Part I. Recommendations for Departments, Programs, and all Courses in the Mathematical Sciences

1. Understand the student population and evaluate courses and programs

Mathematical sciences departments should

- ***Understand the strengths, weaknesses, career plans, fields of study, and aspirations of the students enrolled in mathematics courses;***
- ***Determine the extent to which the goals of courses and programs offered are aligned with the needs of students, as well as the extent to which these goals are achieved;***
- ***Continually strengthen courses and programs to better align with student needs, and assess the effectiveness of such efforts.***

Higher education has changed dramatically since CUPM first convened in 1953. In that era barely 12% of secondary school graduates went on to college, and only about 60% of the eligible cohort were even getting through secondary school. Those who did go to college typically went as full-time students, completing a degree in the traditional four years.

Today, approximately 85% of the eligible cohort complete secondary school, and 25% of secondary school graduates complete at least four years of college.[17] Part-time study is common, especially at state universities. Two-year colleges enroll 44% of the undergraduates in the U.S. and 49% of those undergraduates who identify themselves as members of racial or ethnic minorities.[18] The college population also is more heterogeneous in ways that were inconceivable in the 1950s. Students bring a wide range of mathematical backgrounds to college. Some come with a full year of Advanced Placement calculus or International Baccalaureate mathematics. Many more enter with algebra as their most advanced mathematics subject.

Understand student backgrounds and needs. These changing demographics increase the responsibility of mathematics departments to understand the strengths, weaknesses, career plans, fields of study, and aspirations of students enrolled in mathematics courses. Given the large number of students enrolled, gaining this understanding is a major undertaking. However, it is essential. For example, information about what

---

[17] See `www.ed.gov/pubs/YouthIndicators`, Indicators 23 (School Enrollment) and 26 (School Completion).

[18] See "First Steps: The Role of the Two-Year College in the Preparation of Mathematics-Intensive Majors," by Susan S. Wood, in *CUPM Discussion Papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know?*, MAA Report, 2001. Also see *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus*, American Mathematical Association of Two-Year Colleges, 1995.

mathematics majors do after graduation is critical when designing programs for majors. What percentage go on to graduate programs in mathematics? to graduate programs in other fields? What fraction goes directly into the non-academic job market? Similarly, if students enrolled in a college algebra course will not need the technical computational skills typically emphasized in such a course, they should instead take a course focusing on concepts, applications and general mathematical thinking.

Many mathematics departments use a placement test to sort students into entry-level courses. Such a test may provide useful information on mathematical preparation, but it gives little or no information about the test takers' actual mathematical needs or academic interests.

One way to determine student needs in their major programs and desired careers is to consult with colleagues in other disciplines. Find out *what they think the mathematics department is teaching* and *what they think the department should be teaching*. CUPM has started this process through a series of workshops known as the Curriculum Foundations project, with partner disciplines ranging from information technology to mechanical engineering (see Appendix 2). The results are eye opening. It is a misimpression to think that colleagues in these disciplines only want students who can evaluate a logarithm and solve a differential or quadratic equation. Of course faculty in other disciplines want students to possess the computational skills required for their subjects. But they especially want students to possess conceptual understanding of the required mathematics, to have some experience with mathematical modeling, and to have the communication skills necessary for explaining the methods used to solve problems and the meaning of the solutions. The Curriculum Foundations reports can provide excellent starting points for such discussions on local campuses.

Determine if department programs serve student needs. The information gathered can serve as the basis for evaluating how well the goals of department courses and programs are aligned with student needs and the extent to which these goals are achieved. These efforts should encompass much more than the traditional classroom-level monitoring of student learning; department-level effectiveness of programs and personnel must be evaluated as well. Departments should regularly review course offerings and the topics within these courses, as well as the methods being used to teach and assess them.

The process of aligning the curriculum with student needs should be informed by sources outside of the mathematics department as well as within it. As faculty consult with colleagues in other disciplines about content, they also should discuss means of assessing how well the mathematics curriculum serves the needs of partner disciplines. Mathematics departments should further consult with alumni, employers and graduate programs at other institutions to obtain their impressions of the effectiveness of the mathematics curriculum and how the department could improve its program.

This may sound like a call for constant monitoring and micromanagement, but in fact it can be simple and direct. Faculty can collect much of the necessary information through simple questionnaires and informal conversations (as demonstrated by the examples in Illustrative Resources). Gradual modifications can be made to course syllabi, and information can be informally disseminated at department meetings and discussions over lunch. Faculty mentors can assist less experienced colleagues by initiating reciprocal classroom visits and taking time outside of class to talk about teaching ideas. Textbook adoption committees can use the assignment to revisit assumptions about courses.

In her essay on accountability in mathematics, Sandra Keith calls attention to a number of techniques to make the process non-disruptive and manageable, including starting simply using information already available, looking for "good news" to report and subsequently upgrading the relevant data, and tracking student achievement in courses.[19]

---

[19] "Accountability in Mathematics: Elevate the Objectives!" by S. Keith, in *CUPM Discussion papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know?* MAA Reports, 2001, p. 61.

Strengthen alignment and assess effectiveness. Strengthening programs to better align with student needs requires careful assessment of the success of such efforts. In 1995 CUPM endorsed a set of guidelines for establishing a cycle of assessment aimed at program improvement. They include the recommendation that mathematics departments ask three questions:

- What should students learn?
- How well are they learning?
- What should departments change so that future students will learn more and understand it better?

