, are being less and less fostered in high school. Since the main function of many college courses in geometry is to serve as background for teaching the traditional high school course, which is itself vanishing, its continued viability also seems in doubt.

It would clearly be a tragedy for geometry to fall out of the mathematics curriculum. What might be a viable role for geometry in today's curriculum?

Here is one proposal. It is based on a project I have been pursuing with William Barker of Bowdoin College for several years. I suggest a two-semester sequence in geometry. This sequence can be used for several purposes, and its precise functions will depend on the rest of the offerings of a given department. However, the main goals are:

1) to develop geometric intuition about the real world;
2) to provide experience in careful reasoning;
3) to link 1) and 2); and
4) to demonstrate connections of geometry in the classical sense with broader and more modern themes in mathematics and physics.

Roughly speaking, the first course emphasizes the first three goals, while the second concentrates on the fourth. However, the first course already provides important links to and motivation for abstract algebra, by providing a range of examples of groups, and by showing how transformations can be used to establish theorems of indisputably classical geometric content. On the other hand, the second course maintains a concrete point of view while introducing higher levels of sophistication.

The sequence can be designed to build on the waning geometry course given in high school. If desired, the first semester can serve as a "bridge course", for introducing and providing practice in reasoning and proof for students in need of such an introduction. With perhaps a slightly different emphasis (or not, according to the skill and style of the instructor, and the distribution of students), it could also serve as a general liberal arts course for non-majors. The first semester also has links to more advanced mathematics, especially to abstract algebra, and of course, to the second semester of this sequence. The second semester will connect geometry with modern higher mathematics, expecially Lie groups and differential geometry, but also complex analysis. It makes strong use of linear algebra. It will provide a rich collection of examples of groups, and will show how group theory does indeed control geometry, the insight of Felix Klein summarizing the spectacular development of geometry during the 19th century. It will also serve to connect group theory and geometry to physics, especially the special theory of relativity, and thus might make an attractive offering for some physics students and for ambitious liberal arts majors.

Although the course is ambitious, and may seem overly ambitious to some, it also has considerable flexibility, depending on the preparation and inclinations of the students. For students with limited backgrounds, the earlier topics can be emphasized; for mathematically ambitious students, they can be curtailed. The course has in fact been offered with considerable success to very different audiences, including a class of serious mathematics majors, and others consisting primarily of non-mathematically inclined students.

First Course

Here is an inclusive (meaning overfull) syllabus for the first semester.

1) Recollection of basic geometry. This serves as to solidify student knowledge and intuition about plane geometry, and serves as a crude (not formal, and far from minimal) study of the axioms of geometry.

2) Study of plane isometries. This introduces the idea of transformation, and uses basic geometry to establish the nature of plane isometries. Key results are two structure theorems. The first describes the nature of an individual isometry and provides a crude classification of isometries, based on a factorization into a product of reflections in lines. The second provides a global view of the collection of all isometries. Some language of group theory is introduced, but there is no attempt to study groups *per se*.

3) Extension to similarities of the plane. Here the main facts are that every similarity is (uniquely) the product of a uniform dilation from a point and an isometry, and that every strict (meaning, non-isometric) similarity has a fixed point.

4) Applications of transformations to geometry. There is a great range of attractive geometric results which can be proved elegantly using transformations, including a striking collection of results connected with the 9-point circle. Transformational proofs often provide more insight than do conventional synthetic proofs.

5) Extensions to three dimensions. This provides opportunity to improve spatial visualization abilities. The transformational approach emphasizes the parallels (no pun intended) with the planar theory. Again, factorization of isometries into products of reflections is a fundamental tool. Three dimensions are also needed later on (in the second course) as a setting for the projective approach to non-Euclidean geometry.

6) Symmetric figures. The transformational approach makes a study of symmetric figures in space and the plane natural and easy. One can study finite figures (regular polygons and polyhedra), infinite but discrete figures (frieze patterns, wallpaper patterns and crystals), and continuous figures (lines, circles, helices). One can study symmetry under similarities as well as isometries and include spirals. Of course this topic allows for profuse illustration from nature and art. The varied examples of groups that one sees here, in addition to the extensive use of group-theoretic ideas in the study of isometries and similarities, provide both background and motivation for learning group theory in abstract algebra.

7) Volume, area and dimension. This is a brief introduction to the basic ideas of measure, and the relation between the dimension and the scaling properties of measure under dilations. It allows a foreshadowing of the basic ideas of the real number system and of measure theory, and offers an excursion into fractal geometry. The behavior of various quantities under scaling again offers rich connections with the natural world. This topic is the most optional.

The first course could be given either before or after a course in linear algebra. If given before, and as a bridge course, it would deepen the level at which the linear algebra course could be given, but it would itself have to probably concentrate on the more basic issues. If linear algebra is made a prerequisite, it might appear somewhat artificial for the course to be taught wholly in synthetic mode, which is the assumption of the above outline. The introduction of coordinates, which here is proposed for the second course, might be done earlier.

Second Course

The first course, although it emphasizes transformations, is strongly oriented towards traditional geometry. The second course is much more modern in spirit, and in some sense recreates the developments of 1850–1905 (with, one hopes, some improvements based on hindsight). Specifically, its goal is to show how transformational ideas establish a direct connection between Euclidean geometry and the special theory of relativity, and that this connection passes right through non-Euclidean (hyperbolic and elliptic) geometry. Thus, it closely links all the plane geometries, and links geometry and symmetry with physics, which was a major theme the 20th century theoretical physics. It serves also as an introduction to basic ideas of Lie groups and some of their uses. The course makes heavy use of linear algebra, with numerous concrete calculations, and can serve to solidify a student's grasp of linear algebra. At the end, important links with complex analysis are also made. The course also has strong connections with low dimensional topology, via Riemann surfaces and hyperbolic manifolds, and could be adapted to emphasize these aspects.

Here is an outline of topics for the second course.

1) Coordinatizing the Euclidean plane. This introduces the usual Cartesian coordinate system (or systems, since it pays attention to the data you need to specify a coordinate system). It then studies how isometries transform coordinates. Translations are linked to vector addition, and isometries fixing the origin are expressed in terms of matrices. Then the general transformation represented by a matrix is investigated, and the idea of affine geometry as the geometry of straight lines and parallelism, is introduced.

2) Affine geometry of conic sections. Here the behavior of conic sections under affine transormations is studied. A main result is reduction to standard form by means of isometries. As a corollary, one sees that all ellipses are affinely equivalent, and likewise, all hyperbolas. The consequence, that all conic sections are equally symmetric as affine objects, is investigated, and the natural "distance" on a line, a circle, and a hyperbola is investigated from the point of view of symmetry. This gives a new context to some classical topics of one-variable calculus: the logarithm function, and the analogies between the circular and the hyperbolic functions.

3) Projective and non-Euclidean plane geometries. The projective interpretation of Euclidean plane geometry is given, and used to extend Euclidean geometry to projective geometry. The same scheme is used to give an overview of the non-Euclidean metric geometries. This can be summarized by the "real Klein hierarchy" of subgroups of the group of real 3×3 matrices. The Klein heierarchy embodies one of the major syntheses of 19th century geometry, expressed by Cayley's dictum that "Projective geometry is all geometry." Finally, this picture is used to describe Euclidean geometry as a limiting or degenerate case of either hyperbolic or elliptic geometry, and as separating the two.

4) Special Relativity as geometry of space-time. The main theme here is to interpret kinematic phenomena as reflecting the geometry of space-time. This discussion requires the full strength of Klein's interpretation of geometry as the study of invariants of a symmetry group, and also uses in an essential way the relation between coordinate systems and coordinate changes to symmetry transformations. The ideas of the previous topic allow one to make an analogy between the transition from Galilean-Newtonian space-time to Einsteinian space-time and the transition from Euclidean to hyperbolic (Lobachevskian) geometry. The picture of hyperbolic geometry as a "deformation" of Euclidean geometry allows one to see Einsteinian relativity as a deformation of G-N relativity.

5) Euclidean geometry and complex numbers. Here the essential identity of the group of (orientation-preserving) similarities of the Euclidean plane and the affine group of the complex line is noted, and extended to obtain a “complex Klein hierarchy” of subgroups of the 2×2 complex matrices. This is parallel to, but incompatible with, the much better known real Klein hierarchy of part 3. Euclidean geometry is interpreted in terms of one-dimensional complex projective geometry (aka conformal geometry or inversive geometry) rather than two-dimensional real projective geometry. The relation between the two hierachies can be thought of in terms of the basic figures of Euclidean geometry, the line and the circle. In the real Klein hierarchy, the straight line is the fundamental object, while the circle loses its identity and becomes subsumed under the broader class of conic sections. On the other hand, in the complex Klein hierarchy, it is the circle which is fundamental, and straight lines just appear as circles which pass through the point at infinity.

The existence of both hierachies is perhaps the deepest formulation of the richness of plane geometry. The two heierarchies also correspond to the two most common models of hyperbolic geometry: the real point of view leads to the Beltrami-Klein model, and the complex point of view leads to the Poincare model. The analog of reflections in this situation is the operation of “inversion in a circle”. The discussion can be tied back to classical geometry by considering some of the lovely applications of inversion to classical geometry, including the Feuerbach Theorem and Poncelet’s Porism for triangles and quadrilaterals.

A sequence such as described above has a great deal of flexibility, and has the potential to well serve at the same time prospective high school teachers and majors who aspire to graduate school. Both courses have a great deal of flexibility at both ends, and can be made more basic or more sophisticated while retaining the same core. These courses should also be appealing to a variety of students who are attracted to the philosophical and “big ideas” aspects of mathematics, and especially to physics majors.

Accountability in Mathematics: *Elevate the Objectives!*

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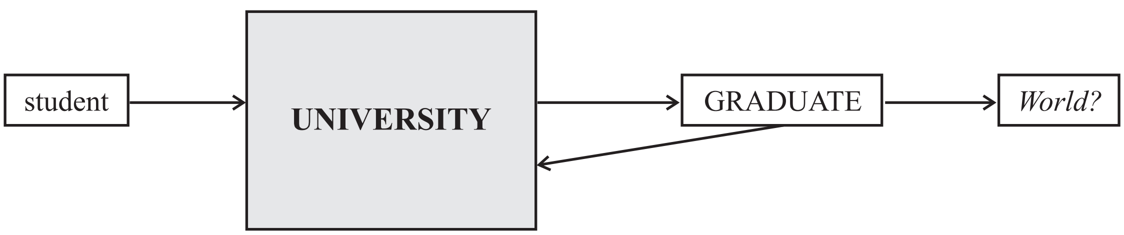
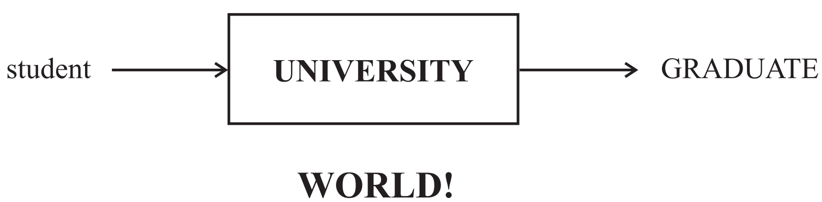
What is the next decade for mathematics going to reveal regarding assessment, or the now-preferred term, "accountability"? In keeping with the subject, we will not predict an outcome, but keep our eye peeled closely on the process by which we might progress through the next decade, in terms of our objectives, the monitoring we do, and the summarizing.

Until recently, this might have been a diagram of our view of a university:



The student was the output from the university "black box," emerging as a kind of variable into the "real world" (or more hopefully, returned at a professorial level to the university itself!) However, "real world" as something that is vaguely "out there" is a term one doesn't hear very much these days. In a more accurate picture of today's educational scene, the university is a porous organism inside the world, allowing and requiring input/output from/to that world:



Factors that have changed the picture

From the public sector

- The public is losing its love affair with higher education. An ever larger fraction of public institutions are in fact no longer public—even state institutions are raising tuition and relying on trustee funding, as with private institutions. (Minnesota's governor, a drop-out from community college, is quotable on this: "If you're smart enough to go to college, you're smart enough to pay for it"—an argument, by the way, with which he seems to have activated the student vote and won.)

- At the same time the public is paying less, it is expecting ***more*** from education, and what it expects is of a more qualitative nature. It is of less concern to the employer that the student got an "A" in a course or that the content of a course included a topic, than that the student is up to the analytical thinking required on the job, can explain technically what is happening in less than a page, will grasp what is basic, be able to read and produce reports, be able to work collaboratively, and be ambitious with technology. In other words, qualitative skills are expected which indicate a student's knowledge, flexibility and potential (depth, width, and height!). And course content is expected to produce these qualities.
- Related and worth noting: industries are looking for students who have an understanding of the feedback process in "*quality control* " (read: assessment) now taught as a subject in some statistics and Computer Science departments.
- While the public is paying less and expecting more, it simultaneously expects an ***ever larger percentage*** of students—say 80%—to experience higher education. This comes at a time when the general mass of high school students in America are notoriously ill-prepared, and competition for admission and job placement is international.

From education programs at state schools (which comprise 60% of the higher education student population) we hear:

- Since elementary and secondary public school systems must assess student learning *openly* and explain themselves to the public, higher education institutions, in turn, should do the same. Courses for secondary teachers are offered *within* a departmental major and these courses are already being assessed by state or national standards in terms that apply not necessarily to majors but to prospective teachers.
- For learning objectives, teacher education organizations in the past have been looking to the national organizations of the disciplines. However, if organizations are not up to the expectations for improvements and feedback coming from the teaching industry, teacher education institutions may design their own curriculum in order to satisfy gaps in qualitative indicators—for example, creating their own course in an area and eliminating another.
- Whereas our habit has been to focus on methods of teaching as an art, there is a growing expectation that funding will be provided based on results. This will make assessment methods inseparable from the practice of teaching.
- Equity considerations at schools are increasingly in the forefront of an institution's concerns, while diverse backgrounds of minority students and international students provide challenges for placement, advising, and instruction.

And from the student:

- In many cases, because of their need to integrate school and work in their lives, students have become more fluid in changing schools. This problem has been trending toward the concept of education as a "televersity" with long-distance learning and virtual lectures—something which is not well understood in terms of its impact on education and the curriculum.
- Lastly, students are demanding the opportunity to find lucrative employment in technological fields that frequently cross over departmental barriers. Meanwhile, the job market is unpredictable and skewed toward technology. As a result, students have begun to critically evaluate programs not just for their quality but for their flexibility, fungibility, and potential to link them with jobs.

Assessment tools provide the best opportunity for departments to react appropriately to these needs and pressures.

Accountability: the need for the examined way

Assessment takes place whether we are conscious of it or not—at worst as a rumor about a teacher or the gradual draining of students from the major. The challenge is then whether we can construct our programs in such a way that permits us control of the assessment environment. And one thing is patently clear: accountability is not something that is intended for others only. A teacher recently told me of the results of a survey at his college: when faculty were asked about the relative importance of various issues for the college and for themselves, personally, the greatest discrepancy was with the issue of assessment. It was deemed of highest importance to the university, yet, the faculty saw no personal use for it at all. Assessment, however, is an *environment*, a *framework* within which education flourishes, and accountability takes place at the level of the student, instructor, department, administration, and national accrediting organizations, and even organizations such as the MAA.

The bulleted list above summarizes some of the forces exerted on a university in general ways, and we have not discussed the impact on ***mathematics*** departments. For mathematics, the concerns go double. Needless to say, it is no longer acceptable to the public that 20–60% of the population will find mathematics unlearnable—a cliff which only the toughest can climb; a filter not a pump. The role of mathematics departments has changed: while the number of students in the traditional major is shrinking, an increasing number of students merely wish to weather the service courses or the developmental and general requirements. And it is no longer enough for us to assess ourselves with gatekeeping measures such as placement tests and pass rates. Teacher education in mathematics has never been more visible, and " lack of preparation" is a term that is batted back and forth from the student (who hates math) to the teacher (who hates math). Students are often desperate and ill-prepared, and simultaneously impatient with the demands of a curriculum that has gone essentially unchanged for half a century.

Where have we come from?

The issues of assessment became noticeable in the education community in the late '80s. Accreditors and other education experts could see that a university operating with the characteristic assessment grid of the times (below) was not demonstrably working up to its full potential.

	Student	**Instructor**	**Department**
Student evaluates		Teacher evaluation forms	
Instructor evaluates	Tests, homework	Committees	
Department evaluates	Data	Committees/Data (research,grants, etc.)	Committees/Data

Moving with this trend, the MAA in 1990 entered the assessment arena with a subcommittee of the CUPM. In 1995 this assessment committee published in *FOCUS* the document, authored by James Leitzel, *Assessment of Student Learning for Improving the Undergraduate Major in Mathematics* [3]. This document gives advice which encourages departments to begin creating an assessment climate by setting goals and using a variety of assessment instruments, which include surveys, evaluation reports, portfolios, essays, summary courses, oral presentations, and dialogue with students. Assessment is described as an ongoing "cycle ":

- goals for teaching and learning are set and communicated to faculty and students,
- questions are gathered and information is summarized, and
- information gathered is made public and used in setting new goals

Strategies for the future

This MAA document of 1995 [3] is commendable for moving the mathematics establishment into new modes of thinking, changing the generally apathetic or antipathetic view of assessment in the departmental trenches to a level of some appreciation. Yet our current job—and this is only in keeping with the recursive nature of the assessment process itself—must be to elaborate the document for aspects that have proved in practice not to be clear enough.

The document offers, as examples, goals for a mathematics program as follows:

- mathematical reasoning,
- understanding of the nature of mathematics,
- mathematical modeling (problems, projects),
- communication (oral and written),
- resourcefulness (group work, presentations) ,
- personal potential, and
- content-specific requirements which may include understanding of technology.

We should stop to clarify here: in education circles one hears the words "***goals and objectives***"—it may help to think that ***goals*** are cognitive (about understanding, as opposed to doing) and ***objectives*** relate more to behavior ("write a discussion of …") . Unfortunately the problem for so many of us is that the goals that have been laid out so far are not so much goals or even objectives, as ***subtopics*** of mathematics. Consider for example, the goal concerning "Nature of Mathematics":

> **Nature of Mathematics**: Students should possess an understanding of the breadth of the mathematical sciences and their deep interconnecting principles; substantial knowledge of a discipline that makes significant use of mathematics; understanding of interplay among applications, problem-solving, and theory; understanding and appreciation of connections between different areas of mathematics and with other disciplines; awareness of the abstract nature of theoretical mathematics and the ability to write proofs; awareness of historical and contemporary contexts in which mathematics is practiced, understanding of the fundamental dichotomy of mathematics as an object of study and a tool for application; and critical perspectives on inherent limitations of the discipline.

*Naturally***,** we say, we teach "understanding"! *Naturally*, we want students to understand the nature of mathematics! And when we see mathematics educational vocabulary attempting to clarify these goals, some of us may be inclined to be confused or feel that we are being made to engage in rhetorical word-play.