These questions provide a foundation for thinking about assessment. *Assessment Practices in Undergraduate Mathematics* (Gold *et al.*, 1999) contains over seventy case studies of assessment in mathematical sciences departments across the U.S. It provides a rich resource of examples to illustrate how assessment can be achieved in practice.[20] With support from the National Science Foundation (NSF), MAA is conducting a series of faculty development workshops in the area of assessment. The three-year project, entitled "Supporting Assessment in Undergraduate Mathematics" (SAUM), includes a workshop series, a volume of case studies and syntheses of case studies on assessment, and an informational website (`www.maa.org/saum/`). Further information about SAUM and other assessment resources developed by NSF and the National Academy of Science is included in Illustrative Resources.

2. Develop mathematical thinking and communication skills

Every course should incorporate activities that will help all students progress in developing analytical, critical reasoning, problem-solving, and communication skills_and acquiring mathematical habits of mind. More specifically, these activities should be designed to advance and measure students' progress in learning to

- ***State problems carefully, modify problems when necessary to make them tractable, articulate assumptions, appreciate the value of precise definition, reason logically to conclusions, and interpret results intelligently;***
- ***Approach problem solving with a willingness to try multiple approaches, persist in the face of difficulties, assess the correctness of solutions, explore examples, pose questions, and devise and test conjectures;***
- ***Read mathematics with understanding and communicate mathematical ideas with clarity and coherence through writing and speaking.***

Terms like "analytical thinking" or "mathematical reasoning" are often used to describe those habits of mind that make employers want to hire mathematics majors, that lead departments and colleges to require mathematics courses, and that entice bright people into the further study of mathematics. Indeed, a striking finding from the Curriculum Foundations workshops[21] is that, contrary to the belief of many mathematicians, colleagues in partner disciplines (e.g., engineers, economists, and natural and computer scientists) value the precise, logical thinking that they perceive to be an integral part of mathematics and would like more emphasis on it in early collegiate mathematics instruction. Here are several such statements from the Curriculum Foundations reports:

- *Biotechnology and Environmental Technology*: "Requiring students to write an explanation of how they arrived at the result/answer and how they interpreted their results should reinforce writing skills and deductive reasoning."

---

[20] The entire volume, along with the MAA Assessment Guidelines, is available online at `www.maa.org/saum/`

[21] See Appendix 2 for a description of the Curriculum Foundations project.

- *Business*: "Courses should stress problem solving, with the incumbent recognition of ambiguities… [and] conceptual understanding (motivating the mathematics with the 'whys'—not just the 'hows')."
- *Chemistry*: "Logical, organized thinking and abstract reasoning are skills developed in mathematics courses that are essential for chemistry. At the physical chemistry level students must be able to follow logical reasoning and proofs, which is enabled by previous experience in mathematics courses."
- *Computer Science*: "Students should be comfortable with abstract thinking… they should have some facility with formal proofs, especially induction proofs."
- *Health and Life Sciences*: "Basic concepts that should be mastered include:… logic and mathematical thinking, generalization, deductive reasoning. At the conceptual level, students should be able to explain … concepts in words."
- *Physics*: "The ability to actively think is the most important thing students need to get from mathematics education.… students should know that being able to integrate is quite different from understanding what integration is.… They must go beyond 'learning rules' to develop understanding."

Participants at the Curriculum Foundations workshop on the preparation of prospective mathematics majors also expressed this conviction in several parts of their report:

> The most important task of the first two years is to move students from a procedural/computational understanding of mathematics to a broad understanding encompassing logical reasoning, generalization, abstraction, and formal proof. The sooner this can be achieved, the better.... There should be an attempt to phase in logical language starting in the freshman year, rather than a sudden jump in the sophomore or junior year.... we should recognize our students' great variety of individual, cultural, and educational backgrounds. Students come with vastly different experience, skills, and learning styles. Introductory courses should provide experiences flexible enough to allow every student the opportunity both to reinforce existing strengths and to fill gaps.

Mathematical reasoning and problem-solving skills take a long time to develop and improvement is incremental. This has contributed to a growing recognition among mathematics departments of the need to consciously help students improve their reasoning and problem-solving skills. One result has been the increasing availability of transition-to-higher-mathematics courses and freshman-level discrete mathematics courses with a focus on logical argument and writing simple proofs.[22] The calculus renewal movement also has emphasized mathematical reasoning and problem solving, and modern statistics courses have been evolving along similar lines. Efforts are now underway to develop pre-calculus and other introductory mathematics courses according to the same principles.

Focusing on the development of reasoning skills from the earliest courses will

- improve the overall understanding of the nature and importance of mathematics, and
- elevate the level of mathematical competence in society.

Such a shift in focus also serves the self-interest of mathematical sciences departments because success will likely

- increase the pool of students capable of succeeding in higher-level mathematics courses, and
- encourage more of them to choose to enroll in higher-level mathematics courses.

[22] It is worth noting that much of the impetus for discrete mathematics courses has come from computer science departments, which have valued the contribution of these courses to students' general intellectual development as much—or more—than the specific topics they contain.

Learn to apply precise, logical reasoning to problem solving. Many students enter college with underdeveloped reasoning skills.[23] When a syllabus requires many topics to be covered in a short time and students' analytical skills are inadequate, emphasis on "problem solving" can easily turn into showing students how to imitate template solutions without real understanding. Even in linear algebra courses, sometimes a first introduction to definition and proof, pressure to teach computational techniques may override the goal of leading students to understand fundamental mathematical relationships.