Recently, my department was given by its math education component a grid to fill out, on which we were to rate eleven mathematics courses which overlapped with the mathematics education major. We were to put an "**X**" in a column which represented a course, if it fulfilled the learning objective of the row. The rows began by stating content objectives (for example, "Understanding the discrete processes from both concrete and abstract reasoning," as well as the somewhat quizzical "Understanding Uncertainty"). But some pages later the goals had moved beyond the content-specific to the affective. A portion of the grid referring to "Mathematical Processes" is reproduced below.

This education grid has my department thinking about what it is doing more clearly than it has been able to do in the last ten years. (And by the way, if we have not been collecting and organizing student work, we are now! Count on mathematics education to put the fire under us!) But as I tried to use the grid, I found myself claiming all six categories for Math 478 since it was difficult for me to imagine a course satisfying any one of these objectives without the others. (This became a touchy issue in a departmental discussion when it seemed that a multivariable course was giving out more **X**'s within the "geometry" sub-category than the geometry course itself which, the professor firmly maintained, did not deal with "measurement"—the power in the course presumably lay in the number of **X**'s it wielded!)

	Math 223	Math 478	(etc.)
G. Mathematical Processes **1. Mathematical Reasoning**			
a.) Examining patterns, abstracting and generalizing based on the examinations, and making convincing mathematical arguments		**X**	
b.) Framing mathematical questions and conjectures, formulating counter-examples, and constructing and evaluating arguments		**X**	
c.) Using intuitive, informal exploration and formal proof		**X**	
2. Communication			
a.) Expressing mathematical ideas orally, visually, and in writing		**X**	
b.) Using the power of mathematical language, notation, and symbolism		**X**	
c.) Translating mathematical ideas into mathematical language, notations and symbols		**X**	

While the grid-system seems to me to indicate that there has been a serious attempt to turn a ***goal*** into a series of more manageable learning ***objectives***, a crucial ingredient is still missing. It is missing as well in the follow-up form from mathematics education (the legislators, actually). Here we were to reify our **X**'s by responding to the following:

(1) What courses meet the particular standard (mathematical reasoning)?

(2) What practices are being applied to meet the standard?

(3) How are we assessing that the standard has been met?.

Mathematicians, do you see the problem? In fact, there is no middle ground here between (2) and (3)! Among other things, what is not being asked is *how and where can we find the learning gradient*! We are to claim a standard, apply it, and then assess it, with no attention to the primary calling of the teacher which is to generate improvement.

Returning to the MAA goals there are more questions we might ask: For example, how will we assess that the student has gained *"an understanding of the breadth of the mathematical sciences and their deep interconnecting principles"*? And if this goal is really not asssessable, is it a goal?

Again and again in educational circles, the discussion seems to be over what we want to do, what we want to offer the student or put on a student's plate. Perhaps we go so far as to collect a bag of artifacts to demonstrate what the student has done, and to justify it in some sort of language. But this doesn't extend to defining in substantial terms, *how students are learning*, or *how they are improving*, or even, *what the student is going to do with the knowledge.* In her first year teaching at Alverno College, one assessment consultant who visited my school described this experience: she had been expressing her delight to a colleague in teaching a certain novel to students—the class was greatly enjoying it, and were highly responsive. She was taken aback when her friend asked, "yes, but *what will the students do with it?*" In all our planning and scheming, we must shift attention to the intellectual and practical issues these questions raise.

How can we do this? An example, please!

Consider another goal from the MAA document:

> **Communication and Resourcefulness:** Students should be able to read, write, listen, and speak mathematically; read and understand technically-based materials; contribute effectively to group efforts; communicate mathematics clearly in ways appropriate to career goals; conduct research and make oral and written presentations on

various topics; locate, analyze, synthesize, and evaluate information; create and document algorithms; think creatively at a level commensurate with career goals; and make effective use of the library. Students should possess skill in expository mathematical writing, have a disposition for questioning, and be aware of the ethical issues in mathematics.

In fact, I myself fall short of the goal. Although I could argue that I meet it with some double-speak, I cannot actually convince myself that my students *understand technically-based materials, communicate in ways appropriate to career goals, or think creatively at a level commensurate with career goals*. Suppose on the other hand, we could elevate the objectives by framing the goal more holistically, as it pertains to learning and usefulness:

- **Who is recommending the assignment and in what courses? Who are the students?**

Most of our students will leave college to enter job situations or graduate school, where these students of mathematics will need to explain abstract mathematical and technical material clearly to others. Writing assignments are increasingly being recommended in the earliest mathematics courses by educators, and mathematicians and legislators.

- **Why will the students need the skill/ability?**

It is expected that students enter the work force with the ability to write clear explanations, present a paper in a conference setting, or perhaps answer pointed questions about their research in a poster session or an interview. Students should be able to read technical books with facility, use library resources appropriately, and be familiar with the advantages of technology (including the web) in demonstrating an argument. On the job, for example, employees must write reports that describe and evaluate the work of a team, design curricula and provide explanatory hand-outs, or even justify the benefits of a mathematics project to a grants organization, or a new curriculum to a parent of a student.

- **What are the trouble spots?**

Students often have the most difficulty writing a convincing opinion piece on something from the newspaper (say, about basic skills testing), although this style of exploration is exactly what would be expected of future teachers, or workers in industry, and in universities. Difficulties may arise with critical thinking, persuasion, understanding the audience, or even organizing a mathematical argument. Often students may assume the audience is familiar with a subject, or assume a fixed position in discussing a controversial topic, or they may be reluctant to find all the necessary information for constructing an argument. Problems of documentation and plagiarism must be explained, particularly with groups working on projects on computers, the web, etc.

- **Are there helping organizations?**

Texts on writing in mathematics, both for the teacher and the student, are available from the MAA. Frequently, English departments provide writing help in skills-centers. Faculty from English departments are often willing to help a teacher in class learn how to discuss papers that students have written.

- **What is the faculty/students experience regarding the objective?**

The complaint from some faculty that these assignments take too much time is counterbalanced by the usefulness of these assignments in producing documentary evidence of student improvement. International faculty faced with teaching writing should be encouraged that they are still *teaching* mathematics, they are merely using *writing* to do so. And that this means that they merely must know whether the written or oral arguments are clear and convincing. Students on their part may tend to feel that writing does not have a place in a mathematics course, and probably will require convincing.

- **What is the learning value?**

Language is the tool with which we think, and good writing is the sign and product of good thinking. It is in writing that we clarify and make explicit what we know and think and how we came to know and think it. In writing, we convey knowledge, and if we cannot express ourselves in words and writing, we create doubts as to what we know.

Revision is an important aspect of learning to write. While continuous revising is expected of students in an English class, a mathematics teacher is much more likely to hand back an essay with a grade and a blur of blue pencil, which students will not read. Writing with revisions makes a good case for documenting improvement in writing.

- **How will you document that learning takes place?**

When drafts are submitted in a portfolio, they can become a significant part of the assessment cycle review. Samples of graded work (excellent, medium, poor) can be included to illustrate standards for grading.

- **Are students themselves learning good assessment practices?**

Students must learn to understand the importance of their own responsibility in the learning process. Sharing papers with the class, presenting colloquia, defending a poster session, and capstone experiences all create situations in which the student is a protagonist, and are useful in helping a student find a "voice."

This list is certainly not complete. Nevertheless, if we could rethink our goals for our departments in this manner, taking account of the student profiles, the purpose, the difficulties, the benefits, the experiences of others, the methods, and how the learning gradient might be measured, among other things—we might be taking a critical first step in making objectives "real" in the minds of both teachers and students.

Get hold of the data!

The MAA document of 1995 does not really discuss an aspect of assessment that is central: an assessment program functions best when it is a dynamic combination of twin processes. Summative assessment deals with data gathering and interpretation after the course or program cycle is completed *(how many students did you place into a course, how many passed?*) and formative assessment deals with tracking progress and creating in-process changes, using the summative information collected. Too often assessment seems to function at one level or another. For example, a faculty evaluation form may provide that the "average score" on all 20 questions about a faculty member is 4.3 out of a possible 5. This is meaningless information. On the other hand, a faculty member who submits a dozen classroom assessment techniques of their own creation ([1]) such as : "What did you learn today?", "What is your favorite subject so far?" will find no administrative recognition of this as assessment.

A major problem is that while the assessment movement counsels the adoption of formative, context-sensitive methods it also requires the need for comparative or universal measures. After all, the health of our programs depends to a large extent on our being able to compare and contrast ourselves using data gathered on other institutions. *How many math majors does the state university system in Illinois serve? Of these, how many are women? How many math majors are picking up a second major or a minor, and in what?*

National and regional cross-institutional data sources are available to a surprisingly limited degree, given the opportunities provided by present computer technology. Currently assessment at a statewide level is often done by accrediting organizations sponsored by the institutions that use their services. These organizations tend to provide on extended cycles (10 years) a yes/no diagnosis—either you pass or you do not. It would be extremely useful if these organizations worked with continuous feedback and were prepared to offer comparative data from other schools. The web raises information exchange and networking to previously impossible standards. If clusters of institutions are having success, we might focus on them, or conversely, avoid their pitfalls. If some departments are receiving financial support for this particular assessment activity or the other, we need to know. There is an argument for seeing accrediting organizations as data managers and networkers; these ideas would require negotiation.

The point is that black-box departments won't survive. As in the graphic at the beginning of this paper, mathematics departments are part of the world. And what we do must be shared with students and other faculty so they can collaborate with us. In this way administrators can know *who we are* by *how we work*, and they will know how to support our needs with the public.

In fact, we might even begin to give credit to the position that summative results should not be interpreted on the straight and narrow, but more descriptively. *Why did students who have had Math 101 perform more poorly on this test than students who have not had it?* Perhaps because the students sampled who had not taken the remedial course were in the Honors program. There is something to be said for "fuzzy" assessment[1] which looks at data not so much as the final answer but as an inspiration to look for the reasons ***why*** the result is happening.

Open up to the public!

Step 3 in the assessment process outlined by the MAA describes the need to go public, and this is discussed above. But mathematics departments have been notoriously private. If a focus group on calculus is held, are the results of that discussion made public? Are other departments contacted about placement results or the ongoing success and problems of the calculus program? How do other departments feel about our curriculum? And how can we possibly operate without this information? If we are to allow ourselves the room to create formative change, we must at least be public about our findings.

Where are we going? A parable of assessment at work

In the above suggestions— (1) elevate the goals, (2) get hold of the data, and (3) open up to the public— we are merely raising the standards for the assessment cycle set out earlier in this paper**.** We have tried to indicate some trouble spots for departments. To make the case that assessment offers *solutions* to problems rather than just additional expense of effort, time, and resources, is a definite challenge. But let's take a journey to Whatsamatta U and like Scrooge's ghost of the future, view the university in the year 2010.

1.) At the U in 2000, the dean has cut back staffing in mathematics—monies will go to aviation (which has commercial ties to the local airport) After all, mathematics has begun to teach lower level developmental courses, which threaten the college's reputation. The department has felt the cuts painfully, as higher-level courses can no longer be justified with low enrollments. The chair has produced data regarding the positions owed in terms of retirements and the overcrowding in classrooms, but the dean is unmoved.

In 2010 the faculty have regularly been holding focus groups which meet with the client disciplines, and the air has been filled with passage of information to and from these disciplines in terms of placement, advising, performance. The sciences have endorsed the work of the mathematics department, and have expressed their collective feelings in a survey about the potential loss of students to them as well, documented in the shared data. When surveyed, students are found to be disappointed that the university has not been true to its advertised promise of small classes. A faculty-monitored web page which routinely airs student opinions reveals that mathematics majors are unhappy about the lack of upper-level courses—their comments come to the attention of the school newspaper, in an article, "Can this Major Hold?" The dean decides another faculty member may be added.

2.) At the U in 2000, members of the retention/promotion/tenure committee are concerned with a junior faculty member, Prof. X. His research is promising but his teaching reveals a lower than average score on the question, "How do you rate the instructor?" The committee have reasons to believe the teacher does a

[1] Philip Keith's term in [2]

good job from well-attended office hours and conversations in the halls. They would like Prof. X to become the vigorous and productive member of the department they feel he could be. But they are unclear what to do, and in their confusion, conclude in their report that Prof. X does excellent research but should work on getting higher ratings next assessment period.

In the year 2010, the committee has suggested well in advance that student evaluations be used in conjunction with other parts of an instructor portfolio. With the mentors available to new faculty, Prof. X has explored his teaching with a variety of instruments. Peer reviews show excellent lecture/demonstration skills and challenging classes, and results of weekly plus-minus surveys to students submitted in the portfolio indicate that students feel they are coming to a better understanding of sequences and are experiencing better problem solving skills and more confidence, although solids of revolution are still troublesome. Prof. X has a somewhat high drop rate, and data on test scores together with examples of graded tests (high, medium, low) and Prof. X's own written comments indicate that by the departmental standards which are available, he is probably grading on the "hard" side. He submits a report that he will be using other methods in future for evaluating learning such as projects, which he has co-designed with an instructor in environmental science. The committee reports that they are encouraged by the readiness of this faculty member to "fit in" with the department, with his promise, and with continued mentoring, they hope that he resolves his issues with the evaluation form.

3.) At the U in 2000, a non-traditional Nigerian student with a college-recognized learning disability is 0.01 points away from being accepted into a Computer Science major. While she has performed at a "B" level in Multivariable Calculus (in part because of the hard work she invested in team projects with software), she has been retaking Calculus I (at taxpayer expense) to improve her grade. To her dismay, her grade for the second time will be a "C"—her overall score, based on 4 exams, is 3 points away from the "B" she desires. She is dejected in part because the extra time she is guaranteed on tests had been denied when the proctor went to lunch. She decides either to drop out of college or to repeat Calculus II.

In the year 2010, the faculty member is aware from departmental goals that tests alone are not considered sufficient evidence of learning, but he has been so busy…! He decides now, on talking with the student and consulting the portfolio of her work (tests, mainly, unfortunately) that there were a few basic skills in which the student showed up poorly throughout the course. He will give the student a take-home test of difficult problems based on these deficiencies and regrade her on that basis. Meanwhile the student is cautioned about the difference between test and project performance, and she is advised that the Student Counseling Center can help improve study skills. Furthermore, tutors are available not only for help with the subject but for consultation about taking tests, etc. It is recommended that the student herself become a volunteer tutor in hopes that by explaining mathematics to others she will become a better communicator—besides, she might come to a sense of her value in the world as a teacher and she might inherit a better sense of "community" while she herself reviews calculus.

The case studies at Whatsamatta U in year 2000 are only too close to this author's experiences—hopefully the predictions will be true, too. But these examples do illustrate how assessment can form a triangulation at the departmental, faculty and student levels, which taken together create a positive assessment environment. The environment removes some of the privacy with which departments and faculty have been allowed to operate—it requires that reporting be open and continuous and feedback given continuously—but the payoff is that, within such a climate, everyone should feel involved in mapping out changes, and confident that there are no secrets or surprises to be sprung. Thus earlier-improvement is substituted for backwards-blame, and problems are turned into questions to be solved. Assessment becomes a natural part of learning and learning a natural part of assessment.

What we have described in the case studies above are techniques of assessment drawn from *Assessment Practices in Undergraduate Mathematics*, by Gold, Keith, and Marion, 1999 [2]. This book, full of examples of assessment at work, cautions that there are no "off-the-shelf," universal methods, and that good assessment techniques are always evolving; nevertheless it supplies enough details to inform the curious about methods in the grid below.

	Student	Instructor	Department
Student evaluates	Classroom Assessment Techniques [1], surveys, homework summaries, projects, writing assignments, autobiographies, portfolios, journals, post-test self-evaluations, student-created goals and reviews	Teacher evaluation forms, quick surveys (e.g. fold a paper vertically and write in the columns the positives and negatives of your learning), writing assignments, journals	Web page, newsletters, exit interviews, alumni surveys, enrollment data, focus groups, general surveys
Instructor evaluates	Tests, homework, written work with revision, ,CATS, surveys, projects with technology, board work, group quizzes, etc.	Portfolio, comparisons, focus groups, committees, peer reviews, mentoring	Committees, focus group, data reports, surveys to client disciplines, team teaching, forums
Department evaluates	Placement, data, comprehensive exams, capstone courses, advising, mentoring, counseling, tutoring, appropriate goals, newsletters, welcoming pamphlets	Data, committees, clear goals and expectations, mentoring, counseling, shared information, and the usual (research, grants, etc.)	Data, committees, focus groups, surveys with client disciplines, external reviews, comparison with other institutions, national organizational data, literature from other organizations

With whom does the responsibility for assessment lie?

Students: Students in general have not been educated in conceptualizing assessment strategies. Yet they do eventually understand when they have learned something, and it is clear that they do not want to waste their time and money. When they can understand they are learning, they are generally appreciative. With the current focus on teacher preparation in so many colleges, the need arises for ***students themselves to learn within their classes good assessment practices***. Thus, departments must be able not only to set an example of accountability, but must teach students how to use assessment as a curricular goal. It is heartening to think that when the day comes that students arrive familiar with assessment tools, they will be able to provide even more dramatic documentation of their learning. On the other hand, students who have emerged from top-down assessment situations which dominantly evaluate the teacher can emerge from that environment severely damaged, finding fault with all aspects of the instructor, expecting explicit directions at all turns, and generally unable to take initiative or to find value for themselves in exploration and experiementation.

Faculty: Faculty may feel that—operating as we do in a worldwide arena of lecture-and-test—we have nothing to learn from exploratory methods of assessment which are time-consuming. Ironically, many instructors who object to formative measures are actually assigning the very essays/projects/presentations that would constitute valid assessment practices if interpretation and explanation were provided as to how these assignments improve learning. In mathematics, there seems to be a gap between doing the right thing and being able to discuss *why* it is the right thing. Indeed, being articulate about a record of thoughtful ongoing assessment strategies can be a teacher's best documentation of teaching effectiveness, and the best defense when things go wrong. There is a flip side: faculty who may be avid subscribers to assessment methods such as CATS [1], might do well to consider assessment methods within the broader framework of classroom research, creating data studies from these methods. Across campus, there is now interest in classroom assessment as research but these ideas have not as yet easily generalized within the mathematics community.

Departments: We know that teaching effectiveness cannot be separated from student learning. But how do we measure student learning? Is it to be judged by "student performance" on homework and tests? Are tests the ultimate affirmation of learning? There is substantial literature exploring this theme; for example, "learning theorists" frequently distinguish between performance and understanding. Meanwhile, CATS, portfolios, projects, writing assignments, cooperative problem solving are alternative ways of measuring and studying how students learn. But do these alternative methods for measuring student learning meet with acknowledgement from *your* department or institution? Furthermore, how is the departmental shelter under which you are operating holding up under the onslaught of new responsibilities and expectations?