Problem solving does not mean following recipes—it requires the application of careful, organized, creative thought to the analysis of complex and often ill-defined questions in a diverse array of situations. Problem solving is thus intimately connected with fundamental mathematical reasoning. Throughout any complex problem-solving process, bits of deductive logic, small proofs, and counterexamples are intermixed with visualization, flashes of intuition, and memory retrieval. It is thus artificial and ill advised to separate instruction in problem solving from other kinds of mathematical thinking.

An instructor can teach logical reasoning by devising classroom activities and assignments that focus on thinking skills. For example, an instructor can introduce a few examples of everyday "if-then" statements with the same logical interpretation as logical if-then statements and then ask the students to produce similar examples on their own. From this exercise students should observe the analogy between the mathematical logic and the everyday statements they have been considering. Or, if a student tries to justify a fact with a few examples, an instructor can give various properties that are sometimes true and sometimes false and have students find instances of both, illustrating the invalidity of "proof by example."

It takes time and effort to help students become better thinkers. Faculty must therefore seek a balance between expanding content and the need to help students improve their analytical, reasoning, and problem-solving skills. Faculty should realize that, while many students enter college with underdeveloped reasoning skills, the effort to help them improve is important. Even modest success makes a difference in students' lives, and when faculty approach the task with courtesy and respect, students appreciate their efforts.

Develop persistence and skill in exploration, conjecture, and generalization. Problem solving requires more than just solid mathematical reasoning — there are broad strategies and mental attitudes that students must identify, master, and internalize. To be successful problem solvers, students must learn persistence in the face of repeated rebuffs and flexibility in the choice of solution strategies. They must replace the question "did I get the right answer?" by "does my solution make sense?" Students must also learn to explore examples and special cases, to let new knowledge lead to new questions, to generalize and pose conjectures, yet to test all conjectures and retain a healthy skepticism toward unproven claims.

Instructors can stimulate students to generate questions and comments in response to readings, exercises, and presentations by modeling good questioning behavior: What do the words mean? What are some non-trivial examples? What motivates the material? What assumptions are being made? How do I know this is right? Through careful choice of problems and dialogue with students, faculty also can lead students to develop a more skeptical stance toward assertions: Does this make sense? Have all assumptions been enunciated?

Students need to be exposed to multi-stage projects that are built on exploration and conjecture and require persistence and flexibility for their solutions. More generally, at least some courses should be restructured to shift the burden from instructor to students for discovering and justifying results. A mathematical modeling course is an especially apt setting in which to make this shift and to raise students' awareness of the need to state problems carefully, articulate assumptions, and apply the mental and strate-

---

[23] See Illustrative Resources for some of the evidence of these difficulties.

gic tools of effective problem solving. Indeed, the value of modeling lies at least as much in the artful and creative thinking and thoughtful interpretation that it requires as in the connections it makes between mathematics and other disciplines.

Read and communicate mathematics with understanding and clarity. When employers are asked to list the qualities they seek in those who work for them, communication skills are invariably put first.[24] Mathematics faculty have an important share of the responsibility to help students improve their ability to communicate about technical matters. Most importantly, requiring students to read, write, and speak about mathematics helps them learn while providing instructors with valuable insight about levels of student understanding.

Some mathematics courses inadvertently encourage poor habits in reading, writing, and speaking. Instructors often do not require students to read the textbook, giving lectures that essentially reiterate it. In such situations, students mostly just scan the textbook for templates relevant to assigned homework problems. In fact, it is clear that many students don't know how to read mathematics. Students don't learn to read mathematics just because faculty ask them to do so. For these reasons instructors in most courses will need to continue to present material that is also in the text. But many instructors are demonstrating that it is possible gradually to teach students to read mathematics (see Illustrative Resources for examples).

Classroom discussion and informal oral presentations are important (but often overlooked) ways to help students improve thinking and problem-solving skills. For example, students can write homework solutions or partial solutions on the board each day at the beginning of class. Lively class discussions can arise from the students' board work, and they can set the tone for the rest of the class period. Many instructors develop problem solutions or proofs of simple statements by working in concert with their students, calling on individual students to explain what needs to be done next. In this way, students build confidence in their problem-solving abilities, while the instructor monitors the general level of understanding in the class.

Group work can further encourage students to verbalize mathematics both in and out of the classroom. Three MAA publications[25] describe the nuts and bolts of organizing cooperative work in a mathematics class. Rogers et al. (2001) includes the results of a faculty survey about the use of groups. Seventy-nine of the 94 faculty members surveyed reported that their students reacted "positively" or "very positively" to cooperative learning.

One of the most significant changes in mathematical pedagogy over the past couple of decades has been the increasing use of writing as a pedagogical tool. Requiring students to write about mathematics helps them learn and gives instructors valuable insight into the nature of their understanding. Some mathematics faculty are reluctant to require writing in the belief that they do not have the training to evaluate students' work. However, as many mathematics faculty are now demonstrating, grading and commenting on certain kinds of writing assignments does not require special training. For example, students can be given mathematical questions and asked to provide short written responses; the product is writing that all mathematicians can reasonably assess. Simply requiring answers in complete sentences can be a first step toward helping students better communicate about mathematical topics. Teaching them to write "From A, I know B because of C" can promote real growth in mathematical thinking. At more advanced levels, mathematics majors can be required to write substantial papers and present their results to faculty and students.

---

[24] See for example the websites `www.careers-in-business.com`, `www.siam.org/mii` and `www.naceweb.org`.