The success of the instructor and the student depends strongly on whether the department sets suitable goals, makes clear its expectations of teachers, offers mentoring, places students appropriately, invests in a curriculum which is suitable to the needs and abilities of the students, provides the necessary support for students as well as teachers, and documents its own activities and assesses the outcomes of its assessment program. While departments generally fight the need to assess, here are a few techniques to make assessment non-disruptive and manageable:

- Start simply, using information you already have. Review enrollment figures, or the average of the previous five years; call it assessment and report it.
- Start looking for good news that you want to report on, like scores of graduates who took the GRE. In the following year, plan to upgrade the data with better surveys. What is important is continuity of involvement and the usefulness of the process.
- An assessment might be a small-scale project by a new faculty member or even a graduate student. Or it might be to review a problem or substantive issue in a course cluster—writing assignments, use of computer, etc.
- Student learning data are the gold coin of the realm. Track test scores in course sequences and portfolios of representative capstone experiences.
- The primary audience for departmental assessment should be the department, and assessment should provide confidence that programs are doing things they are intended to do.

Administrators: With the lack of consensus or clarity concerning assessment of a more formative nature, it is difficult for faculty to feel confidence in these methods, and old methods, such as student evaluation forms, can be inhibiting: a user of Classroom Assessment Techniques [1] should not have to ask a class the question, "*Were the lectures organized?*" The more temperate assessment climate requires the understanding of cooperative administrators. While faculty need to approach the problem of assessment with the same seriousness they expect to use in their *research*, administrators need to give faculty the same amount of freedom and respect they do in *supporting* faculty research.

As we have explained, administrators often have a tendency to view assessment solely in conventional summative terms: as data on test scores, job and graduate school placement, earnings, grants received. This misunderstanding accentuates "ratings-type" summative data and downplays the value of formative data as evidence of program effectiveness and improvement. Just as statistics can make "damn liars" of us all, summative assessment can be dangerous and can adversely skew an institution if it creates the wrong emphases or feels like law enforcement. Too few administrators act on the opportunity these methods provide for interaction among faculty for program development. Record keeping on computer permits tracking of students by means of grades and surveys and later performance in the service disciplines. Imagine then, data from a department's calculus program published on a web page—the dean makes a note of numbers to use in his/her biweekly report at the academic council, and then reports to the chamber of commerce on what is currently going on in the college.

Conclusion

It is clear that assessment has created an aura over campuses, and the next decade will allow us to observe in what directions it has transformed mathematics programs. Rather than get tangled in a reflex mode of frustration with assessment, we must focus on what use we can make of the process of assessing within the limited time and resources we have. As yet we are not being asked to do useless things. However, if departments do not learn to self-assess on a regular basis, we will easily become targets for disabled data simply because we have yielded up the front court. If our perception is that assessment is trivial, surely the legislatures and other overseeing agencies will make decisions based on their own externally imposed surveys or tests. Entering the next ten years, we may well find ourselves forced to compete with alternative education systems, distance learning and virtual universities. With directives coming from the sheltering organization of the MAA, my guess is that colleges will be free to experiment in enlightened ways. If we don't, we will find ourselves working against the odds in the economic and political storms to which we have become vulnerable.

1. "CATS," standing for "Classroom Assessment Techniques" is the term of Angelo and Cross—in Angelo, T.A. and Cross, K.P. *Classroom Assessment Techniques: A Handbook for College Teachers*, 2nd Ed. Josses-Bass, San Francisco, 1993.
2. Gold, Keith, Marion, eds., *Assessment Practices in Undergraduate Mathematics*, MAA Notes, 1999.
3. Leitzel, et. al., "Assessment of Student Learning for Improving the Undergraduate Major in Mathematics," prepared by the MAA Subcommittee on Assessment, CUPM (Bernard Madison, chair), *FOCUS*, 1995.

The Mathematics Major at the Start of the New Century

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Introduction

Against the backdrop of dramatic progress in the mathematical sciences over the last 40 years, several signs concern those members of the mathematical community interested in mathematics education:

- recent drops in the number of students taking advanced courses in mathematics departments
- concerns that students interested in mathematics will not choose to attend small colleges (or will transfer out of them) because of a marginally viable mathematics major
- highly publicized factionalized disputes about how mathematics should be taught in grades K–12
- increasing shortages of mathematics majors who want to teach in secondary schools (especially in large urban areas)
- relatively low numbers of students educated in America who complete a doctorate in mathematics
- concerns that students with a serious interest in mathematics are choosing to major in computer science because they feel that their interests in mathematics can be pursued via this route and that expertise in computer science will bring a more lucrative career
- ongoing concerns about the small number of minority and women students who major in mathematics and complete doctoral degrees in mathematics
- ongoing lack of understanding concerning the importance of mathematics and its applications by the general public.

What follows are some personal thoughts about how individuals and the mathematics community in general can make positive changes that affect these issues.

Perhaps the biggest area of concern of the mathematics community is the general withering of Departments of Mathematical Science serving undergraduates. This withering is taking the form of fewer students completing the major, of fewer strong students (and students with well-developed mathematical skills) choosing mathematics as a major, of canceling of advanced courses due to insufficient enrollment, and of the need to offer advanced classes less often due to decreased demand.

There appear to be three major causes for this phenomenon:

1. Students with interest in mathematical science feel that they can achieve their goals by majoring in a computer-related program rather than a mathematics program.
2. Public universities are being starved for funds by elected officials who find saving costs by cutting expenditures for public higher education a politically attractive position.
3. Students entering our colleges are unaware of the full range of career opportunities open to those with an undergraduate degree in the mathematical sciences.

Some of these phenomena are not subject to direct control by members of the mathematics community. Nonetheless it seems as if it should be possible to do much more to promote interest in undergraduate mathematical sciences programs and improve the ability of these programs to serve a changing audience. If nothing else the public airing of contentious debates within the mathematics community seems not to be productive.

Self-destructive activities include:

1. Continued backbiting concerning the pros and cons of calculus reform and the role of discrete mathematics in the mathematical sciences curriculum.
2. Highly contentious public debates about new pre-college mathematics curricula.
3. New chapters in the pure/applied mathematics wars including fractious debates about “open-ended problems” and “mathematical modeling.”

The reason such activities are very dangerous is that for a public not well disposed towards mathematics in the first place, these public fights confuse parents about how to interact with teachers involved with their children’s mathematics education and encourage parents to mindlessly fight changes in the mathematics curriculum even when these changes are for the better. These fights confuse undergraduate mathematics majors who may “bolt” to computer science, business, or some other major.

Background Perspective

What has really changed for higher education from 1960 to the present?

Clearly the most dramatic change has been the growth in the number of students who attend college. Have Americans become so much smarter that all these students can take the same advantage of higher educational opportunity as the (de facto) elite who were educated in 1960? Have colleges changed to meet the challenge of our changed audience? A second dramatic change is the decline in jobs that are available for students with no more than a high school diploma. While in 1960 there were many jobs for telephone operators, for receptionists, for low level secretarial jobs, and for jobs in heavy industry, the number of jobs available now to people with no more than a high school education has declined dramatically. Furthermore, many employers who have jobs that in principle require no more than a high school diploma prefer college graduates when they can get them.

New technology has resulted in the disappearance of many positions that formerly required only a high school degree. Within the last year, by way of an example, phone companies have put on line new voice recognition software that automates the process of finding phone numbers. Only tricky cases are forwarded to a human being for attention.

Professions that traditionally required only a high school degree, such as a policeman or fireman, now require some college experience. Agents to sell tickets for concerts and transportation services (token sellers) are rapidly becoming obsolete due to the development of automated fare systems for transportation and web-based technologies for ordering movie, concert and show tickets. At my local supermarket a recent remodeling converted a sizable number of lanes that had had human checkout clerks to an automated system where the shopper checks himself/herself out. Self-service gasoline stations have sharply reduced the number of jobs for people to pump gas. ATM machines have diminished the number of jobs for tellers in banks. Changes of these kinds are likely to accelerate and continue to erode the jobs available for people with only a high school education.

In essence, leading a middle-class life in the America of today requires a college education. Society in effect has asked the higher education community to provide a college education for larger and larger groups of students. Instead of educating students who were in the top 20 percent of a “general” ability curve, colleges are now being asked to educate perhaps the top 60 percent of students. While high profile schools (e.g., Ivy League, MIT) can easily get very strong students, community colleges and four-year colleges have expanded their student bodies and in many cases with remarkably little concession to the fact that these students are not “elite.” All too often the response of colleges to weaker students is to remediate them rather than assist them in finding ways to put their talents to advantage. Another complication is that after a relatively long period of not overly high levels of immigration America is again experiencing a large wave of immigration. Many new immigrants seek higher education so that they can live the "American dream." However, for immigrants beyond the childhood years whose first language is not English, their problems in reading and writing English serve as an obstacle to maximizing their higher education experience.

As America has become a richer country in the past 30 years, the gap between the rich and the poor has grown. This has meant that more middle class individuals have been able to afford to send their children to private colleges; public universities have needed to serve economically less well off people and students who are recent immigrants or the children of recent immigrants. In some states politicians have decided to cut the budgets of universities that serve these less politically influential clienteles.

One consequence of these changes is that mathematics teachers are seeing a growing diversity in ability levels in the students we teach. This new reality will require that we adopt teaching methods that allow all these students to have the most fulfilling mathematical experience possible.

Mathematics and Computer Science

Starting in the mid nineteen seventies colleges and universities began to offer graduate degrees in computer science and undergraduate schools started to offer degrees in computer science outside of mathematics departments, where the courses were being taught by scholars with degrees in computer science (and initially mathematics and electrical engineering). For a variety of reasons computer science programs which were originally housed in mathematics departments broke off and formed their own departments. (In some cases these splits arose due to the way faculty with interests in computer science not directly connected to mathematics were treated.)

Many of the leaders of the mathematics community were educated before the discipline of computer science emerged, either as an undergraduate major or as an area of graduate study. Before computer science was born there was no section of newspaper want ads that listed jobs for mathematicians. Those who majored in mathematics as undergraduates and took that degree as a terminal degree could not look in help-wanted ads under mathematics and find job offerings. Yet it was true that mathematics majors were often in demand. The kinds of companies who were looking for mathematics majors included engineering firms, defense contractors, banks, and insurance companies. However, the world has changed a lot in 40 years! During the first flush of demand for computer specialists in both academia and the business and industrial world, computer specialists were paid a premium for their capabilities. Although there have been brief periods of diminished demand in computer science, job opportunities in the information sciences are currently very robust and are likely to stay that way for some time.

Computer science has become a very broad subject. For many parts of computer science one does not require high levels of mathematical ability or accomplishment. This means that students with strong mathematical abilities have a choice in selecting between mathematics and computer science as a major. However, mathematics departments all too often do not call to students' attention that one can major in mathematics with the goal of pursuing a career in computer science. This is in part because, historically, mathematics college faculty have all too often taken an interest primarily in students who plan to seek higher degrees, especially the doctoral degree. I believe a good source of additional majors would be students interested in computer science with high levels of mathematics ability even though such students might not seek advanced degrees in the short run. It is imperative that students who want to major in mathematics be sought out and given across-the-board support whether or not they have plans or the ability to seek higher degrees.

Keeping Our Students Informed

Mathematics and Computer Science are both very large subjects. They overlap in many areas. The public perception is that there are many jobs for computer trained specialists. Jobs that require the creation of a web page and maintenance of a web page do not require much in the way of mathematical skills. It is entirely possible that a mathematics major or a computer science major could do the work, though it is possible that people who major in either computer science or mathematics would not find such a job of interest no matter how high the pay or how secure the work.

By contrast, the job of designing and setting up a data base or the development of software to support the human genome project might be a job that could be done equally well by a mathematics major or a computer science major. The point is that in many cases mathematics majors would be at least as attractive as employees for many jobs, provided that the mathematics major had some skills in computer science. Yet, at many colleges little is being done to inform students that mathematics is not only an alternative for them to majoring in computer science but a superior alternative for many students.

Why do mathematics departments not make this information available to their students? In some cases the departments are the same so they do not see that there is a difference. In other cases they are ignorant of the facts and the consequences of inaction.

Principles of Mathematics

The development of the digital computer has had a dramatic effect on the content of mathematics. While Calculus continues to be a major tool of applications and leads to many theoretical results, vast new parts of mathematics which do not directly involve Calculus have emerged in the last 40 years. These areas are not only important for computer science but have also found a myriad of applications in areas ranging from biology to business. Currently, students majoring in mathematics or one of the sciences or engineering begin their studies with Calculus, and often continue with Calculus for 2 or 3 semesters before they see other parts of mathematics such as linear algebra, combinatorics, probability theory, discrete mathematics, etc. Given the growing diversity of students in goals and ability levels, this is most unfortunate because it does not allow students to see the full range of mathematics that they might find interesting or for which they have talent. This problem could be redressed by the teaching of a Principles of Mathematics course rather than the Calculus to beginning students.

Data suggests that starting with the first semester of calculus in college, there is a rapid decline in students who enroll in the next course in the sequence. It is unclear if a Principles of Mathematics course would have a different attrition level, but even if the attrition level were to remain as high, students would come away from a semester or year of a Principles of Mathematics course with a much richer tool kit of mathematical ideas than is currently the case.

Adopting a Principles of Mathematics course as a beginning course in mathematics has a host of problems, mostly having to do with coordination problems for students who wish to major in chemistry, physics or engineering. I do not believe these challenges are insurmountable; majors in the sciences and engineering would also benefit from students seeing ideas from probability, linear algebra, abstract algebra, and combinatorics much earlier in their mathematical careers than is currently true.

Courses for Client Disciplines

Computer science departments that split off from mathematics departments soon realized that calculus was less important to computer science majors than to mathematics majors. They clearly saw that linear algebra (which can be pursued with little background in calculus), probability theory, and the emerging subject of discrete mathematics served the interests of their students better than traditional calculus. Mathematics departments responded by creating "service" discrete mathematics courses for students who wanted to become computer science majors, but often forbade students from taking these courses to fulfill credit for the mathematics major. Mathematicians who used calculus in their work sometimes became defensive about allowing students who majored in mathematics to take a route through the mathematics major that did not include 3 semesters of calculus, differential equations, and advanced calculus/real analysis. In recent times some computer science departments, finding mathematics departments unresponsive to their needs in terms of the topics treated in discrete mathematics courses taught in mathematics departments, are moving in the direction of teaching these courses within their own departments.

Regrettably, mathematics departments have a long history of being asked to teach "applied" service courses (e.g., business calculus, engineering calculus, linear programming, statistics, computer science, and operations research, etc.) which have been taught without ongoing interactions with the departments being served by teaching these courses. When the content being taught by the mathematics departments becomes too tangential to the needs of these departments, they are now teaching their own versions of these courses for their own majors. This pattern has been followed concerning a spectrum of courses (e.g., game theory (initially taught by mathematics departments, now increasingly taught in economics departments, business departments, or political science departments), operations research (now often taught in business and economics departments), etc.) This behavior is little short of suicidal. What we should be doing is working closely with departments that need courses taught with significant mathematics content and having these courses taught by people in mathematics departments. Another alternative is to have the course listed within the mathematics department but request the appropriate department (business, economics, etc.) to have an interested member of its faculty teach the course. Not only does this give us more courses to teach but it may also encourage students to take other mathematics courses or even to major in mathematics.

Service Courses

Many departments (e.g., psychology, business) which do not require lots of mathematics for their students have taken to requiring specific courses as part of their majors that must be taken in the mathematics department. Rather than giving their students a choice within the framework of a general education requirement for mathematics, these departments require a specific choice of course within the mathematics courses available as part of a general education requirement. Examples of such courses are the statistics course often required of social science and nursing students and the finite mathematics course required for business and accounting majors. Some departments offer more choice but not a full range of choice. For example, an economics or geology department might allow statistics or calculus as a choice for their students. In the context being discussed here discrete mathematics is a service course for computer science degree programs where it is required, though discrete mathematics rarely can be used to meet a general education course requirement. Another issue with service courses is that occasionally they are used by other departments who wish to filter their students. For example, in some cases disciplines such as medical technology have required calculus of their students despite there being only a tangential need for this particular course by their students.

Mathematics departments often do not pay as much attention to these courses as they deserve. Not only is there no reason why people who take a finite mathematics course because they are business students might not become an ambassador for mathematics, but also students who have a good experience in such courses may return to take other courses (even advanced courses) or choose to change their major to mathematics.

Involvement with Our Colleagues in other Departments

It is imperative that we increase consultation with members of other departments concerning examples of the use of mathematics that should be covered in the courses being taught as required courses for these majors external to mathematics, so that these departments will be pleased with the courses that are being taught to serve their students and feel some involvement. Ideally, every mathematics department should have a mathematics club that invites speakers from departments outside of mathematics to give talks about how they use mathematics in their work. Doing this will broaden students horizons and at the same time build badly needed bridges to other departments.

Prerequisites

Mathematics by its very nature makes it very tempting to organize its development in a very hierarchical way. However, in these days of falling enrollments and declining numbers of majors, it is not in our best interests. For example, rather than set calculus prerequisites for such courses as linear algebra, graph theory, combinatorics, or geometry, as a way to "guarantee" mathematical maturity, we should allow interested students to take these advanced courses and make them as self-contained as possible. Often we might attract a few extra students to take our courses and the price to be paid is merely presenting some brief bits of material which the student might not have seen had another course been listed as a prerequisite.

Liberal Arts (General Education) Mathematics Courses

Although historically mathematics departments have paid little attention to the courses that were not service courses for science, computer science, and engineering students, they should do so now. The same is true for courses required of liberal arts students (LA students) taking general education requirement mathematics courses. First, some LA students may choose to change their major to be more overtly involved with mathematics. For example, they might choose to direct an interest in teaching towards teaching mathematics in high school or junior high school. Second, LA students will become parents and the attitudes they have towards mathematics and the extent of their knowledge about the applications of mathematics will affect the values and attitudes they convey to their children about mathematics. Third, some LA students will become the political leaders, entrepreneurs, and decision makers of the future. Funding for mathematics research, mathematics education, etc., will depend on the perception of these individuals about the importance and value of mathematics. Courses that "turn off" LA students poorly serve the mathematics community.

Historically, many courses for LA students have been designed to correct the fact that students arrive in college lacking "working ability" with skills traditionally taught in high school. Such courses are truly ill advised. If students lack such skills on their arrival they already have usually selected future courses of study which do not heavily use mathematics. Furthermore, trying once again to develop working knowledge of skills students have been exposed to in the past with little success is unlikely to succeed at the later time. In effect, students who take this type of course are often happy with any passing grade and do not truly try to come to grips with what they did not learn in the past. Rather, such courses entrench negative attitudes. Very recently there has been much debate in the mathematics community concerning what is called quantitative literacy. Sometimes this has been used to suggest that college graduates should have a "minimum" tool kit of mathematical skills when they graduate college. To the extent that quantitative literacy involves teaching technical algebraic and trigonometrical skills that students were exposed to in high school and for which they did not develop working skills in these techniques, it seems unwise to require students to master these skills unless they choose majors for which such skills are specifically required. Certainly, students should be exposed to mathematics in college, regardless of their majors; however, such courses should be more tailored to the needs of the student than has been true in the past.

Teaching mathematics courses for LA students which mention and/or demonstrate the importance of mathematics for the development of new technologies (e.g., genome research, wireless communication) as well as the role that mathematics has in supporting the sciences is valuable. Such courses contrast with those designed to build skills that students lacked when they were admitted to college. There is no reason why students who take required mathematics courses in college should not emerge from such courses as ambassadors for mathematics rather than mathematics phobic!

It is imperative that we mention recent ways that mathematics has been used to develop new technologies in all of our mathematics classes. New technologies are not merely a gift of physics and engineering to society but a gift from mathematics as well. Furthermore, in cases where we can be explicit in showing how things they are learning in their courses assist with these recent and emerging technologies, this should be done. For example, if we are teaching linear algebra, we should be mentioning the role of linear algebra in linear programming, structural engineering, and so on. If a faculty member truly feels he/she does not have time to discuss applications in class because there is so much theory to get to, how about having a writing-across-the-curriculum assignment that requires students to learn about applications of linear algebra and write a brief paper about what they have learned? Such issues should be addressed in every course we teach! Here are some other examples:

Number theory: when discussing congruence arithmetic, take some time to show students how error check systems based on modular arithmetic is used in the Universal Product Code, for ISBN numbers, etc.

Geometry: Show how finite planes can be used to construct error correction codes, how convexity ideas show up in computer vision and robotics, etc.

Abstract algebra: Show how symmetry issues arise in art, architecture, and engineering; how codes based on polynomial rings are being used for a wide variety of applications, etc.

Thematic Approaches to Mathematics

Historically, mathematics has been taught by teaching mathematical tools and, if the student is lucky, applications outside of mathematics to which these tools can be put. This approach to mathematics tends to emphasize the special role that symbol manipulation plays in mathematics. While this approach is certainly necessary and warranted for majors (and to a lesser extent client discipline courses), it is not the only alternative for students' LA courses and courses for teachers at various levels. An alternative approach is to view mathematics as being able to provide insight into a variety of "themes" and show how this insight can be accomplished. Shown below is a list of the techniques mathematics has historically emphasized and an alternative packaging of content based on themes that I view as being useful in all mathematics courses, even though driven primarily by technique considerations.

Techniques:
0. Arithmetic; 1. Geometry; 2. Algebra; 3. Trigonometry; 4. Calculus (Single Variable and Multivariate); 5. Differential Equations; 6. Linear (Matrix) Algebra; 7. Modern Algebra; 8. Probability and Statistics; 9. Real Variables; 10. Complex Variables; 11. Graph Theory; 12. Coding Theory; 13. Knot Theory; 14. Partial Differential Equations

(Many more!)

Themes:
1. Optimization; 2. Growth and Change; 3. Information; 4. Fairness and Equity; 5. Risk; 6. Shape and Space; 7. Pattern and Symmetry; 8. Order and Disorder; 9. Reconstruction (from partial information); 10. Conflict and Cooperation; 11. Unintuitive behavior

One advantage of a themes approach to the teaching of mathematics is the very natural way that treating applications of mathematics grows out of this approach.

Geometry

Although the public identifies mathematics strongly with symbol manipulation, the public is probably unaware that the best symbolic manipulators are not human mathematicians but computer programs such as *Maple* and *Mathematica.* This fact should also serve to provoke thought among mathematicians. Computers have proved much less valuable in the mechanics of geometry than in algebra. Not that geometric applications of computing (morphing and computer graphics) are not transforming society in many ways, but it is also true that computers can not be given an image and then print out a simple description of what the computer "sees." This suggests that there is much undone work in the area of geometry that is yet to be mined. Since geometry has the charm of being quick starting (compared with Calculus and algebra) and has an increasingly growing list of applications (computer vision, robotics, communications technology, etc.), it deserves more attention at all levels of what we teach in our mathematics courses.

Primary School Teaching

Many states have been doing away with the practice of permitting students who wish to teach in primary schools to major in "education." States are often now insisting that students major in a subject area and as part of such major take the courses that are required for "state certification." In the past, future teachers have taken a mathematics course within the education department as part of their major. In light of the changes going on, mathematics departments are being approached in many cases to mount appropriate courses within the mathematics department which would be required of majors in other disciplines to complete their major.

It is strongly urged in such situations that mathematicians adopt a forward-looking approach to curriculum. Historically, the content of primary schools has concentrated on rote treatments of arithmetic and little more in the way of geometry than being able to name a variety of shapes. But why shouldn't future primary school teachers see material about tilings, polyhedra, make a Moebius band, study symmetry patterns on a strip and the plane, etc., even though these topics are not traditional? Might not 4th graders learn to play sprouts along with Nintendo?

Last year my 3rd grade son came home with "math minutes." These are systematic problems that involve addition and multiplication problems and there are more of them than "super-mathman" can do in a minute. The object is to time oneself to see how many problems one can complete in one minute. I wonder what lesson this teaches youngsters about the value, importance, and nature of mathematical work. My son tells me that he has to do math minutes in school as well as at home. Might the time spent doing math minutes be equally valuably spent doing something else mathematical? This can only be done if the courses we offer primary school teachers lay the foundation for something beyond "mathematics minutes." We should strive to turn future teachers into ambassadors for mathematics. In particular, such teachers should be well versed in the role that mathematics plays in the development of so many modern conveniences.

Secondary School Teaching

America's secondary schools, especially those in large urban areas, are approaching a crisis state in being able to fill positions for secondary school mathematics teachers. Although one of the prime reasons for this is the depressed salaries and respect that secondary school teachers receive in American society, there are, none the less, actions the mathematics community can take to help with this situation. First, those of us who teach can set an example by taking teaching seriously and by talking about the satisfactions that being a teacher gives. Second, we can call to our students' attention the growing opportunities in teaching in secondary schools. Our teaching makes a difference and our example can lead to a situation where secondary school students in the future will have a cadre of well-trained and enthusiastic teachers. Third, we should not regard mathematics majors who plan to teach in high school as "lesser vessels." Like students planning to get doctorates in mathematics or who plan careers in the actuarial sciences, they are students with a special set of needs. We should aim to help them fulfill their goals in the best ways we can. Certainly our goal for these secondary school teachers should be to turn them into ambassadors for mathematics. Fourth, though breadth and depth are important for all mathematics majors, it seems especially important for those planning to teach in high school that they see as wide a range of mathematical ideas as possible, so that they can make the students they will teach in the future aware of all that modern mathematics is concerned with. In particular, it is especially important that these future teachers learn about the many applications mathematics has in modern society and in emerging technologies.

Conclusion

We live at a time in the college and university communities where a number of academic disciplines (examples include philosophy and classics) have been marginalized. I believe that unless the mathematics community takes appropriate actions, mathematics in college could also become marginalized. This would be a sad occurrence for mathematics and America. I believe that decisions our community makes in the next few years will govern the outcome. We should act wisely.

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Business View on Math in 2010 C.E.

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What should an undergraduate student do who is interested in mathematics but who is also interested in seeking a career outside the teaching profession? My counsel to young people has always been: Choose what you love most to do. This makes sense not only

- to undergraduate mathematics majors whom I may meet as summer interns or coop students or whom I may meet at the annual meetings of the Illinois Section of Mathematical Association of America (ISMAA), but also
- to students enrolled in mathematically-intensive majors, to those interested (in love with?)
- life sciences (such as microbiology, ecology, health, medicine),
- physical sciences (such as astronomy, chemistry, physics), or
- the business world (such as banking, computing, data analysis, engineering, finance, insurance, and many others).

This essay is intended to help stimulate mathematics faculty to make curricular decisions on the undergraduate program in mathematics for the benefit of young people who are trying to figure out what they want to do for a living and for whom mathematics is a key success factor. The following six (6) questions will be addressed from the perspective of a thirty–year career in business, focusing on the mathematical issues involved.

- What does it mean to be prepared for a career in business?
- What nonmathematical skills are expected?
- What mathematical skills are expected?
- What is the importance of proving theorems?
- What career opportunities may be expected?
- What are the implications for the curriculum?

Before addressing these six questions, a little information about my educational background and professional career may orient the reader to my point of view and lend some credence to the material presented. The point of view of the subsequent discussion is, what contributes to a successful career in business. The recommendations offered for the undergraduate curriculum in mathematics are intended to help students become attractive as job applicants to potential employers in business.

Autobiographical data

While completing my graduate studies at the Illinois Institute of Technology in Chicago, I held a position as Instructor of Mathematics at North Park College, also in Chicago. To help make financial ends meet, I began what was to become my primary vocation by working in the Academic Computer Center at the University of Illinois, Chicago Circle, on a part–time basis, as a scientific programmer, developing mathematical software. Upon finishing my doctoral program at I.I.T, I obtained a position at G.D.Searle & Co. in the corporate systems and data processing department. I have been an employee of Searle and its lineal business descendents ever since.

At G.D.Searle & Co., my career began with an initial apprenticeship involving commercial applications and writing programs in COBOL–IDS (an extension of COBOL for a network database system, a precursor to SQL). My first decade at Searle consisted primarily in supporting biologists, chemists, MDs, and statisticians. The second decade brought major changes in customer base and in the whole company. The Searle family hired Don Rumsfeldt, former U.S. Secretary of Defense, to serve as President and lead the company through its transition from a family-owned business to public operation. The philosophy of a centralized management of computer systems was replaced with a decentralized model. This phase ended with the purchase of Searle by Monsanto. During this period, my primary focus shifted from statistical software support to IBM system software, and back again. For the last decade and a half, scientists and their research have been my primary focus. Initially, I held various positions in the statistics department writing software used directly by statisticians or building application systems intended for use by their clients. About halfway through this fifteen year period the management of all scientific programming for Searle Research and Development was centralized in the R&D computer department, in which I've been ever since.

Along the way, I found that staying active mathematically helped me in my day-to-day activities at work. About three years after Searle hired me, I joined the American Mathematical Society. At that time I thought, you taught for three years, now you have been programming three years. What are you, a math teacher or a programmer? I decided that I was a mathematician who happened to have a job in a computer department. More recently, I have added, "in a pharmaceutical company." Before coming to Searle, I had joined the MAA when I completed my masters degree program and the Association for Computing Machinery at the time of my work at the University of Illinois.

The notion of staying active mathematically, and how to do so, took hold in a gradual fashion. It was not unlike trial and error. The more mathematically fit I was, the better my performance on the job. It started with extensive serious and recreational mathematical reading plus solving problems in the *Monthly*. The next step was attending the Annual Meetings of the Illinois Section of the MAA.

Thanks to John Schumaker, I began to get involved with service and leadership in the operation of the Illinois Section, first in Membership and Public Relations, then editing *Greater Than Zero*, the ISMAA newsletter, and culminating with service on the ISMAA Board of Directors as President of the Section. Thanks to Len Gillman, I started to attend national meetings, the first being the 1976 meeting in Toronto. Sitting in the audience behind Coxeter, as he quietly said that he wished that what was offered as a proof of the Four Color Conjecture had not been done by computer, drew me to the January meeting in 1977 in St. Louis. I have been coming ever since.

I found that the combination of being exposed to the best in exposition, in print and in the spoken word, coupled with the opportunities for service, was highly stimulating, rewarding, and growth inducing. At the national level, I have served on, and chaired, the AMS Short Course Committee, and the MAA Committee on Mathematicians Outside Academia. I have been a member of the MAA Council on Human Resources and have served as a Governor–At–Large on the MAA Board of Governors representing mathematicians in business, industry, and government (BIG mathematicians). Currently I am serving on the MAA Short Course Committee. I have also been asked to participate in several projects by the United States Department of Education, the Mathematical Sciences Education Board, and in the revision of the Illinois State Learning Standards.

In addition to developing a passion for certain parts of mathematics by virtue of staying in touch with trends in mathematical research, I have found that the committee and projects work have had a somewhat consistent focus. I have been led to think seriously about the communication and use of mathematics in nonacademic contexts. For me, the mathematics curriculum is very important, for several reasons. First, the people I meet at work today are products, or beneficiaries, of the undergraduate curriculum in mathematics.

Second, what will happen in the future will be determined, to a large degree, by the contents of today's undergraduate curriculum. Third, most typical applications of mathematics turn out to be essentially elementary in nature, only requiring undergraduate mathematics. Fourth, the undergraduate curriculum fosters the development of various ways of thinking mathematically. Mathematical modes of thought, or habits of mind, that are extremely effective in applied contexts include the ability to recognize patterns and symmetry, to use induction and recursion, and to reason logically with abstract systems, such as systems of algebraic symbols.

1. Preparation for a business career

What does an employer look at when considering an applicant for a job in business? Clearly, the first things evaluated include the applicant's letter of introduction, resume, and the actual interview. What information is the employer seeking? How does the employer evaluate what has been learned from the applicant during the interview process, during the first six months on the job? The employer is looking for knowledge and the ability to apply it. However, mostly the employer is just trying to answer the question: Would it be a (really) good idea to put this person to work on Monday? After a person has been working in a specific job for a while, the question becomes, is it still really a good idea to keep working with this person to improve their performance?

When evaluating a resume to decide whether or not to extend an invitation for a job interview to an applicant, the first thing I do is look at the extent of formal education and relevant training. My advice to the high school student: Select your college or university with an eye on how the school is perceived by potential employers.

Next, I look at the employment history, with special attention to the performance levels and responsibilities that have been achieved by the applicant. In addition, I consider the nature of the work environments experienced and how they relate to the position in which the applicant would be placed, if hired. A plus would be job experience very similar to the kind of work the applicant would be expected to do right after being hired. My advice to prospective applicants: While a student, get work experience in the same line of business in which you wish to seek employment after graduation. Jobs held during high school years, before college, or while in college, such as part–time jobs, summer internships, or coop programs, all make an applicant much more attractive. It is easier to gauge how a person will fit into a specific work environment if that person has already taken advantage of opportunities to explore similar ones.

Finally, I look at the inventory of skills, such as fluency with programming languages, computer environments, word processors, foreign languages, and relevant business knowledge, that relate directly or indirectly to skills the applicant will need to exercise if hired. Whether I am filling a summer intern or coop position or looking for a person to fill a full-time permanent position, applicants who do not have the potential to be productive within two weeks of being hired are not even considered for an interview. How the applicant built his or her skill base taking courses or gaining work experiences is taken into consideration. The amount of time the applicant devoted to developing skills is also a factor. What also is taken into consideration is evidence that the applicant's existing skills would translate or transfer or that the applicant can acquire needed new skills quickly without expending much time. The advice to the applicant is two–fold. First, deliberately expose yourself to new and varied situations, in part with the intent of acquiring skills that would interest potential employers. Second, get actual, practical field experience in your area of interest. For example, if an applicant wants to become a statistician or actuary, then practical fieldwork with data analysis would be beneficial.

2. Nonmathematical skills

Interpersonal skills are an essential part of any career for continued success and a rewarding experience, none the more so for a career outside academia in the "real" world, than for one in academia. Interpersonal skills, in addition to study, can be developed by on–the–job coaching efforts and by developing the ability to learn from experience, either from personal experience or from the experience of others.

To communicate well is one of the most important skills for people working in business. A keystone skill is the ability to listen well, such as having a good ability to understand verbal communications. For example, most typically, expectations are conveyed to individuals verbally. When the need for clarity is paramount, written communications are always presented to the employee in a forum that allows people to ask questions of the person making the presentation. Even if the presenter does not know the answer to a particular question, the process allows other people to provide additional information that may clear up any prior ambiguity or incompleteness in the initial communication.

In addition to having good listening skills, people need to be able to get their own ideas and viewpoint across to others. Without effective presentation skills, an individual contributor will not be able to communicate to others on the project team, or to management, what has been accomplished or what needs to be done. One piece of advice I find myself giving repeatedly is to remind people that when they present an idea to others they need to keep in mind to whom they are trying to deliver the message. Once a presenter clearly focuses on the intended audience, the details of the presentation fall into place.

Whether or not an individual has good decision–making skills often decides what opportunities they have for autonomy and for exercising control over their own projects and assignments. In most complex situations in business, it is often the case that there is no single, best option. The ability to apply sound judgment between trade–offs, in which the various alternatives are distinguished by distinctly different criteria, is extremely crucial.

Nearly everything worth doing in business takes place in the context of one or more teams. Teamwork skills, such as self–confidence, self–reliance, and a firm understanding of the psychology of group work, are essential. Often teams are constituted with only one person selected from separate disciplines, with each individual relied upon to function as the expert in his or her designated area. Typically, a team would have at most one person with an advanced mathematical background. One mathematician is sufficient! Such an individual needs to work well in a multi–disciplinary context with others whose mathematical background is not so extensive.

In addition to those interpersonal skills just discussed, individuals will frequently draw on other such skills throughout their career, such as leadership skills, flexibility and the ability to respond appropriately to ambiguity, stress, and change. It seems that there is always something new to learn, or at least to improve. For example, I have been surprised to learn over the last year exactly how beneficial acquiring, improving, and employing time management techniques can be. Learning how to handle several competing, overlapping project timelines helps widen the door, giving individuals many more opportunities to get involved in diverse and substantial projects and activities and enabling them to complete them all in a satisfactory fashion. Continuous self–improvement, and staying alive and attuned to what is new in your chosen profession, makes for a very rewarding career in industry.

3. Mathematical skills

In terms of what students can do to prepare for a career in business, what comes first to mind is taking certain courses to learn specific mathematical "facts," such as those offered in courses in differential equations, statistics, and so forth, or gaining application–area expertise, say by taking courses in accounting, biology, chemistry, or data processing. However, mathematical modes of thinking, or habits of mind, are of even broader utility outside academia than knowledge of any individual mathematical "fact" (algorithm, formula, or theory) that may be acquired through coursework. The following discussion describes several ways people think mathematically that are of specific value outside academia to employers in business, industry, and government.

One can always employ a mathematical viewpoint when solving any practical problem. This is so even if the contribution is limited to a simple, complete and logically organized enumeration of all the possible alternatives. Specific mathematical problem-solving techniques, as described in *How to Solve It* by George Pólya, for example, provide an excellent example to an employer of the potential benefit and effectiveness of a mathematical approach to solving real world problems.