[25] *A Practical Guide to Cooperative Learning in Collegiate Mathematics*. Nancy L Hagelgans et al. eds. Mathematical Association of America, 1995; *Readings in Cooperative Learning for Undergraduate Mathematics*. Ed Dubinsky, David Mathews, and Barbara E. Reynolds, eds., Mathematical Association of America, 1997; and, *Cooperative Learning in Undergraduate Mathematics*. Elizabeth C. Rogers et al. eds. Mathematical Association of America, 2001.

The time needed to read and evaluate student writing can be reduced by a variety of techniques: cutting back on the number of assignments, asking for just a few sentences rather than lengthy essays, or having students write group reports. Some institutions make use of graduate or even undergraduate assistants to help with writing instruction and evaluation. Sessions at national and regional MAA meetings provide opportunities for faculty to learn additional ways to make student writing and presentations both beneficial and efficient. Examples of both modest and ambitious writing assignments can be found in Illustrative Resources, along with information about other useful resources.

3. Communicate the breadth and interconnections of the mathematical sciences

Every course should strive to

- ***Present key ideas and concepts from a variety of perspectives;***
- ***Employ a broad range of examples and applications to motivate and illustrate the material;***
- ***Promote awareness of connections to other subjects (both in and out of the mathematical sciences) and strengthen each student's ability to apply the course material to these subjects;***
- ***Introduce contemporary topics from the mathematical sciences and their applications, and enhance student perceptions of the vitality and importance of mathematics in the modern world.***

Mathematics is built from a rich variety of topics, perspectives, and applications. It includes the study of number systems, limits and calculus, algebraic functions and geometric transformations, deterministic and stochastic processes, topological shapes and geometric spaces, the separating of pattern from noise in real-world data, and the nature of logic itself. Great leaps of intuition are balanced by precise logical arguments. Practical problem solutions and the careful analysis of data are supported by powerful theoretical structures. Mathematics is inspired by the problems inherent in understanding the world, and it is strengthened and renewed by finding new ways to frame and answer questions about this world, both the physical world and the world of the intellect. Every mathematics department, in each of its courses, has a responsibility to communicate this mathematical richness, power, and beauty.

Within individual courses instructors should seek opportunities to present a broad range of perspectives on the material. It is impossible to incorporate every single viewpoint within every single course, but for every institution it is possible to include a broadly representative range of mathematical ideas and viewpoints within the full set of undergraduate courses. Departments should offer opportunities for students at various levels of mathematical sophistication to explore analytic, algebraic, geometric, discrete, statistical, and applied questions at a depth that is both accessible and challenging to them. And departments should ensure that their offerings include exploration of contemporary topics and applications, to communicate that mathematics today is alive and ever changing.

The phrase "strive to" in Recommendation 3 means that every course should make progress toward reaching the specified goals. Some of the goals are more accessible than others, and thus the following discussion elaborates not only the rationale for each goal but also the challenges to achieving it.

Present key ideas and concepts from a variety of perspectives. Key ideas and concepts should provide the intellectual and pragmatic themes that unify a course. These themes need to be developed from several perspectives for students to achieve the depth of understanding required for application of the material.

In recent years the undergraduate mathematics curriculum has broadened to include a greater variety of content perspectives. This trend needs to continue and deepen. The curricular emphasis in traditional mathematics instruction was on correct theoretical development of the material, with an emphasis on algorithms based on symbolic algebraic manipulation leading to exact solutions. Such a perspective on mathematics was, and still is, important. But there are other important perspectives that have been underrepresented in the traditional curriculum.

Every mathematics course—from introductory courses for liberal arts students to courses for mathematics majors—should offer opportunities for students to explore mathematical ideas from this variety of perspectives. Students differ in how they tackle problems and process information. The more varied the discussion of a topic, the better the chance that each student will find something that makes the idea understandable and memorable. This is not just good pedagogy; it enriches students' understanding of the nature of mathematics.

The theoretical perspective, based on logical accuracy in statement and verification, is a bedrock foundation for mathematics. But an overemphasis on this perspective can have negative effects on student understanding and motivation, especially in introductory courses. Developing intuitive conceptual understanding is a practical perspective that needs more exposure. Determining and achieving the proper balance is not always easy, but neither perspective can be ignored.

Developing appropriate symbolic manipulation skills will always be a goal of mathematics instruction. But in addition graphical and numerical understanding must be developed. Viewing topics from these analytical, graphical and numerical perspectives is necessary for true understanding. For example, in calculus courses students should experience geometric as well as algebraic viewpoints and approximate as well as exact solutions. In linear algebra students should learn to "see" the null space of a matrix as well as to compute a basis for it. In statistics students must recognize the geometry behind linear regression as well as how to apply it to answer meaningful questions about real data.

The traditional curriculum does not make clear the rarity of situations in which exact solutions are possible, and hence the need to balance their study with that of approximate solutions. Exact solutions are important precisely because they are rare, and this needs to be communicated. At the same time, students need to be equipped to handle situations in which exact solutions are not readily available. Evaluation of integrals is one example of a situation in which exact solutions are not common. Students need to understand the importance of the exact techniques that are available. At the same time, more emphasis is needed on proficiency with techniques of approximation.

Employ a broad range of examples and applications to illustrate and motivate the material. Examples and applications can often illustrate, explain, and justify the logical structure of a mathematical subject better than an unadorned theoretical development. Applications help students move beyond an algorithmic approach to mathematics to grapple with what concepts mean and how they are used. When students look back on a course, it is often the examples and illustrations that are most memorable. Authentic and interesting (and sometimes surprising) applications can be powerful hooks drawing a student's interest into the mathematics under study. For these reasons, every course—from the most basic to the most advanced—should include a rich assortment of examples and applications.