Earlier this year I was engaged in coaching a junior staff member whose problem-solving skills were woefully inadequate. It eventually occurred to me that this person would benefit by soaking up the information in *How to Solve It.* In preparation for telling this employee about the book, I went to `www.Amazon.Com` to get all the information the employee would need to order a copy. As I was doing so, I came across the following wonderful example of a nonacademic employer's perception of the benefit of general (mathematical) problem-solving skills. I was amazed and delighted to discover the following anonymous review, dated July 7, 1996, provided by a reader of *How to Solve It*: "Microsoft, for instance, used to and may still give this book to all of its new programmers." After the employee ordered a copy, started to study the material, and began to use the newly acquired mathematical problem-solving skills on the job, it was very gratifying to see an overall improvement in his performance.

An individual who is able to recognize patterns, to generalize and to abstract, is also able to work at a systems level. The capacity to improve, modify, adapt, and extend a business system is greatly aided by the ability to see patterns, to recognize them for what they are, and to understand their essence, significance and relevance.

Being able to generalize is, perhaps, another way of being able to imagine. Too often, opportunities for introducing improvements with important, far–reaching ramifications are overlooked. Unfortunately, this happens even when the individuals involved are familiar with the necessary viewpoint, ideas or facts.

The ability to abstract in a particular problem situation guides direction and focus. Being able to recognize the common aspects of different objects as instances of the same thing permits the development of systemic problem solutions, such as the development of computer software systems. As an example of a fairly abstract object, consider a systematic collection of coordinated activities. In this case, there is no physical object in the ordinary sense. What constitutes the "object" is human perception: a collection of ideas.

In addition to generalization and abstraction, facility with the notions of isomorphism and homomorphism provides further examples of pattern recognition skills. To be able to perceive an underlying algebraic, analytic, geometric, or topological structure, for example, and to see two different situations as either being the same or as standing in the relation of one being embedded in the other, allows one quickly to perceive what is essential and what is not.

Understanding the relationship between various models of a situation and the actual situation itself is a critical element when solving problems with other people. More important than only providing customers answers to mathematically formulated questions, a mathematician can also give customers real insight into what is actually happening in a given situation.

Applying what you know makes all the difference. The crux here is the application of mathematical facts or ways of thinking in diverse situations other than those in which the mathematics was first learned. This is a crucial ability for a mathematician who works in business, industry, or government.

For example, what comes up very frequently when applying notions from calculus and differential equations to specific scientific situations is the need to distinguish between the content, versus the appearance, of the algebraic symbolism employed. In each field of scientific endeavor, specific symbols have been historically employed to represent various physical quantities. In physics the "dot" notation may be used instead of the Leibniz notation dy/dx. In biology, acronyms, such as VOL, CONC, Vmax, will be employed, instead of the traditional mathematical notation x, y, z.

What is interesting is that the formulae appear much larger and more complex when expressed in traditional notation from a scientific field, for example, biology. The biologist cannot understand the point that a mathematician wishes to make if the mathematician uses "mathematical," as opposed to "biological," formalism. However, the biologist will oftentimes not be able to manipulate the formulae expressed in "biological" form (too complex). Even the mathematician may get distracted by the "biological" symbolism, finding it much easier to simplify the notation to reveal the underlying mathematical relationships between the various quantities involved. Nevertheless, after the mathematician has identified what is key, the resulting solution will have to be expressed in "biological" form to make sense to the scientist.

Be careful not to throw out the baby with the bath water. That is, keep traditional subjects in the curriculum. Algebra is key, with its focus on symbols and their manipulation. Analysis supports reasoning precisely with approximations, which is also key in business, where exact data are rare. The list of subjects given in the section on curricular implications, may seem to stem from the undergraduate mathematics curriculum of three or four decades ago, but in reality, the needs of science and engineering, plus the ability to work objectively and quantitatively with scientists and engineers, still keeps them in the running. However, given the time constraints in the undergraduate curriculum, it is clear that choices must be made.

4. Proving theorems

One of the prerequisites for many upper–division undergraduate mathematics courses is "mathematical sophistication." This prerequisite also comes into play in the graduate mathematics curriculum. Mathematical sophistication may be regarded as a certain form of experience, namely the experience of developing a deep, fundamental understanding of mathematical situations. To the employer in business, this amounts to the recognition of the power, utility, and effectiveness of mathematical reasoning. The benefit of acquiring, possessing, and maintaining mathematical reasoning skills to the employee is in being better prepared to participate in and contribute to larger, more complex and interesting projects in a more creative and fulfilling way.

Employers sorely need people who reason well. Employers need people who can use accurately complex (written) information. Employers need people who can learn on their own. Employers need people who can create, and who can follow, complex, abstract, large arguments. Employers need people who can persuade.

Coming up with a proposed solution to a complex problem in business, along with a convincing explanation for adoption, is very much the same kind of intellectual work as proving a theorem. The greatest need is for people who not only can conceive appropriate technical solutions, but also can give convincing explanations to others, that is, who also can *prove* to others that what they claim to be is so. To be sure, knowledge of what the intended audience knows is vital. Nevertheless, knowledge of how to frame appropriate arguments in a convincing and effective manner is also vital.

Writing a proof involves a combination of logical analysis, creative synthesis, and perseverance, coupled with the ability "to know a proof when you see one." The same also applies to reading, or "really" understanding, an argument offered as a proof. For example, one of the more common shortcomings that often undercuts the successful prosecution of large, complex business projects is failing to provide a general, comprehensive approach that covers all the options and potential scenarios. This shortcoming is often described as "lack of planning", but is really reflective of not having a complete identification and logical analysis of all the possibilities in a given situation. It is often better to take risks of which one is aware, than to risk omissions of which one is not aware.

5. Career opportunities

What are the prospects for a career in business in the coming decade? For a person who wants to do pure mathematical research, there are few opportunities outside academic circles, now or in the foreseeable future. On the other hand, for a person who has a strong mathematical background, who likes to solve problems and enjoys using mathematical facts or modes of thought, and who also has strong interests outside of mathematics, the prospects are excellent. A major factor here, however, is the assumption of the continued growth and expansion of the world economy. What is not so clear now is the nature and direction such growth will take.

The phrase "information age" may be trite, but it is also descriptive of where we have been, where we are now, and where we are going. That means that opportunities for working with abstract, electronic forms of information will be ever more prevalent in the business world. Data analysis, software engineering, and scientific issues impacting society, such as environmental issues, the biodiversity crisis, and the very survival of humanity and much else, will be placing ever increasing demands on the educational system for graduates who can work well with quantitative information.

6. Curricular implications

The undergraduate mathematics curriculum should provide the student with opportunities to apply mathematics in real-world situations. Student intern or co–op programs allow individuals to gain relevant experience in the work force while there is still time to make potentially beneficial choices in coursework, before graduating. Another possibility is special modeling or project coursework. For example, Robert Fraga, former of the MAA Board of Governors, initiated an MAA Undergraduate Student Consultancy Program in Wisconsin at Marquette. A curricular design that facilitates student employment would also be helpful.

There are ongoing efforts, such as the use of writing in the mathematics classroom, cooperative learning, and the use of student teams, that should be continued and expanded. Broader encouragement for students to participate in contests, especially in teams, whether specifically arranged by a college or university or sponsored by other organizations, such as contests held for students at MAA section meetings, would also be desirable and help build nonmathematical skills needed for employment in the business world.

Traditional mathematically–oriented constituencies and needs must still be addressed. Curricular changes should not be implemented that preclude a student from acquiring specific mathematical knowledge serving the needs of graduate school, engineering, or the sciences. Specific topics/subjects/courses that should not be “thrown out with the bath water” include:

Calculus: finding, or proving, limit results (Chain Rule, l’Hospital’s Rule)
Probability and other purely mathematical models
Data analysis, statistical inference and hypothesis testing
Complex variables
Wavelets, harmonic analysis
Linear and abstract algebra
Foundations of mathematics (sets, numbers, operations)
Geometry (differential, projective, non–Euclidean)
Computer graphics

In addition to such specific content is the matter of rigor and depth. The curriculum should require students to study at least one challenging area of mathematics in depth with a high degree of rigor. It is only with such study that students will be able to develop the powerful mathematical reasoning skills needed for success in business.

Mathematical analytic or synthetic skills, mathematics as a way of thinking or approaching problem solving, are best developed by studying a particular mathematical subject in depth. Which particular mathematical subject is studied does not seem to matter as much as that the subject itself is challenging and that the amount of effort invested to explore the subject is extensive, intensive, and thorough.

Providing students with the opportunity throughout the curriculum to develop as deep, fundamental, and profound an understanding of mathematical proof as is possible for undergraduates to attain is mandatory. Courses designed specifically to communicate the axiomatic method as well as upper-division courses in which proof plays a prominent role should be part of every undergraduate mathematics curriculum, now as well as in 2010.

The Mathematics Major Overview

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Gone are the days when majoring in mathematics meant either you were going to teach in elementary or secondary schools, or go on to graduate school. Today, the undergraduate mathematics major provides a rich opportunity for students to enter a wide variety of career fields. The computer industry, financial and investment firms, actuarial and risk management institutions, and federal and private research laboratories or agencies all have come to recognize the value of a person skilled in analytic thinking, problem solving, framing conjectures, data analysis, and modeling. For these same reasons, the mathematics minor as an accessory to degrees in engineering, sciences, economics or business is also gaining importance.

Given the above, what does this imply for the curriculum? At many colleges or universities, especially the major ones, the mathematics major is often fuzzily pigeonholed into three options or tracks: *Pure Mathematics; Applied Mathematics*, possibly with sub-specialties such as classical applied mathematics, operations research, computing, finance or actuarial, or biological; and *Teaching*, intended for prospective secondary teachers. In small departments there may be just one track, *Mathematics*, with a core curriculum and various elective choices reflecting some or all of the tracks above. *Statistics* may be another option, depending on whether there is a separate statistics department.

There is nothing intrinsically wrong with the structure described; it has served the discipline well. But it is essential that the boundaries between the tracks must be permeable, allowing the utmost flexibility for a student to sample or even switch. A pure mathematics major may become fascinated with dynamical systems and want some courses in differential equations or computing. An applied mathematics major may discover cryptography and wish to elect some courses in algebra, or number theory. Any enrichment prospective teachers can acquire will surely enhance their future classroom experiences and performance.

On many campuses there is an *Honors* track within the various majors, which is reserved for students having shown exceptional academic attainment and qualifications. In mathematics departments this track should not be regarded as mathematics exemplifying the purest of the Pure, but should be seen as an opportunity to offer enriched courses across the mathematics panorama to bright students. This is good for the department morale since it allows faculty in more applied areas the wonderful opportunity to teach an advanced course to a highly motivated class. Furthermore, it must be kept in mind that Honors students will be highly sought after by prospective employers, and may decide to delay or not attend graduate school. Hence, the broader their training the better.

In addition to being flexible, the curriculum for the mathematics major must be dynamic, reflecting current trends in mathematics, educational technology, career opportunities, and graduate training. It must be a major responsibility of the entire department, and be considered as much of the planning process as the allocation of departmental fiscal and personnel resources. Delegation of the responsibility to a vice chair or an undergraduate studies committee is appropriate, but they should ensure that the faculty is informed of decisions and future plans. Requirements for the major, upper-division curricula, use of educational technology, and textbooks should be examined at least every three years. The basis for that examination should include course enrollment data, comparisons with requirements for the major at other comparable schools (easily obtained on the web), graduation and post-graduation information, and faculty research interests complementary to possible credit carrying research experiences for undergraduates.

The advances in mathematics, coupled with those in scientific computing, are having a huge influence in other areas of the sciences, social sciences, and engineering. The major in mathematics must reflect this trend, and this can be accomplished by interdisciplinary collaboration ranging from something as simple as a team-taught course or undergraduate research seminar, to a sequence of allowable electives outside the department, to a fully developed joint major. Mathematics is still the queen and servant of science, hence the mathematics major must reflect her ever-expanding domain, and the faculty must be her ambassadors.

Preparation of Teachers

The predicted huge shortage of mathematically trained teachers at all grade levels, the growing levels of state-mandated assessment and testing, and the perceived weak preparation in mathematics of entering college freshmen all point to the fact that teacher preparation must be one of the highest priorities of *all* institutions of higher education, and especially the state-supported ones. Faculty must face the fact the while the students who elect to go into teaching are usually not as mathematically capable as other mathematics majors, those students will teach the incoming freshmen of the future. The task of preparing them for their classrooms is even a greater challenge than teaching an upper division honors course, and cannot be left to colleges of education and adjunct faculty.

All departments, and especially those in the top rank, should make greater efforts in giving visibility to the *Teaching* track of their undergraduate program. Many entering students considering mathematics as a major have an interest in teaching, and they should find an encouraging and supportive atmosphere in the department. The shortage cited above must be met by positive actions, not neglect.

International comparisons indicate that America's fourth grade students are on a par mathematically with other European and Asian countries. By the eighth grade they have fallen behind, and by the twelfth grade they are near the bottom. Mathematics departments could protest and rail against the state education and accreditation agencies, the national teaching organizations, and the Department of Education about standards, curricula, and certification. But most faculty as agents of change have very short half-lives, and their energies would be better spent ensuring that the future teachers they produce have the best possible training and strong, conceptual understandings of mathematics. Hopefully, those teachers will be the leaders of reform by working within the educational system through mathematics and leadership workshops, becoming master teachers, and exemplifying the best in content mastery and pedagogy. Their undergraduate training will be a cornerstone of that effort.

Elementary Teachers

In the sequence of courses prospective elementary school teachers will take at a college or university, there will usually be a package of two or three courses entitled something like *Mathematics for Elementary School Teachers (MEST).* The prerequisites for such courses will be something like a college algebra course, maybe a one-semester calculus course, and sometimes an elementary statistics course. Very often these prerequisites may have been taken at a two-year college, and the MEST courses may be taught in the College of Education. If they are taught in a mathematics department the teachers may be lecturers or adjuncts, who are usually talented and dedicated people, not well rewarded for what the faculty often regard as laboring in the salt mines.

In some university mathematics departments there may be regular faculty members who usually have doctoral degrees in mathematics education, and who supervise the preparation of teachers. Sometimes, their battles to climb up the tenure ladder are colossal, mainly because their research is regarded by the tenured faculty as far below the quality expected of a “research” department. They are accepted largely because they are doing a job no other faculty member wants to do. This must change.

Faculty assigned a MEST course, akin to a term at Devil’s Island, will be warned by their colleagues that this will be the worse course they have ever taught, the students are terrible and hate mathematics, and the textbook is a huge, multi-author supermarket of motivational projects, games and examples. Unfortunately, there are germs of truth in this assessment, and it is probably true that a majority of the students wish in vain that they will never have to teach mathematics. Many of the students believe that the professor’s job is to teach them how to teach mathematics, whereas the main purpose of the course is to teach them structure and content. This can cause real problems for any professor accustomed to teaching regular mathematics courses and it biases student course evaluations.

The reality is that the students will teach elementary mathematics, although some might eventually end up teaching first-year algebra. Furthermore, they have taken some education courses where the emphasis is pedagogy and classroom management, and their formal coursework in mathematics is minimal. Consequently, to emphasize rigor is a lost effort, but to emphasize conceptual understanding through lecture, discussion, inquiry projects, etc. can have rich rewards. Teachers must develop the ability to creatively answer the ever present, ubiquitous question, “Teacher, why can’t we do it this way?”

This document is not intended to show how to teach MEST courses, but only to recommend that mathematics departments must regard these courses as much a part of their mission as the courses for their majors. The courses and their content should bear the same level of review, and there should be cognizance of state and national curriculum standards. Regular faculty who first teach MEST course must be prepared to be frustrated quite often; a partial antidote is to remind themselves that some of the students may end up teaching their child or grandchild. Such faculty will be well advised to meet frequently, even weekly, with other instructors of the course and the more experienced instructors to go over the best teaching strategies in covering certain topics—the theorem-proof approach can be left outside the door.

Secondary Teachers

There are two groups of students in this category: the regular mathematics majors who elect the *teaching* option, and, in some states, students with other majors who seek secondary certification to be able to teach high school mathematics in addition to their specialty. This certification is usually awarded based on their performance on some state-administered test. All the students will have taken the usual introductory calculus and linear algebra courses: it is in the upper-division courses that their curriculum may differ from that of the regular majors. All students will take some courses in the College of Education.

An examination of the current national secondary mathematics scene makes one thing lucidly clear- a large number of these students will be teaching high school calculus. This is largely the result of the push to consider Advanced Placement (AP) Calculus no longer as a special course for well prepared, high achievement students to be able to obtain college credit; but instead it is regarded as the mark of an exemplary school, with financial incentives and bonuses to encourage the establishing of AP courses. This means that many schools and school districts will be hard pressed to find teachers qualified to teach calculus, much less really understand it except from a formulaic viewpoint. If the pressure is too great any warm body who has taken some calculus may do.

International comparisons indicate that American twelfth grade students are much weaker in geometry than comparable students in other Western and Asian countries. This is not surprising since geometry is a subject that is being squeezed in the American curriculum. Trigonometry shares a similar fate. Some school systems are creating courses with titles like Pre-AP Eighth Grade Algebra, and the trend to push algebra into the eighth grade comes in part from the vision that this will allow students to take calculus in the eleventh grade! If college and university mathematics departments do not follow these trends closely, leaving them to the Admissions Office, they may be unable to explain or fill the gaps they find in the mathematical preparation of their entering freshmen.

Before the '60's a student walked into the first calculus class having taken two full years of algebra, a year of geometry, a year of trigonometry and precalculus, *and* a semester of analytic geometry. Consequently, his or her kitbag of mathematical techniques and structures to which to apply the calculus was very full. Today, that kitbag has shrunk to a coinpurse, clipped on to a graphing calculator, and this is the reality into which our prospective secondary mathematics teachers will enter when they take on their teaching responsibilities. It is they who will prepare the freshmen of tomorrow.

The previous discussion clearly impacts the upper-division mathematics course menu offered to prospective secondary mathematics teachers. In times past they were usually required to take advanced calculus and algebra, sometimes one semester of each instead of two, then filled the rest of their program with courses recommended for prospective teachers only. Examples would be History of Mathematics, Topics in Geometry, and maybe Elementary Number Theory; the unstated rationale being that those courses were needed because the prospective teachers couldn't compete with the regular majors in the standard courses, plus some long outdated idea that they would need these courses in their teaching. At some schools less rigorous advanced calculus and algebra courses are given instead, because the prospective teachers are even worse than the regular majors at trying to prove theorems. The unanswered question is, "Does a high school mathematics teacher need to be able to formally prove anything to do a good job in the classroom?"

Understanding concepts and being able to connect those concepts across mathematics and to the physical world in their classrooms should be the foundation of the curriculum for the prospective mathematics teacher. There is no list of courses provided here; departments should study their course offerings and the interests of their faculties to develop the curriculum. Worthwhile would be to invite outstanding high school teachers, such as the state's Presidential Award winners, to consult with the faculty and help shape the curriculum based on their own experiences. But they must be treated as peers, not inferiors. Mathematical modeling is an excellent course to accomplish the goals and provides opportunities for collaborative learning, regarded as an important component of today's educational experience. It is gaining importance in the high school curriculum, and colleges and universities could have a great deal of influence by providing guidance.