A growing number of textbooks now include such examples and applications. It is not necessary to be an expert in another discipline or spend substantial time providing background for students in order to incorporate them.

For instance, exposing students to numerous examples of the accumulation of a varying quantity as a generalization of a simple rule about accumulation of a constant quantity helps them understand the definite integral. Such examples include generalizing the volume of a cylinder to volumes of solids of revolution, generalizing "work equals force times distance" to work as an integral when the force varies, or generalizing simple interest calculations into an integral formulation for the present value of an income stream. Radiocarbon dating illustrates how an initial value problem (IVP) can model a real world situation, and the solution of the IVP then yields obviously useful and interesting results. It also is impressive to learn how a simple system of differential equations predicts the cyclical population swings in a predator-prey relationship, how the singular-value decomposition of a matrix facilitates the compression of information, and how modular arithmetic is used in cryptography and the transmission of encoded information.

Non-trivial applications can also lead to unanticipated mathematical questions. For example, when constructing a bio-economic model for the Maine lobster fishery based on the logistic equation, one is faced with having to model a population for which no direct population data are available: the available data consist of the tonnage of catch each year along with the number of traps in use each year. Drawing useful conclusions from these unusual data poses interesting mathematical and statistical problems.

Many examples and applications fall naturally under the rubric of mathematical modeling, and it should receive increased emphasis. Modeling requires students to translate back and forth between verbal descriptions and mathematical expressions, to make their assumptions explicit, and to decide which details are important and which should be ignored. Exposure to modeling will help students consolidate their mathematical understanding and empower them to use their understanding in other courses. Modeling also gives students practice in framing questions, trying multiple approaches, and interpreting results. Further, it provides effective topics for student writing.

According to the summary report of the Curriculum Foundations project, "The importance ascribed to mathematical modeling by *every* disciplinary group in *every* workshop was quite striking."[26] While the relative emphasis on applications will vary from one course to another, every course in the undergraduate mathematics program—from the most basic to the most advanced—should strive to include meaningful applications that genuinely advance students' ability to analyze real-life situations and construct and analyze appropriate mathematical models. It should be emphasized that modeling need not be thought of as restricted to major projects but can occur in small ways throughout a course. Often insights that instructors take for granted as "obvious" are far from obvious to their students.

Good examples don't always have to come from applications. Puzzle problems couched in everyday language also can be very effective. Solving the Tower of Hanoi puzzle for increasing numbers of disks is an excellent illustration of recursive thinking and can demonstrate the usefulness of proof by induction. Rubik's Cube is a rich source of group theory problems.

Examples can also come from mathematics itself. For instance**,** students may enjoy calculating pi by numerically estimating a definite integral or by finding partial sums of a series. After computing characteristic polynomials, students can use the power method to converge on the largest eigenvalue of a big matrix. And as shown in George Pólya's classic video "Let us teach guessing," students' creative imagination may be engaged in trying to determine the number of spaces created by five planes randomly positioned in space.

However, more substantial applications appropriate for instructional use are also needed, and they are neither trivial to find nor easy to frame effectively for instruction. Indeed, substantial applications—though valuable—can be quite difficult for students because they must not only learn the mathematics, but also learn the relevant areas of the discipline from which the application is drawn.

Good applications and examples should be disseminated to others in the mathematical community. Published texts and papers, such as those produced by the MAA and COMAP, provide part of the solution, and dissemination via Internet sites, such as the one for the Journal of Online Mathematics (JOMA), is of growing importance.

Published texts and papers provide part of the solution, but dissemination via Internet sites is of growing importance. The mathematical community as a whole should put more effort and resources into development

[26] "A Collective Vision: Voices of the Partner Disciples," *The Curriculum Foundations Project: Voices of the Partner Disciplines*, p. 4.

and dissemination of applications. And individual faculty should contribute when they can and take advantage of what is available when they cannot.

Make connections to other subjects and apply the course material to these subjects. Students often are totally unaware of important connections between separate mathematical subjects and between mathematics and other disciplines. This is a serious deficiency for both the understanding of the mathematics and for the appreciation of its importance and use.

Also, students are surprisingly reluctant or unable to apply the knowledge they obtain in mathematics courses to other disciplines. This "transfer problem" is a serious weakness in current mathematics instruction. It is not reasonable to expect students to become sophisticated modelers and applied mathematicians with a few courses nor to expect that there are guaranteed solutions to the transfer problem. However, there are strategies that are commonly understood to be helpful.

For instance, one reason students encounter difficulty in applying mathematics to problems in other disciplines is that they have trouble identifying appropriate mathematical procedures when problems are expressed with different symbols than those used in the mathematics classroom. To counter this problem, textbooks and instructors can go beyond conventional x, y notation to use a larger collection of symbols for both constants and variables. It is tempting to stay with x's and y's because extra class time is often needed to help students become accustomed to alternate symbolism. But the very fact that this is the case underscores the importance of incorporating the activity in courses.

Students' failure to recognize connections with other disciplines extends to mathematics itself: undergraduates (including mathematics majors) often see mathematics as a collection of separate, isolated subdisciplines with little interplay between the separate areas. This is a flawed and limiting picture of the discipline. The unity of mathematics must be conveyed via clearly presented connections between mathematical fields and the symbiotic interplay of theory and application.

Even upper-level courses that are traditionally limited to one corner of the mathematical landscape—abstract algebra, for example—should offer more than a narrow perspective. For instance, through symmetry groups and their subgroups, group theory is intimately tied to geometry, and applications to coding theory are elementary enough to be discussed in an introductory course. Moreover, the structures included in an abstract algebra course can be used to help students understand the basis for the mathematics they studied in elementary and secondary school. (Also see C.3 and the corresponding Illustrative Resources.)

Mathematics faculty need to be consciously aware of the need to build connections and make this a higher priority in their instruction.

Introduce contemporary topics from the mathematical sciences and their applications. Mathematics is one of the few disciplines in which many undergraduates complete degrees with little idea of what has happened in the field within the past hundred years. Consequently, although students are well aware that research is ongoing in the natural and physical sciences, they often are totally unaware that research is done in mathematics. This is a highly inaccurate and misleading perception that can decrease student interest in the field. Mathematics must be presented as a discipline with intellectual challenges and open questions, not as a subject whose vitality ended long ago. This requires presentation of contemporary topics and at least glimpses of current fields of research.

Although much modern mathematics requires extensive preparation to be fully appreciated, many contemporary topics can be included in the undergraduate mathematics curriculum, either as courses, units within courses, student projects, or simply as examples used to enliven interest. The power and ubiquity of mathematics in the modern world is an important "hook" to engage student interest. Even students taking a single course can learn that mathematics underlies technology, like CD players and secure electronic communication, that they use every day. Mathematical sciences departments have a responsibility to see that all students, definitely including mathematics majors, have opportunities to learn about contemporary topics in pure and applied mathematics.

Here is a sampling of topics that can be included in various ways and in courses at varying levels:

- Dynamical systems, including chaos and fractals. These topics can be treated in introductory, intermediate and advanced courses.
- Resampling methods. Computers have radically changed the way statisticians approach data. Statistics courses should convey to students the excitement of newly developing techniques.
- Wavelets. The common wisdom is that this topic cannot be properly presented until students have mastered Fourier analysis. However, junior/senior level undergraduate courses do exist which successfully address this material.
- Error correcting codes. Besides being excellent applications of linear and abstract algebra, error-correcting codes have important uses in new technologies such as CD-ROMs and cell phones, as well as for analyzing data-transmission from outer space.
- Complexity theory. The "P vs. NP" problem (answer 'easy to find' vs. 'easy to check') is well within the range of undergraduates' understanding. Pointing out that the Clay Mathematics Institute is offering a million dollar prize for the solution helps to pique students' interest.

Other examples include number theory and cryptography, computational algebraic geometry, computer graphics, Monte Carlo methods, game theory, knot theory, quantum computing, and algebraic combinatorics.

Thanks in some cases to judicious use of technology, new topics have become appropriate for undergraduates and existing undergraduate courses are ripe for rethinking. The examples mentioned above are only a few. Textbooks and mini-textbooks that support inclusion of these topics are just starting to appear. More are needed.

4. Promote interdisciplinary cooperation

Mathematical sciences departments should encourage and support faculty collaboration with colleagues from other departments to modify and develop mathematics courses, create joint or cooperative majors, devise undergraduate research projects, and possibly team teach courses or units within courses.

Towards Excellence urges departments to "Build strong relationships on campus. Faculty should make building strong relations with other departments and the campus administration a conscious department goal."

The Curriculum Foundations workshops demonstrated a substantial willingness of faculty in other fields to work with mathematicians to develop programs and courses and to team-teach courses or units within courses. Such interdisciplinary programs and courses are labor intensive and often costly but they are needed because of the rapidly changing nature of the disciplines that use mathematics.

Interdisciplinary programs energize both the students and faculty who participate. They entice students from other disciplines to learn more mathematics and to learn it in a context that is important to them. They open new ways of looking at mathematics for mathematics majors, including possible career options or avenues for further study. And they provide an opportunity to communicate some of the cutting-edge work of mathematics. Interdisciplinary collaborations also enrich the faculty who participate, affording them opportunities to learn powerful applications and ideas from other disciplines.

The Curriculum Foundations workshops also demonstrated that many faculty in partner disciplines are eager to provide input to help mathematicians revise existing courses to support current disciplinary needs more effectively. Communication about such courses has the added benefit of increasing awareness of their content, which improves the ability of faculty in other disciplines to build upon the mathematical backgrounds of the students in their own courses.

Joint efforts can cement relations between a mathematical sciences department and other disciplines. Lines of communication are kept open so that the mathematics department is sensitive to the needs and concerns of these partner disciplines. This has the paradoxical effect of reducing pressures on the department to conform to the wishes of others by making it easy to spot the small adjustments that can be most beneficial to them.

Effective dialogue with colleagues in other disciplines is not always easily achieved. Especially in a large institution with numerous departments requiring mathematics for their students, a mathematical sciences department may hear conflicting requests. Constrained resources limit the capacity of departments to tailor separate courses for specific partners. Priorities must be determined and compromises are necessary. But small cooperative successes can breed larger ones, and the benefits of genuine cross-disciplinary cooperation far outweigh the potential difficulties.

In order to promote the creation of interdisciplinary programs, the development of new courses, and the appropriate revision of existing ones, mathematical sciences departments should identify and encourage faculty who are particularly interested in interdisciplinary cooperation and are willing to invest the effort required to bring it about. Encouragement should include rewards such as release from other obligations, additional summer salary or professional development funds. Many institutions are suffering severe budget cuts that make funding initiatives difficult, but steady advocacy for these rewards moves them higher in the priority list when resources become more available. Moreover, even in times of budgetary constraint, departments should ensure that such activities are viewed favorably when tenure and promotion decisions are made.