Consider that when the newly minted teacher steps into the high school classroom it will likely be the first time he or she has seen the subject matter since his or her own high school days. Consequently, a valuable capstone course in the prospective secondary teacher's course of study could be one which connects the high school courses in algebra, geometry, pre-calculus and calculus to the mathematics experienced by the students in their college or university upper-division curriculum. That could be done by sequentially taking the state or national standards for those courses and pulling them apart, then connecting the pieces to the mathematical concepts underlying them. It would provide excellent opportunities for discussions and presentations, collaborative projects, and inquiry based learning, as well as an introduction to some of the computer and information technology available to support and enhance the classroom experience. But its principal goal must be to make the conceptual connection between what is learned and what will be taught. To do this well will require some creative thinking and teaching from the faculty giving such a course, but it will have far-reaching benefits to the prospective secondary teachers.

The Curriculum — A Planning Document

The previous discussion indicates that the undergraduate curriculum, and especially the upper division and teacher education parts, must become a broad, ongoing discussion among the faculty. It can be the major responsibility of an undergraduate studies committee or vice chair, but recommended changes or revision must be broadly aired. For instance, if there is dissatisfaction with a textbook, the opinions of all faculty who have taught from that book should be solicited before a selection committee makes its choice. If a change in the requirements for the major or in the choices of electives is recommended, that is a matter for departmental discussion. It is surprising how many faculty have only the vaguest idea of the structure of the major, and only complain when their favorite course is canceled for reasons of insufficient attendance.

In many mathematics departments the only hot topics guaranteeing well-attended faculty meetings are salaries, the hiring picture, and tenure or promotion considerations. At research universities changes in the graduate curriculum or qualifying examinations may also draw a crowd. The undergraduate curriculum is left to “those who care about such matters” or the “MAA types”; this is a serious shirking of responsibility. What is worse is that certain “top researchers” may never deign to teach an undergraduate course, yet their presence in undergraduate classes may actually increase the number of majors and certainly enhances their learning experience. Bright students recognize high quality.

The job market for mathematics majors has greatly expanded, reflecting the change and growth in technology, especially computer technology. The opportunities for graduate study are no longer solely restricted to pure or applied mathematics, because of the development of a wide variety of interdisciplinary studies with mathematics a key player. The teaching of elementary and secondary mathematics is constantly evolving and increasingly using the new technology.

What all this means is that the undergraduate curriculum must become part of the departmental planning process, heretofore largely devoted to budgetary and hiring questions. The undergraduate curriculum designed for the freshman mathematics major of today, may not be the right one four years from now. More options or electives will need to be added, and some will have to be eliminated. New courses and interdisciplinary programs will have to be developed. All this requires planning, taking into account the department’s strengths and weaknesses, its future plans, enrollment and graduation data, the elementary and secondary education panorama, and graduate study and employment opportunities.

Such planning may definitely impact the hiring process, and consequently it requires strong leadership from the department chair, and possibly the dean of the college. At major research universities departmental hiring priorities are often ramrodded by well supported research groups who wish to add another star to their galaxy. This is the acknowledged way to climb up the departmental rating ladder. But this could conflict with the undergraduate curriculum plan which plans to recommend a new hire with a different research specialty because of an envisioned new program, or to fill a serious gap in an existing one. This is now in the domain of playing hard ball, but the playing field will be more level if every faculty member regards the training of undergraduates as much their intellectual responsibility as their research specialty.

Another area that requires curriculum planning as a guide is the acquisition of computer technology and software intended for use in the undergraduate program. The fiscal planning is an obvious one, but what must be studied is its use in the classroom. This cannot be left to the more or less laissez faire approach which worked pretty well when there was just a course and a textbook. A scenario like— “I use a lot of MAPLE in my ODE course”, “I don’t use it at all”— is unfair to students. There must be established guidelines, and possible curricula or modules, easily adopted by faculty, which indicate the power of computing and computer graphics in gaining mathematical insight or finding solutions, or their approximations, of highly complex problems.

The domain of the queen and servant of science is constantly and dynamically expanding, and to traverse it her ambassadors, the faculty, need charts. The undergraduate curriculum plan is one of them.

Equity Issues — A Different View

This section will not address the situation of women in the undergraduate mathematics major, which is greatly improving with many colleges and universities having fifty percent of more female majors. The situation that is of concern is the relatively few Ph.D’s in mathematics and the sciences being granted to African American, Hispanic, and Native Americans in the past decade, supposedly the era of affirmative action. Their numbers have hardly changed at all, year after year, and their numbers at the major research universities is almost constant and virtually negligible. But faculty positions at these institutions represent the power elite, the leaders in the field, and consequently the leaders and models in their community.

Restricting the discussion to mathematics and the undergraduate major, what can be done? The time-honored affirmative action strategy was to recruit *qualified* minority students into the college or university, provide financial aid and possibly special tutoring or mentoring programs, then congratulate ourselves on a job well done. But what is needed is the tougher job of recruiting *competitively qualified* minority students who will successfully climb the academic ladder to its highest rung.

Those very bright, mathematically gifted minority students are just as scarce in their own communities as similarly endowed majority students are in theirs. But what they often lack, as distinct from the majority student, is the confidence and willingness to take the risks needed to succeed. There are a number of socio-economic and background factors coming into play here, but that discussion is beyond the scope of this paper.

What is needed when that competitively qualified minority student arrives in the mathematics department is the mentoring needed to build that super confidence, characteristic of all talented mathematicians; it is mentoring of a very individual kind. It will not likely be done by a faculty member of the same minority background, although it is almost a sure bet that kind of mentoring was done by a high school mathematics teacher or counselor who recognized the student's ability and probably encouraged him or her to go to college.

The faculty mentor can approach the task in variety of ways: a congratulatory note on the margin of a well-done examination, calling the student in and praising a clever solution to a tricky problem, discussing the possibility of a summer research experience at a national laboratory or another school (minority students are very apprehensive about leaving their home turf, so this will take some convincing), encouraging the student to enroll in an advanced course or a research seminar, and eventually, discussing graduate study at schools best suited to the student's abilities and interests. Individual attention, caring about the student's education, is what counts.

This kind of mentoring need not be a chummy kind of relation; it should reflect professionalism and the counsel a faculty member gives a potential future professor. All the strategies mentioned above, and many more, are intended to build that confidence that minority students need when they enter graduate school. There they will recognize that although their majority member classmates may have undergraduate degrees from the elite schools, they can compete with the best of them. In graduate school some other faculty member will take over the needed coaching job, perhaps the Ph.D advisor, but the real work has already been done. Congratulations!

Mathematics and Mathematical Sciences in 2010: The How of What Graduates Should Know

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Introduction

I recall a visiting professor in my department once saying, "Knowledge is the residue that remains after the facts are forgotten."[1] This phrase comes to mind often as I think about teaching and learning. I visualize a glass container or vase filled with a liquid substance. The substance is made up of many different ingredients whose individual identities have been lost by being soaked or dissolved in the liquid. This filled vessel perhaps represents information in a student's mind toward the end of the semester, integrated and coherent. After a summer or semester break the liquid seems to evaporate leaving a residue on the inside walls of the vase. As educators we must concerned ourselves with the content and quality of the "residue that remains." If the appropriate residue remains, when this knowledge is again needed students can add liquid, shake, and reconstitute the original mixture.

Mathematics educators and the curricula they implement influence the two critical components of the residue remaining after a mathematics-intensive undergraduate experience, the content and the quality of knowledge. This paper addresses the quality of the residue. It is important to note I use the term *quality* in its broadest definition, "that which makes something such as it is," rather than the more specific definition of "the degree of excellence; relative goodness grade," although we are certainly interested in the latter. While other authors in this volume will specifically address the content of the residue, the what, its content and quality are necessarily coupled. The content dictates much of the quality of knowledge — hence the title of this article. Prior to specifically addressing the "how" of knowledge of the mathematics-intensive major, the following provides background discussion on knowledge itself and the process of knowing.

Background — Knowledge and the Process of Knowing

In general there are two types of knowledge, conceptual and procedural. In the past conceptual and procedural knowledge were viewed by many as two separate entities, often resulting in a dichotomy between understandings and skills. In mathematics education, pedagogical discussions centered around which should be emphasized, the actual mathematical skills or the understanding of mathematical concepts. Hiebert and Lefevre [1] propose that although a distinction between conceptual and procedural knowledge is useful for thinking about mathematics

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[1] Phrase was stated by Professor George Rosenstein, Franklin & Marshall University, while visiting at the U.S. Military Academy during academic year 1988–89. Others have told me of hearing it elsewhere.

learning, it does not provide a classification scheme for all knowledge. Some knowledge possesses a combination of conceptual and procedural qualities, while other knowledge may have neither. In any case, the distinction does provide a useful scheme to investigate the structures in student learning and understanding.

Procedural knowledge of a particular concept refers to knowledge an individual has about that concept which has linear relationships to other knowledge. Conceptual knowledge of a particular concept refers to the knowledge an individual has about that concept which is rich in relationships [1]. These relationships are complex in nature and may be thought of as forming a web or network. For example, consider the concept of the derivative. Individually, rules of differentiation may be thought of as procedural knowledge. Given a polynomial function, apply the power, sum and constant multiple rules of differentiation to determine the derivative function. With a linear diagram I could depict this procedure step by step. The concept of differentiation itself, however, is rich in many relationships, to include a relationship with each of the various rules of differentiation. The concept of differentiation is also related to the concepts of function and integration. Additionally, there are various interpretations of the derivative that lead to further relationships with many other concepts. If you were to visually depict all of the relationships involved with the concept of differentiation, the result would be a very complex non-linear web. In examining the quality of our residue, we want it to contain both linear and non-linear concept relationships.

Conceptual and procedural knowledge are critically linked and mutually beneficial. The ability to carry out procedures without understanding the overall concepts and connections is very limiting, as is the ability to visualize a concept or idea without the processes required to solve problems or provide results [1], [2]. The relationships necessary to develop procedural and conceptual knowledge structures must somehow be formed through the activities of teaching and learning. I propose that mathematics educators adopt a constructivist perspective on teaching and learning to support the formation of these relationships.

Constructivism has been the major guiding philosophical basis for pedagogical change within mathematics classrooms in the last decade. Constructivism is said to be a theory of knowing rather than a theory of knowledge. One can trace the roots of constructivist ideas to general research in cognitive psychology done in the latter half of the 20th century, and more specifically to the work of Piaget. Piaget believed that action and knowledge are inextricably linked [3]. Constructivists generally agree that the learner constructs all knowledge and that cognitive structures are under continual development. Constructivists believe it is purposive activity that induces the transformation of existing structures, and that the environment presses learners to adapt [4]. Once an instructor acknowledges a constructivist perspective as a cognitive position, "methodological constructivism" follows. "Once a constructivist perspective is adopted, the day-to-day life in the classroom is profoundly and significantly altered for both teacher and students [5, p. 314]." One does not, however, need to understand the deeper cognitive aspects of constructivism to teach in what would be considered a constructivist manner. Many teachers are natural constructivists, guided by an intuitive feel for what is most beneficial for their students — a more interactive and engaging classroom.

Through a constructivist lens we see that the quality of the residue is synonymous with the quality of the cognitive structures that the individual has constructed. Humans have a highly developed capacity for organizing knowledge and will continually develop and transform cognitive structures through purposive activity. Therefore, if we see these cognitive structures as knowledge, we can infer that they are the result of the purposive activity we call learning. Learning can be defined as a change in disposition or capability, which persists over time, and results from experience. This definition provides three important guidelines. In learning change must occur — change in cognitive structures. This change must persist — persist to become part of the residue. This change results from experience — experience from participation in something. What is the something? This is where teaching comes in. Using this definition of learning, teaching can be defined as arranging experiences.

Focus on Mathematics-intensive Majors

The mathematics graduates of the twenty-first century must possess much broader abilities than their predecessors. They must be prepared as problem solvers and mathematical modelers in a rapidly changing world.

They must adeptly use technology and creatively draw from the many aspects of the mathematical sciences. They must comfortably work in other disciplines and in teams. If they specialize or narrowly focus their studies and are unable to adapt, they will be left behind [6, p. 41]. And yet, at the same time, these graduates must be proficient, both conceptually and procedurally, with specific mathematical concepts to serve as mathematical experts when called upon.

What types of experiences must teachers arrange in the undergraduate classroom to prepare these future graduates? How can departments support curricular and pedagogical change that will improve undergraduate experiences? How can the mathematics community support departments in their efforts to improve these experiences? This paper will focus on how activities at the classroom level can influence and support mathematics-intensive majors' undergraduate experiences, and will also address support at the departmental and mathematical community levels.

Classroom Environment

Undergraduate students of the twenty-first century have experienced adolescence in a world shrunk by technological advancements and yet more complex than ever. They play computer and video games, not board games. They've never known a world without cellular phones, pagers, personal computers and the Internet. They do science reports and research papers using strictly electronic sources. These students read and digest information directly from a computer screen. They are comfortable working with information in electronic form without the need to print out paper copies. They use computers and calculators to do mathematics symbolically, graphically and numerically. In this world the teaching of mathematics must change from many current practices or students will lose interest. In 1996 an advisory committee to the National Science Foundation stated "Too many students leave SME&T courses because they find them dull and unwelcoming [6]." Mathematics classrooms must adjust to motivate learning by the twenty-first century student. Even mainstream popular news magazines are addressing issues of mathematics education. *Newsweek* recently highlighted Williams College professor Edward Berger who is known for exciting even his 9 a.m. calculus class. While nationally about one percent of graduating seniors are mathematics majors, at Williams it's almost nine percent [7]. Teachers must arrange experiences that excite and challenge students to learn.

Teaching with a constructivist perspective is characterized by active learning and student centered instruction. Active learning is a process that focuses on developing learners' analytical and critical thinking, and in which the students are active collaborators in their own learning. Student centered instruction is a broad teaching approach that includes substituting active learning for lectures, holding students responsible for their learning, and using self-paced and/or cooperative (team-based) learning [8]. In general, a constructivist approach moves the focus of classroom activities from the teacher to the learner. On the one extreme, only the teacher speaks or lectures while on the other extreme a teacher may not even be present as students independently work through the course material. I propose that neither of these extremes used exclusively serve the learner well. A constructivist perspective allows teachers to move along a continuum of approaches within these extremes using a variety of techniques in their classroom. Instructional strategies influencing movement along this continuum include questioning or discussion, applied problem solving, integrating technology, explorations and discovery, multiple representations of mathematics, writing, variety in assessment instruments, small section sizes, and cooperative or collaborative groups. The bottom line of the constructivist value of any strategy is that the students are engaged in the learning experiences. Teachers choose techniques as appropriate for their student population, facilities, class size and course objectives, as well as their comfort level [9].

One example of a recent change in undergraduate mathematics teaching is an increased emphasis on mathematical modeling. The mathematical modeling process includes making assumptions, building a model, solving the model, verifying the conclusions and communicating the results. These activities contribute to student understanding of mathematics and its applications [10]. Presenting real scenarios and problems of interest to the students motivates them to learn the mathematics in order to solve the problems. Students quickly become engaged in the project and seek out the necessary mathematics. The modeling process replicates the future real-world situations our graduates will encounter. Many different fields require graduates to solve problems by developing and analyzing mathematical models, and then communicate their solutions by providing the user their analyses in an understandable form. A modeling approach to mathematics learning early in the undergraduate curriculum can excite students into pursuing a mathematics-intensive major.

In arranging experiences teachers must concentrate on the important mathematics. What and how we teach must reflect what is important and how it is important. Recall the image of the residue. The procedural and conceptual knowledge that remains is a result of what is learned and how it is learned. Therefore, what and how we teach must also be reflected in what and how we assess. Classroom assessment serves several purposes in the learning environment. Assessment should provide feedback at several levels. First, assessment provides feedback to the student. It also provides feedback to both the instructor and the program. Finally, assessment can be used for evaluation purposes to assign a value to the work. In any case, all assessment instruments should focus on the important mathematics and require demonstration of procedural and conceptual understanding that reflects how the mathematics was experienced. The important mathematics is that which is critical to the objectives of the course. How the mathematics was experienced dictates how the assessment should be designed. For example, few inferences about knowledge can be made if classroom activities stress purely theoretical or procedural approaches and then the assessment requires modeling and problem solving, or vice versa.

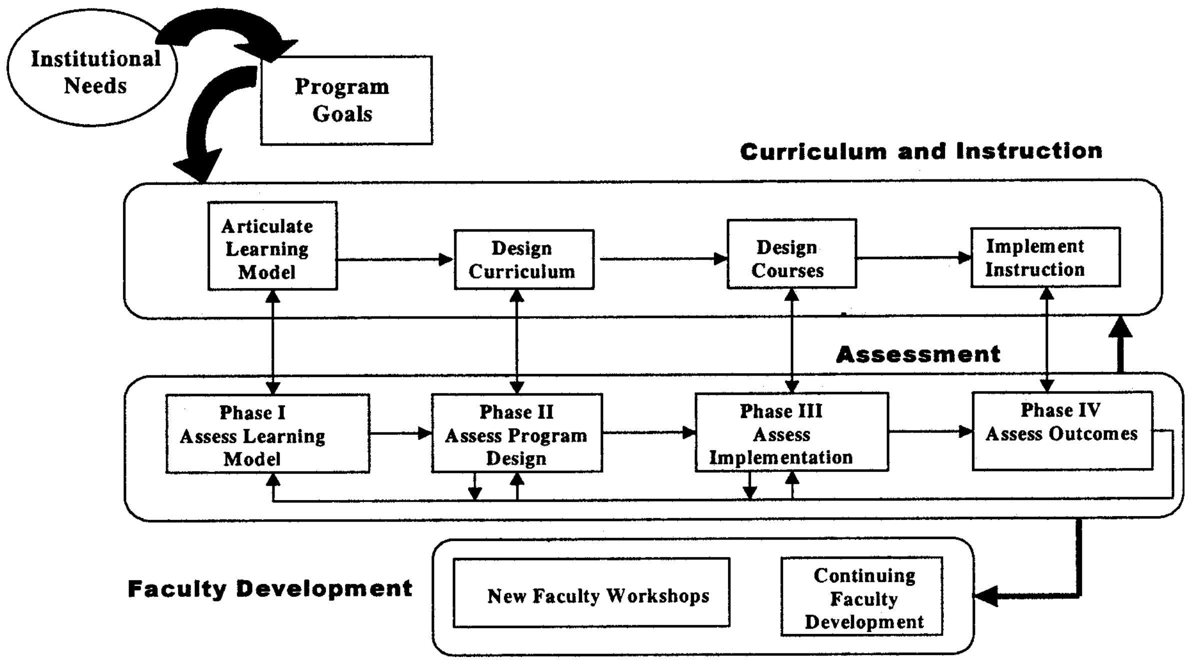
An assessment plan designed to provide inferences about student knowledge should include a variety of instruments. Examples of instruments include quizzes, exams, applied or interdisciplinary projects, problem solving activities, essays, journals and problem sets. These instruments should include a variety of requirement types addressing both procedural and conceptual knowledge. These types may include fill in, short answer, calculations, graphical analysis, numerical analysis, explanations and modeling. Assessment instruments can be done both in- and out-of-class and should appropriately integrate technology. In order to most accurately infer student understanding, assessments must include a variety of problem types and presentations [11].