When a department reaches out to and interacts with other departments, it sets an example that college and university administrators will recognize and encourage. Collaborations are more than a service to students and colleagues. They serve to enhance the visibility of a mathematical sciences department and ensure its importance in achieving the mission of the entire institution.

5. Use computer technology to support problem solving and to promote understanding

At every level of the curriculum, some courses should incorporate activities that will help all students progress in learning to use technology

- ***Appropriately and effectively as a tool for solving problems;***
- ***As an aid to understanding mathematical ideas.***

Mathematical sciences departments should look for opportunities to make effective use of technology—desktop and hand-held computers—at every level of the curriculum. Every instructor teaching every course should consider whether and what uses of technology are appropriate to the material and to the students' needs.

The use of technology can help students develop mathematical skills and understanding.[27] However, the use of technology must be focused on students' needs rather than on the capabilities of the technology itself. Instructors must first decide what mathematics is to be learned and how students are to learn it. The answers to these questions will determine whether and how students should use technology.

---

[27] Adapted from the 2001 reports on technology from the MAA Committee on the Teaching of Undergraduate Mathematics and the Committee on Computers in Mathematics Education and also the report of the statistics workshop of the Curriculum Foundations project.

The answers to these questions will also be affected by the technology available for student use. For example, in a beginning calculus course, the existence of graphing utilities may affect the amount of time spent on graphing by hand as well as opening the door to different approaches to graphing families of functions. Similarly, the existence of computer algebra systems can affect the treatment of techniques of integration, and the availability of a differential equation solver can permit the early introduction of modeling with systems of differential equations. As with all teaching strategies, the effectiveness of the use of technology should be evaluated throughout its implementation, and modifications should be made until the desired goals are attained. The preferences of faculty from partner disciplines should also be taken into account. For instance, participants from many disciplines at the Curriculum Foundations workshops expressed a desire for the use of spreadsheet software, when appropriate, in mathematics courses.

There should be a variety of introductory courses that make some use of technology. For example, technology can enhance student learning in introductory courses in modeling, whether with calculus or not. Most calculus students, especially those who may take only one semester, profit from the use of a graphing utility and a tool for numerical integration. In 1992 the Joint Committee on Undergraduate Statistics of the MAA and the American Statistical Association (ASA) recommended use of statistical packages for introductory statistics courses, as well as more advanced ones. Introductory (and advanced) discrete mathematics courses can take advantage of software for the manipulation of discrete graphs, theorem provers, and functional and logic-based languages.

Technology use should also be present in a variety of intermediate courses. Graphics packages enhance multivariable calculus. Most modern texts on differential equations make use of a differential equation solver. Some intermediate “bridge” courses include computer experimentation to motivate conjecture and proof. Linear algebra courses can use technology for matrix manipulation or for visualizing the effects of linear transformations in two or three dimensions.

Software exists that enhances topics in a variety of advanced courses including geometry, probability, complex analysis, algebraic geometry, and group theory. Illustrative Resources includes descriptions of effective use of technology in courses at a variety of levels and in varying institutions.

Beyond these instructional uses of computer technology, there are others. For instance, Thomas Banchoff points to the power of emerging “super texts” that use hypertext to make texts (and courses) more flexible, permitting both instructors and students to pick and choose. Wade Ellis describes the potential of tutorial software to offer remedial help to students. Gilbert Strang, who has been involved in experiments with videotaped lectures and an on-line “encyclopedia” to offer “just-in-time” help in basic mathematics courses, elaborates on the potential benefits of such strategies.[28]

In order for technology to be useful in mathematics instruction, students should be able to focus on the mathematics rather than on how to use the technology. Introductory tutorials, training sessions, and on-line help libraries can assist students to overcome difficulties with the technology. Using the same software in several different courses also can shorten the total technology learning curve. In addition, the *MAA Guidelines for Programs* specify that when a department decides to use technology in a course or program, it has a responsibility to offer appropriate training for faculty in that technology and its effective use in instruction. If resources (equipment, personnel, training) are not adequate, departments should press for their improvement.

Using technology has costs as well as benefits. Besides the obvious hardware and software costs, there also are substantial human costs, not just for support personnel (if a department is fortunate enough to have

[28] See “Some Predictions for the Next Decade” by T. Banchoff, “Mathematics and the Mathematical Sciences in 2010: What Should Graduates Know?” by W. Ellis, and “Teaching and Learning on the Internet,” by G. Strang in *CUPM Discussion Papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know?*, MAA Reports, 2001.

them) but also for faculty who must learn to make effective use of the technology, spend time developing suitable course materials, and oversee the facilities available to students. Learning to use technology means time for students as well, time that might otherwise be spent on mathematical content. And when misused, technology can become a crutch, used for tasks that students should be able to do by hand and providing only an illusion of accomplishment.

The reality of such problems with technology should not cause mathematics departments to avoid its use. Rather, they indicate the care and effort needed for effective implementation. The potential benefits of technology for student learning are worth this care and effort.

Use technology appropriately and effectively as a tool for solving problems. Technology can help strengthen students' problem-solving skills by encouraging them to utilize multiple problem-solving strategies (graphical, numerical, algebraic); it is especially valuable for visualization. Data sets stored in local computer files or available on the Internet provide the information that makes it practical to address problems rooted in real data. This allows students to apply their mathematical knowledge to real situations, motivating them to learn and understand mathematics in the context of real-world problems. Statistical packages and spreadsheet programs free students from tedious or non-illuminating computations, perform extensive computations that are unrealistic to do by hand, produce results for students to interpret, and depict results visually as graphs, histograms, and diagrams. Technology makes large linear systems tractable in linear algebra and complicated dynamical systems accessible in introductory differential equations courses.