Assessment should also be done at the program level. This type of assessment may address a single course or the entire undergraduate mathematics curriculum. As mentioned above, classroom assessment provides feedback to programs and therefore can be part of a program assessment plan. In doing program assessment at various levels, the department and/or school can determine if its programs are resulting in the appropriate residue based on its outcome goals.

Department and School Support

Program assessment at the department and school levels is critical to improving undergraduate mathematics education. At the program level, schools must determine what residue should remain after the four year integrated experience of a mathematics-intensive major. Program assessment provides feedback to examining the department or school's outcome goals. Therefore, prior to conducting any program assessment the department and school must determine their outcome goals. The major purpose of this CUPM document is to assist departments in planning the undergraduate program in mathematics intended for mathematics-intensive majors. Departments and schools can use this volume to assist in defining the outcome goals within the constraints of their own institution. Once goals are defined, the department can design a program assessment to determine how their current program is meeting these goals.

A program assessment is much more difficult than simply one professor assessing the results of one section of a course. Program assessment data can come from classroom assessments (and evaluations—grades), but may also include surveys, interviews, classroom observations, partner and client discipline/department feedback, and post-graduation employer feedback. Once assessment data are collected, departments analyze the data against the outcome goals. Departments and schools recognizing the need for improvement must support change through curricular and faculty development initiatives—activities that address the what and the how. The primary purpose of academic program assessment is to evaluate programs in terms of their stated goals in order to enhance learning. Additionally, program assessment provides information to measure academic outcomes, as well as to answer to outside agencies such as trustees or accreditation associations. The following diagram depicts an example of a program assessment and program planning model [12].



Departments and institutions that extensively include faculty in the processes of goal development, assessment, curriculum development, and faculty development cannot help but improve their chances of a successful program. Unfortunately, many institutions do not reward faculty members for putting time and effort into this type of venture. Departments and institutions interested in change must support faculty members engaged in making teaching and learning a priority and use these faculty members as leaders of change in their departments. In its review of undergraduate education, the advisory committee to the National Science Foundation states [6, p. 46]:

> Frequently, discussions of this issue lead to the realization that we need to inculcate an institutional culture that elevates learning to a position of importance on par with the discovery of new knowledge. Progress here would facilitate (among other things) the development of research programs within existing academic structures that focus on the learning effectiveness, thereby creating valuable knowledge that can be employed to help make informed choices about new curricula and teaching methods. The lack of adequate rewards for improvements in education is seen by many as the fundamental problem that makes more difficult the widespread acceptance of responsibility for the learning of undergraduates, and puts the burden on single, committed individuals.

Departments truly interested in developing the appropriate curriculum for mathematics-intensive majors in 2010 must commit themselves to assessment and improvement initiatives that encourage and support faculty to make teaching and learning activities a priority. Volumes such as this one, funded curricular projects, and undergraduate mathematics education research are critical in informing institutions about current curricular and faculty development initiatives.

External Support

Professional societies, publishers and funding agencies must support institutions in their efforts to improve undergraduate programs. *Shaping the Future: New Expectations for Undergraduate Education in Science, Mathematics, Engineering and Technology* provides action-oriented recommendations to the National Science Foundation for improving undergraduate education, as well as makes recommendations to "mission oriented Federal agencies, business and industry, academic institutions and their administrations, professional societies, private sector organizations, state and local government, and to other stakeholders in undergraduate education"[6, p. i]. The National Science Foundation has provided considerable funds to support innovative projects designed to improve undergraduate education, but this report calls for a significantly higher level of commitment and funding.

The MAA has been at the forefront in providing support and guidance to the mathematics community through the work of committees such as the Committee on the Undergraduate Program in Mathematics and the documentation of such work in the MAA Notes series. Of note for our purposes are *Reshaping College Mathematics* [13], *Heeding the Call for Change* [14], *Research Issues in Undergraduate Mathematics Learning* [15], *Models That Work* [16], and *Confronting the Core Curriculum* [17]. These volumes provide a wealth of information and thoughtful analysis of the undergraduate mathematics curriculum and teaching.

Most recently, the MAA recognized the importance of conducting research in undergraduate mathematics education in announcing the formation of the Association for Research in Undergraduate Mathematics Education (ARUME) as a Special Interest Group of the MAA (SIGMAA). Several other actions are also supporting the growth of the field of Research in Undergraduate Mathematics Education (RUME). The American Mathematical Society (AMS) has joined with MAA to form a committee focused on research in undergraduate mathematics education that has published several volumes of research papers. Professional mathematics meetings include sessions for reports of research in this area and journals are beginning to include undergraduate mathematics education research reports. Research is emerging that addresses questions about what effects various curricular and pedagogical decisions have on undergraduate students' understanding and their attitudes toward mathematics.

Conclusion

In their efforts to improve programs for mathematics-intensive majors, institutions must continually focus on the content and the quality of the residue that remains — the knowledge that a graduate takes away from the program. Pedagogically this requires a focus on the student and on student learning. Culturally, teaching and learning must be made a priority. At the classroom level, instructors must challenge and excite our twenty-first century students. Department curricular and faculty development initiatives are critical and imperative for successful programs. Institutions must encourage and reward these activities. National agencies and professional societies must support these efforts toward improvement.

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Teaching and Learning on the Internet

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This paper comes from personal experience rather than philosophy. My experience is with MIT's linear algebra course and it is ongoing. You will see that I have recently gotten myself into some kind of box (a new box instead of the usual one). As a result I don't know exactly what to do in the linear algebra lectures this fall. Writing this paper in the summer of 2000 gives me a small chance to think through this rapidly approaching problem. It involves using the Internet, and videos in particular, in combination with ordinary lectures and homework. I believe that the reader will encounter the same problem, in some form, soon.

In spite of the opening sentence I suppose there is a "philosophy" that underlies my teaching of mathematics. Many of the students are learning engineering and science, and they care first of all about applications. This seems to fit with my approach. I get a lot of pleasure from showing them examples, and connecting with their interests, and convincing them that mathematics is directly useful. It is true that I use the words "beautiful" and "wonderful" to call their attention to ideas that are especially neat. But the beauty is alive and not frozen.

The only theorem that I mention by name is the Fundamental Theorem of Linear Algebra. I would not want the rest of the faculty to know how seldom I complete a proof in the lectures. An example can be much more memorable anyway. Two examples are totally convincing! (My favorite proof remains the one I found in a book by Ring Lardner: "Shut up" he explained. But I use this in class only when desperate.)

Let me come directly to the recent events that present new problems.

1. My linear algebra lectures and review sessions last fall were videotaped live. They are on the web page `web.mit.edu/18.06/www`, and they can be viewed with (free) Real Player software. The compression makes my own motion a little jumpy, but the blackboard is surprisingly clear. So all students are going to have the lectures available when they want them (not only MWF at 1).

2. Independently of the videotaping, I joined with David Jerison and Haynes Miller in a proposal to a new funding source within MIT (established by a gift from Microsoft). Our proposal was to introduce 'new communication links' in calculus (Jerison) and differential equations (Miller) and linear algebra (Strang). It was nearly successful but in the end was not funded for next year. I hope that some of the ideas might be of interest to readers of this article.

Those ideas are speculative and very much in flux. A second proposal, more directly involved with the structure of lectures and recitations, was awarded a planning grant from a different fund. Haynes and David are testing new possibilities in the calculus lectures. Eric Mazur's website `mazur-www.harvard.edu/education/educationmenu.html` has been a source of inspiration. They have very properly suggested that this paper should concentrate on my own part of the original proposal, which was to create an "online encyclopedia" of short and specific pieces of undergraduate mathematics. These will be quite different from complete lectures, but a camera will still be involved.

Note added in proof: I have now seen a device that records as you write on a whiteboard, without needing a cameraman (it transmits the writing but not the speaker). Combined with audio, this may become useful in communicating mathematics in real time.

The videotapes in linear algebra

I can explain first about the videotapes in 18.06. The year before, when Gian-Carlo Rota died suddenly, I expressed to the class my regret that we had no permanent record of his lectures. They were exceptional in every way. The conversation in class moved toward more ordinary things, but several students emailed me afterward. They suggested that I contact the Center for Advanced Engineering Studies. I discovered that the Center was embarking on a large-scale videotaping project in physics (with Walter Lewin). Eventually we realized that for a small additional cost the cameramen could stay in the lecture room and tape the 18.06 lectures.

The original tapes were digitized and compressed (and saved) by David Mycue, in between his work on engineering classes that are running jointly with Singapore. This was all a part of MIT that I had never seen.

I insisted on only one point, that the lectures must be freely available to everyone. Modulo congestion on the web (which depends on the viewer's modem), this is now the case. I had no idea what use might be made of the tapes, it just seemed a good thing to try. I still have no idea! Readers of this paper are very welcome to make suggestions. I can mention two developments within MIT, and I hope there will be more outside:

1. In the semester of videotaping, there were a few classes that I had to miss. Those tapes were made in advance without an audience. I asked the students whether they would prefer to have a substitute teacher, but they firmly chose the tape. Apparently they did come to class and watch quietly.

You will realize the implications. One is that I can be away more and more — leaving a shadow of myself behind. On the other hand (and more seriously), the students can be away more and more. Let me come back to this new freedom, which is partly desirable and partly alarming.

2. The MIT Lincoln Laboratory learned about the tapes and decided to offer a linear algebra course this summer. The volunteer students are mature scientists and engineers, who watch two tapes each Thursday afternoon. My best teaching assistant, Peter Clifford, is there to answer questions. I went three times, to be part of the group and ask for their reactions. I frankly thought it would be a horrible experience to watch the tapes with the class, but it wasn't. They are seeing the uncompressed form, not so different from a live lecture. The volunteers at Lincoln Lab continue to attend and their comments are very positive.

I now realize more clearly and urgently that students will have an alternative to attending lectures this fall — or possibly the tapes will be more a supplement or a complement than an alternative. My question now is what to do in class when they can watch the lectures at their own convenience. I could vary the examples, and I certainly will. But I can't vary the mathematics...

18.06 splits into ten or more recitation sections, one hour per week, to discuss homework problems and unanswered questions. The lecture hours could now be more interactive (subject to the limitations of a large class). I am very much in favor of active learning, and I mix in questions as I go. I don't always wait for the answers! Students are hesitant to stand out in a large anonymous group, but I am learning about the successful use of flash cards and class votes.

In general it will not be possible to assume that students have watched the lectures in advance. Do I want to assign specific lectures as part of their homework (and risk developing a habit that will lead them to skip class)? The new situation offers more freedom, but with it comes change and uncertainty. Every innovation implies an altered set of rules. Most definitely, students have learned to deal successfully with the old rules. After long acquaintance, those rules are more or less accepted as fair. Any alteration implies that somehow or somewhere, an extra effort is required. This is likely to be unwelcome.

In the present case, a more active lecture hour might succeed. I have no means of compelling students to attend, and don't want any. I do already try to make the hour more productive for the class than an hour spent reading the textbook (which unfortunately I wrote). I was already competing with myself, and now even more so! Where I previously offered a focus on the more important points, and used the medium of speech to bring home those points, now the video lectures offer speech too, at all times of the day.

Will a live lecture three days a week be preferred to a videotape available at all hours, seven days a week? In the long run I really don't know. Perhaps in the short run, inertia (and maybe the lack of anything better to do) will bring most of them to the classroom.

And there is another question. Could the videotapes affect linear algebra classes around the country? The text is widely used. I am hopeful that instructors will welcome the availability of lectures on the web, as a supplement to their own courses. This is really a key question that will surely arise throughout our teaching — how to make the Internet into a "TA."

In short, I took this videotaping step in the belief that it could only be useful — not knowing exactly how, but certainly knowing that lectures on the web are sure to come. They will come in different forms, from different sources. If it becomes clearly helpful to add summaries of the lectures, or answers to frequently asked questions, or additional examples (all probably to be prepared on transparencies), I will try to do that. First I hope to learn how these videos can be used. I will be extremely grateful if readers of this article send thoughts and suggestions by email (to `gs@math.mit.edu`).

The Internal Proposal Within MIT

The second part of this paper will describe some aspects of the proposal that Haynes Miller, David Jerison, and I made in December 1999. We offered to create new on-line possibilities for the students, without a radical change in the existing lecture system. We were unwilling to destroy something that is pretty good (certainly not perfect). The new ideas would definitely need testing and adjustment, and a lot of work.

Our overall goal is to make the experience of freshmen and sophomores more active and positive. In every society, whether on the scale of a nation or a university or a family, there is tremendous constructive energy. Very often this is potential energy, and it is never released. To convert that stored potential into kinetic energy is a central goal of teachers (and of leaders wherever they are).

One tool we proposed to use is the Internet. We know that experiments are going forward in this direction in many mathematics departments. Undoubtedly the results are mixed — the same would surely be true for us. I can reproduce here a substantially edited summary of two ideas, from the proposal that the three of us prepared:

A. We will create an on-line encyclopedia to offer quick help in our basic mathematics courses. This material could extend beyond freshman courses to help all students who have access to the Internet.

The advantage of video and multimedia presentations over textbooks is that the added dimension of time can convey complicated (and also simple!) mathematical ideas more effectively. A critical advantage over lectures is that the information can be delivered in packets at the moment when it is needed. That moment can be the point at which a student gets stuck on a homework problem. It can also be the time in the next month or the next year when a specific piece of information is needed again.

B. We will establish Chat Rooms in which students will be able to discuss ideas and problems. They can manipulate the graphical tools that we plan to develop, and they can have have fun working together. We want MIT students to appreciate the active and cooperative elements of mathematics. Mathematics depends on communication.

The chat rooms could develop into a new feature of MIT education. Students already form study groups to do problem sets and to review for exams. We do not wish to lose the value of these interactive groups as we move to exploit the power of computer education for every individual. On-line chat rooms allow students to interact even if they cannot do so in person, or if they do not have access to a compatible group. A group can use graphical and computational tools jointly, to go further with discovery than separate individuals. The transcripts of the chat rooms will be useful in monitoring the whole process and understanding where students get stuck.

Like most universities, MIT relies on lectures by experienced teachers. It would not be wise to overturn this framework tomorrow — the quality of those lecturers (and their dedication) is too valuable. But when we consider the whole experience from the viewpoints of the students, we do see new ways to communicate with them individually—and new ways in which students can communicate with each other.

It is changes in the student experience, made possible by the revolution in faculty-student and student-student communication links, that motivate this proposal. The first project is the most straightforward — a direct way for students to access (on line) essential concepts of each course. The second is the most novel — a way for students to talk to each other. We want them to work together, because in some unexpected way that promotes individual learning. Thus our presentation concentrates on these elements:

- Direct on-line access to quick help with the fundamental ideas of each course and their applications (with many examples).
- Student-student interaction on homework problems as well as central concepts. The magical moment of understanding generally happens outside the classroom.

May I return from the proposal to this MAA paper, for several comments. One is to repeat that in the actual development, changes in these ideas would be absolutely certain. I will add some details to the descriptions given above, but those changes are already coming. And there was another key part of our proposal that is not reflected in this paper — the idea of modules in differential equations, where the biologists and physical scientists and engineers are often interested in totally different examples.

Here are elaborations of those two ideas A and B (on-line encyclopedia and homework chat room) also edited from the original proposal.

A. Direct on-line access to quick help.

Students need a fast direct source for information about key ideas. This means definitions, examples, and applications, as well as theorems. We believe that the web provides a new communication link between faculty and students that we can establish and develop. Our plan is to present specific topics on the web, and we give ten examples from calculus:

Second derivatives: Maximum, minimum, inflection points
The idea of a limit
The chain rule
The exponential function
The Fundamental Theorem of Calculus
Integration by parts
Length of a curve
Polar coordinates
Taylor series
Lagrange multipliers

For every topic we will give immediate feedback. Effectively, we will be creating an on-line encyclopedia. This is in reality the format that students find most useful — the "look-up" format rather than the "expository" format. Both are needed and the text already gives one. Our plan is to create the other.

Each topic will be presented on its own. There will be links of course. But it is the individual concepts that we will focus on, using our teaching experience to guide our work here. Every presentation will aim to have four ingredients:

1. A videotaped description of the concept and its context;
2. Transparencies following the video to recapitulate the key ideas;
3. Graphics during the live video;
4. Exercises and examples: some solved in detail but not all.

The viewer can move quickly and easily from one stage to another and back. Some viewers will prefer to see the examples first, or the summary. Those will all have sound as well as image, not only to emphasize what is important but also to maintain a human contact that carries the signal to the listener.

B. Student-student on-line interaction.

Student learning often comes in bursts — frequently while working on problem sets. These are crucial times outside the classroom, and we can supply extra help. They are the moments we must concentrate on, because the students are concentrating. We will provide video clips to the students. But we are convinced that their communication with each other is a powerful force in learning. We will also provide graphical and computational tools that form integral parts of the problem set. The next paragraphs discuss three types of on-line support: videos and graphical tools and student chat rooms.

1. Videos. The advantage over textbooks is that video, with the added dimension of time, can convey complicated (and also simple!) mathematical ideas more effectively. The critical advantage over lectures is that the information can be delivered in packets at the moment it is needed. Homework problems will in some cases be explicitly linked to video clips. Successful animations will also enliven our lectures.

2. Graphical Tools. These make a basic concept visual, and they also teach methods of discovery. In using Newton's method, calculus students can discover rates of convergence and periodic orbits and especially basins of attraction. At the same time they will reinforce their basic understanding of linear and quadratic approximations. By varying the coefficients of a quadratic function of two variables, they will see saddle points and maxima and minima. Graphical tools will be heavily used in the "technology-enabled" classrooms of the future.

3. Chat Rooms. Students already form study groups to do problem sets. But this includes only some students — it is good, but not fully inclusive. On-line chat rooms allow students to interact even if they cannot do so in person, or if they do not have access to a compatible group. A group can use graphical and computational tools jointly, to go much farther with discovery than separate individuals.

The transcripts of the chat rooms would be recorded. This makes it possible to see where students got stuck. To make better use of the transcripts, we propose to require students to cite their sources of help, with no penalty for accepting help. This will make it possible to track down abuses, such as wholesale copying of homework solutions, and (more important) to track down successful interactions.

Chat rooms should heat up before exams. Students will have access to support at late night crunch times. And especially at exam times, teaching staff could participate. In some cases the chat room can become an on-line recitation. Transcripts of these pre-exam interactions will be particularly helpful in course design. The TA could clip pieces and discuss the interaction with the lecturer.

An interesting problem is how students will communicate with each other in chat rooms if the keyboard is the only link. Mathematicians generally use an informal version of Donald Knuth's typesetting language TeX, where \int_a^b f(x)dx means the definite integral of $f(x)$. There may be a better language for on-line mathematical conversations. And keyboards are presently inadequate for conveying pictures. The Holy Grail is the functional equivalent of a blackboard — which would support distance learning everywhere.

One can object that we are proposing to encourage last minute learning. But students learn when they are receptive, and not before. Any mechanism that makes learning easier must have its place. We may also include problems that specifically develop reading and listening skills, because lectures and a textbook are still central to the course. We could make the chat rooms off limits for certain homework exercises.

Furthermore we will use technology to help with learning after the last minute, in the form of post-testing. Most students do not go over their homework to see what was wrong. Even worse, many of them don't review their old tests. Teachers are guilty of encouraging this behavior, by racing to the next topic. This is a crucial learning opportunity and it is frequently lost. No experienced teacher expects everyone to learn subjects the first time. We aim to develop the habit of going back to learn and to reinforce learning.

We can teach our students more successfully if we deliver information to them when they need it and are ready for it. Students can work cooperatively on line or in recitations. With their friends also on line, students will find it easier to learn to use these tools, to make progress on problems, and to have fun doing it — without the hesitation that we all recognize in a lecture room.

Our main goals are to engage the students more fully; to make better use of their time; to provide more inspired teaching; to encourage them to manipulate graphics in their own mathematical experiments; and to offer easy access to background information at the moment when it is needed. We hope for, and we expect, changes in style and substance.

That concludes the part of this paper which is drawn from our joint proposal. I owe thanks in so many ways to Haynes Miller and David Jerison. And over a very long period, too long to contemplate, I have learned from an army of students. To be truthful, I have seldom thought deeply or carefully about theories of education — it has always been more instinct in a classroom, a feeling for what students might understand and enjoy. And I read aloud (quietly) everything I write, so the same instincts are in control there too.

The arrival of the Internet has opened tremendous new possibilities.

Just in time.

First Steps: The Role of the Two-Year College in the Preparation of Mathematics-Intensive Majors

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American Mathematical Association of Two-Year Colleges

The two-year college is the institution of choice for many of today's students enrolling in postsecondary education. While some students at two-year colleges prepare for immediate entry into the workplace, others select a major and earn an Associate's Degree in order to transfer to a four-year college or university to complete the requirements for a Bachelor's Degree. Included in this group are mathematics-intensive majors, who complete their important collegiate foundation content courses at the community college. The purpose of this paper is to describe the two-year college's role in the preparation of mathematics-intensive majors. Forthcoming recommendations from the Committee on the Undergraduate Program in Mathematics must include attention to the mathematics-intensive majors who begin their mathematics preparation at two-year colleges, and therefore to the two-year college mathematics curriculum.

The Two-Year College

Enrollment in our nation's more than 1100 two-year colleges has increased from 3.9 million (Fall 1975 headcount) to 5.5 million (Fall 1998 headcount). These enrollments have stabilized at around 5.5 million during most of the 1990's, and are projected to increase in the coming decade. Two-year colleges enroll almost half (44%) of the undergraduates in the United States, and 46% of first-time freshmen. Forty-nine percent of all undergraduates who identify themselves as members of racial or ethnic minorities attend community colleges. Significant diversity is found in two-year college student populations, with:

- 46% of all African American students in higher education;
- 55% of all Hispanic students in higher education;
- 46% of all Asian/Pacific Islander students in higher education;
- 55% of all Native American students in higher education.

Indeed, "community college enrollments reflect the rich diversity of the nation," as reported by the American Association of Community Colleges.

Students choose two-year colleges for a variety of reasons. For some, geographic accessibility, low tuition and fees, or the open admissions policy are key reasons. Community colleges are located in the communities where students live and work, and offer high quality postsecondary education at an average annual tuition rate of $1,518. Other students are attracted by small class size, the focus on teaching and learning, and the availability of faculty office hours. Still other students need the preparatory work offered through developmental courses and benefit from the wide range of student support services offered by the two-year college. Students who have attended college previously, perhaps earning Bachelor's or more advanced degrees, may engage in “reverse transfer,” attending the community college to gain necessary training for a change in career.

Data from a survey conducted by the Conference Board of the Mathematical Sciences (CBMS) show that in Fall 1995, about 1,384,000 students were enrolled in undergraduate mathematics courses offered in two-year college mathematics programs, as compared with about 1,469,000 students taking such courses in four-year institutions. While the total enrollment in mathematics courses at the undergraduate level remained constant from 1990 to 1995, the decrease in four-year college and university enrollment was matched by a corresponding increase in two-year college enrollment. The Fall 1995 CBMS Survey states that “by the turn of the century, mathematics enrollment in two-year colleges will equal or exceed enrollment in four-year colleges and universities.” The CBMS Survey conducted in the year 2000 should provide the necessary data to support this enrollment prediction.

In many two-year colleges, adjunct faculty teach roughly half of the mathematics courses offered by a department. While the varied experiences of adjunct faculty can be of value, adjunct faculty should understand the mission and purpose of the two-year college and be integrated into the life of a department. Adjunct faculty should be included regularly in discussions of teaching and learning, textbook adoption decisions, and professional development activities. Just as new faculty benefit from mentoring relationships with exemplary faculty, so can adjunct faculty (and in turn their students) benefit from being paired with outstanding full-time faculty members.

The Curriculum

The American Mathematical Association of Two-Year Colleges (AMATYC), in its 1995 publication *CROSSROADS IN MATHEMATICS: Standards for Introductory College Mathematics Before Calculus*, stated a vision and standards for two-year college mathematics. In its 1989 *Standards* document, the National Council of Teachers of Mathematics (NCTM) recommended changes in school mathematics. Recommendations for the college curriculum were proposed by Mathematical Association of America (MAA) publications including *A Curriculum in Flux* and *Reshaping College Mathematics.* National Research Council reports addressing changes in mathematics education include *Everybody Counts* and *Moving Beyond Myths*. Building on these reports and publications, as well as the calculus reform movement, the AMATYC Standards address standards for content, pedagogy, and intellectual development as follows:

- **Standards for intellectual development** address desired modes of student thinking and represent goals for student outcomes.
- **Standards for Content** provide guidelines for the selection of content to be taught at the introductory level.
- **Standards for Pedagogy** recommend the use of instructional strategies that provide for student activity and interaction and for student-constructed knowledge.

Mathematics-intensive majors take a variety of mathematics courses offered by two-year colleges, ranging from developmental (or preparation) courses, precalculus, introductory calculus, multivariate calculus, linear algebra, differential equations, discrete mathematics, and statistics. Students needing preparatory mathematics courses before enrolling in college-transfer courses study “the Foundation,” a common core that meets individual student needs, provides multiple entry points and a mathematical basis for several career paths, and equips students with the ability and confidence to study higher level mathematics. Content focus in the Foundation emphasizes number sense, symbolism and algebra, geometry and measurement, functions, discrete mathematics topics, probability and statistics, and deductive proof. The *CROSSROADS* offers particular guidance for students selecting mathematics-intensive programs at two-year colleges. Faculty must demand quality performance from mathematics-intensive majors, who will become the mathematicians, scientists, engineers, and economists of the future. These students must engage in sustained work in challenging courses that offer a rich variety of mathematical experiences. Many students at two-year colleges will need a precalculus course before entering calculus. As indicated in *CROSSROADS*, the content in the precalculus course should emphasize the understanding of function: linear, power, polynomial, rational, algebraic, exponential, logarithmic, trigonometric, and inverse trigonometric. Elements of discrete mathematics leading to modeling and problem solving are important. The use of real data, coupled with the study of probabilistic models and statistical inference should be included in statistics courses offered at two-year colleges. Two-year college students interested in mathematics-intensive programs will build upon precalculus coursework with a calculus sequence including the study of limits, differentiation, integration, linear algebra, multivariate calculus, and differential equations. An emphasis on the use of technology in instruction is common at two-year colleges—community colleges help narrow the digital divide by providing computer skills to a substantial number of students.

The writers of the National Research Council report *Transforming Undergraduate Education in Science, Mathematics, Engineering, and Technology* articulate a vision for undergraduate courses in mathematics and science that provide "diverse opportunities for all undergraduates to study science, mathematics, engineering, and technology [SME&T] as practiced by scientists and engineers, and as early in their academic careers as possible." In particular, Vision 2 states that:

> SME&T would become an integral part of the curriculum for all undergraduate students through required introductory courses that engage all students in SME&T and their connections to society and the human condition. (p. 25)

In concert with the AMATYC *Crossroads*, faculty are challenged to design introductory courses to meet the needs of a diverse student population through intellectually challenging courses. In both content and approach to the discipline, introductory courses should encourage many students to continue their study of SME&T, serving as "pumps" rather than "filters." Inherent in the open access principle of two-year colleges is course design that accommodates a diverse group of students who exhibit differing educational backgrounds, experiences, interests, aspirations, and learning styles. Students' experiences with mathematics in introductory level courses are pivotal and influential in determining career choices.

Research opportunities, also recommended by the *Transforming* Report, are also available at two-year colleges. Although disciplinary research by faculty is not generally a part of the mission of most two-year colleges, individual mathematics faculty may engage in research as part of their regular professional development. Opportunities for two-year college students to engage in supervised research in mathematics or a mathematics-related area are available at some two-year colleges.

Two-year colleges are prime locations for developing a scholarship of teaching, critical to providing excellent undergraduate instruction. An article in "The Landscape" feature of the November/December 1998 issue of *Change* reports the results of a survey of two- and four-year college faculty conducted by the Carnegie Foundation for the Advancement of Teaching. The survey results cite the successful history of two-year colleges in focusing on teaching and suggests that community college faculty would be excellent in developing an environment that supports "innovation, reflection, and conversation about teaching and learning across all colleges and universities." Community colleges—"the nation's premier teaching institutions"—can lead postsecondary faculty in the identification and development of best practices.

Recruitment Issues

Two-year colleges, with their significant minority population, are prime sources for the recruitment and preparation of a diverse work force of the mathematicians, scientists, and engineers of tomorrow. Grant-funded opportunities such as those available through the National Science Foundation's Alliance for Minority Participation Program demonstrate successful collaborations between two- and four-year colleges to increase the numbers of minority students involved in the science, engineering and mathematics disciplines. AMATYC has adopted in its Strategic Plan for 2000–2005 the objective to promote classroom, department, and campus environments that encourage members of underrepresented groups to succeed in and further their study of mathematics. This objective will be implemented through strategies that permeate the organization's activities—through its committees, its professional development opportunities, and more. It is incumbent upon all those involved in postsecondary mathematics to encourage and nurture ***all*** students in their study of mathematics, particularly those who are from underrepresented groups or who are considered at-risk.

Teacher Preparation

Many future K–12 teachers enroll in two-year colleges and transfer to a college of education at a four-year institution to complete their degree and certification requirements. The critical introductory level courses in mathematics and science are taken by these prospective teachers during their enrollment at the two-year college. The quality of introductory mathematics and science courses at two-year colleges will influence the knowledge and skills these future teachers take to their own classrooms. Prospective secondary mathematics teachers who start their postsecondary education at two-year colleges will complete the necessary building blocks in their understanding of mathematics while enrolled in the two-year college. Mathematics faculty should model the pedagogy that students will later use in their classrooms and incorporate the recommendations of the recently-released NCTM *Principles and Standards for School Mathematics*.

Connections with colleges of education are critical. Two-year college faculty in mathematics should establish partnerships with the institutions to which they feed students for the benefit of their students who are future teachers. These partnerships should involve arts and sciences faculty as well as faculty in colleges of education, and should revolve around the goal of providing quality programs at two-year colleges and a smooth transition to the four-year institution for prospective teachers. In particular, two-year colleges can offer students early field experiences such as Teaching Apprentice Programs with guided classroom observation in area school districts, establish mentor relationships with experienced schoolteachers, link with peers at four-year institutions, and initiate future teacher clubs.

The National Science Foundation report *Investing in Tomorrow's Teachers: The Integral Role of Two-Year Colleges in the Science and Mathematics Preparation of Future Teachers* addresses a vision and recommendations for the involvement of two-year colleges as full partners in teacher preparation discussions. The report calls for two-year college action in these areas: recruitment of prospective teachers, strengthening of undergraduate SME&T courses, pre-teaching experiences, in-service activities, liaisons between two-year colleges and four-year institutions, and connections with business and industry, professional societies, and other organizations. In-depth information about eleven two-year colleges recognized by the National Science Foundation in 1998 for their exemplary activities in teacher preparation is presented in the Spring 1998 issue of *The Journal of Mathematics and Science: Collaborative Explorations*. In addition, National Science Foundation-funded projects such as the Collaboratives for Excellence in Teacher Preparation have linked community colleges with their four-year partners in successfully encouraging students to choose mathematics or science teaching as a specialization. (See the NSF website, `www.nsf.gov`, for descriptions of the collaborative projects.) With their geographic accessibility in the communities (particularly urban and rural) where teachers are needed most and their focus on teaching excellence, two-year colleges are uniquely positioned to recruit and prepare future teachers of mathematics. Four-year institutions must recognize, and in turn act upon, the important role of two-year colleges in providing prospective teachers, particularly minorities.

Distance Learning

Mathematics instruction is no exception to the recent popularity of distance education courses. Many college and universities, including two-year institutions, are turning to distance learning for a variety of reasons, among them the challenge to reach and meet the needs of new student populations and the ever-present institutional concern about fiscal efficiency. A report issued by the National Center for Education Statistics cites data from 1997–98 indicating that:

- 48% of the postsecondary institutions offering distance education courses are two-year colleges;
- 43% of the approximately 1.66 million students enrolled in distance education courses are in two-year colleges, and 30,830 are enrolled in college-level, credit-granting mathematics (or statistics) courses at these two-year colleges;
- 860 different college-level, credit-granting mathematics (including statistics) distance education courses were offered by public two-year colleges, as compared to 600 at public four-year institutions.

Virtually all disciplines are being affected by distance learning initiatives. The full impact on mathematics-intensive majors has not yet been realized. Students have many choices in where and how to take postsecondary mathematics—institutional and departmental survival may hinge on the ability to teach in virtual classrooms of the future, with nontraditional calendars. As calculus and other courses are delivered via distance means, it is critical that high quality be maintained in order to ensure the promotion of effective learning environments for all students studying mathematics.

Articulation

Articulation issues are particularly important in two-year colleges. The interface between high school and college as well as that between two-year and four-year institutions are of concern. An Articulation Workshop convened in February 2000 by the Mathematical Sciences Education Board engaged professional societies and other representatives in discussions centered around three key principles: alignment of expectations across institutions and with assessments, openness of expectations (including assessments), and collaboration across and within institutions. Recommendations for further action clustered in the areas of curriculum, assessment, and policy, including the possible development of a set of common expectations for mathematics in grades 11–14 and the collection of information about placement tests and their uses.

Two-year college students who transfer to four-year institutions can choose from among many four-year institutions, each with its own requirements for mathematics-intensive majors. It is indeed difficult for the two-year college curriculum to meet the needs of many transfer institutions that do not agree with each other on the preferred curriculum for their native students. Many states, regions, or local higher education alliances have developed articulation agreements to benefit transfer students. Such agreements enable students to transfer in with junior status provided they have graduated from a participating two-year college with an Associate in Arts or Associate in Science degree in an approved major. These agreements on the surface seem beneficial, but their implementation is not always smooth. The courses reflected in articulation agreements are typically general education courses, and do not necessarily prepare students or meet prerequisites for upper division coursework in a particular major. Effective transfer policies will advance students towards the bachelor's degree. Students not completing an associate degree who choose to transfer courses piecemeal may find transfer more problematic, as not all courses they have completed at the two-year college will necessarily be accepted for credit towards graduation at the four-year institution. Similarly, the student may be required to take freshman- or sophomore-level courses offered by the four-year institution as an upperclassman if such courses do not have a natural equivalent at or are not offered by the two-year college.

An issue related to articulation is that of technology usage by students and faculty in mathematics courses. While two-year colleges are frequently on the cutting edge of the use of technology in mathematics courses, the particular type of technology used is different from that used at four-year institutions (for example, graphing calculator vs. computer software). An emphasis on technology as a tool in learning, not as an end in itself, will help minimize difficulties encountered by students who transfer between institutions using different forms of technology.

As four-year colleges revise existing courses and develop new courses, two-year colleges may not be able to respond in a timely way that benefits students. For example, four-year colleges that offer interdisciplinary approaches to general education for their native students may find that incoming transfer students from two-year colleges have separate disciplinary courses that do not align with the interdisciplinary courses.

While data from some four-year colleges show that two-year college students who transfer in perform as well as (or better than) native students, the perceptions among four-year college faculty may reflect an opposite view. Thoughtful and well-publicized research based on accurate data on student performance of two-year transfers vs. four-year college native student performances will aid in aligning perception with reality.

An ingredient contributing to successful articulation programs for two-year college transfer students is strong collaboration between two- and four-year colleges. Successful articulation programs often reflect positive, professional working relationships among faculty at the designated institutions that promote understanding and foster development of policies that do not hinder students' progress. The goal of seamless transitions from high school → two-year college → four-year college or university, high school → four-year institution, or high school → two-year college (for those students not transferring) requires much more attention. The new CUPM recommendations should address the development and implementation of successful articulation policies that facilitate transfer of mathematics-intensive majors from two-year colleges to four-year institutions. Novel approaches, such as focusing on outcomes and competencies in the disciplines that students can achieve in a variety of course packages, should be considered.

Summary

President Bill Clinton has said, "Community colleges are America at its best." The two-year college provides unparalleled access to high quality postsecondary education in the communities where students live and work. Mathematics-intensive majors who begin their collegiate work at two-year colleges receive mathematics instruction at the introductory level from faculty who are committed to teaching, in small classes, and with access to technology and support services. The recruitment potential for the next generations of mathematics-intensive majors from the diverse two-year college student population should not be minimized. The AMATYC *Crossroads* document addresses the curriculum for introductory college mathematics courses before calculus. Provisions should be made within two-year colleges for students who want to teach secondary mathematics in order to ensure that they receive rich content delivered using effective pedagogical techniques and up-to-date technology. Partnerships with four-year institutions are important not only for the smooth transfer of future K-12 teachers, but also for effecting articulation of all transfer students. Further attention is necessary to collaboration among high schools, two-year colleges, and four-year institutions, for the purpose of furthering articulation issues. Two-year college mathematics faculty must be partners in the important discussions and resulting formulation of recommendations for the undergraduate program in mathematics.

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