Students should learn to choose the appropriate means for the problem being solved. Some problems should be solved by hand. Others should be solved using technology such as calculators and spreadsheets. Still others require more substantial computer power. Students should learn to distinguish between thinking, which is a human activity, and computing, which machines do well.

Students especially need to learn how to be intelligent consumers of the answers technology provides. They should know how to make order of magnitude estimates for numerical calculation and to do trial calculations for small cases where answers can be checked by hand. They should use qualitative analysis to test quantitative results for reasonableness. For example, if approximating the integral of a positive-valued function, students should recognize that a negative answer cannot be correct.

Use technology as an aid to understanding mathematical ideas. Students should learn to visualize geometric objects, to relate graphical objects to their analytic definitions, and to see the graphical effects of varying parameters. Technology allows students easy access to the graphs of planar curves, space curves and surfaces, and more specialized software gives access to other geometric objects. Visualization also helps students understand concepts such as approximation of integrals by Riemann sums or functions by Taylor polynomials. The ability of technology to handle even symbolic manipulations allows students to focus their attention on understanding concepts. Use of technology, whether a high-level programming language or a spreadsheet, can also help students learn to think algorithmically by giving them experience working with algorithms.

Technology—especially dynamic tools—can promote students' exploration of and experimentation with mathematical ideas. For example, students can be encouraged to ask "what if?" questions, to posit conjectures, to verify or refute them, and to use technology to investigate, revise, and refine their predictions. Specific examples include studying the effects of manipulating parameters on classes of functions and fitting functional models to data. When students see that some patterns persist and others eventually fail, they begin to understand the meaning of mathematical truth, and they can learn to value proof as a source of both justification and explanation.

6. Provide faculty support for curricular and instructional improvement

Mathematical sciences departments and institutional administrators should encourage, support and reward faculty efforts to improve the efficacy of teaching and strengthen curricula.

CUPM Guide 2004 lays out a vision of excellence in the undergraduate mathematics program. The *MAA Guidelines for Programs* specify the mathematical sciences department's responsibility to promote excellence in teaching and to create programs to achieve and sustain it. Excellence can be accomplished only with sustained faculty effort, and this effort is most likely to occur in an environment where it is encouraged, supported, and rewarded.

Creating such an environment must begin with the department administrators, with deans and provosts who establish expectations for the department, and with those members of the department who enjoy prestige and influence. "Departments should ensure that senior faculty assume a leadership role in the undergraduate program by participating fully in teaching, curriculum development, and student advising."[29] These leaders should find ways to show faculty that teaching is respected and excellence in education is a departmental and institutional goal. A simple way of conveying this message is to publicize significant achievements in curriculum development and program improvement along with those in faculty research.

Department leaders should direct faculty to information about effective innovations, how they work, and practical means of implementing them. With college administrators, they should provide the necessary budgetary support to bring in speakers or workshop leaders and to make continuing professional development available to all members of the department. Adjunct and part-time faculty and graduate students are important contributors to the teaching mission of many departments. As stated in the *MAA Guidelines for Programs*, "Departments that employ part-time instructors should provide them with all of the resources necessary for teaching that are provided to full-time instructors." In particular, means should be found to provide them with mentoring, training, and professional development opportunities.

The departmental leadership should actively seek to identify and encourage those faculty who are or could be positioned to make a substantial improvement in some aspect of the educational mission of the department. This is in line with the recognition in *Towards Excellence* of "the importance of identifying the right person to lead a department initiative and giving that person the support needed to create a successful program."[30]

However, finding someone to lead an initiative is not enough. No program will long survive if it represents the work of a single individual. For long-term sustainability, initiatives must be team efforts, with faculty in supporting roles who can be prepared to expand or take over the leadership of the program. At institutions that require research for promotion and tenure, untenured faculty should not be expected to take on roles that would seriously hamper their scholarly development. However, means should be found to involve all faculty in improving the curriculum and its instruction. This is especially true for younger faculty who often bring enthusiasm and openness to innovative ideas.

There are many ways that support can be made tangible. These include release time for key faculty, clerical support, and funds for speakers, programs, or special materials. As noted earlier, advocacy for this tangible support is warranted even in times of fiscal constraint. For many effective innovations (especially those involving changes of course content or pedagogy), support includes the promotion of broad departmental engagement. Effective initiatives must be seen to be appreciated and recognized by the leadership of the department as important contributions to the mission of the department.

---

[29] *MAA Guidelines for Programs and Departments in Undergraduate Mathematical Sciences,* MAA, 2001; `www.maa.org/guidelines/guidelines.html`. See 2g.

[30] *Towards Excellence*, page 33.

The salary structure of educational institutions must respect the full range of contributions to the department's missions. Faculty should see concrete expressions of support for their educational efforts in the form of appropriate salary increases, as well as benefits such as travel allowances, funds for course development, and support for co-curricular activities. *CUPM Guide 2004* supports the recommendation of the AMS Task Force on Excellence:

"There should be clear standards of excellence for those whose greatest achievements are in teaching or other educational activities, and faculty who meet those standards should share in faculty rewards, both financially and through promotion in rank."[31]

[31] *Towards Excellence*, page 35.