PREGRADUATE TRAINING

The Panel on Pregraduate Training was appointed in 1959 to study the needs of, and to recommend programs for, undergraduate students who intend to study mathematics at the graduate level. The initial efforts of the Panel were concentrated upon the construction of an ideal curriculum for students of outstanding ability. Course outlines designed to lead the undergraduate rapidly toward the frontiers of mathematical research and the Ph.D., purposely overlooking local problems which might be caused by inadequate preparation at the secondary level or by lack of staff at the college level, appeared in the 1963 publication Pregraduate Preparation of Research Mathematicians.

Despite many misunderstandings regarding its assumptions and intent, this report served effectively as a basis for discussion and planning at many institutions. It was reprinted in 1965, together with some additional comments on constructive use of the booklet and admonitions to the effect that misinterpretation of the spirit of the outlines might result from a lack of knowledge of the Panel's basic assumptions and objectives. Perhaps the report's chief value lies in showing what is regarded as ideal preparation for graduate study in pure mathematics by a very distinguished group of mathematicians.

Having completed its work on an ideal undergraduate program for the future research mathematician, the Panel turned to the urgent practical task of recommending specific undergraduate curricula for colleges which would be unable, for any of a variety of reasons, to achieve quickly the goals of its original report. These recommendations were drawn up after consultation with representatives of about 25 of the leading graduate mathematics departments. They appear in the 1965 document Preparation for Graduate Study in Mathematics, together with outlines for upper-division courses in Abstract Algebra and Real Analysis.

PREGRADUATE PREPARATION

OF

RESEARCH MATHEMATICIANS

A Report of

The Panel on Pregraduate Training

Goals for pregraduate programs in mathematics. Schools for which these goals are not immediately attainable are referred to the Panel's companion pamphlet, Preparation for Graduate Study in Mathematics.

May 1963

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ABOUT THIS BOOKLET

In the first year of publication over 7,000 copies of these recommendations have been distributed to departments of mathematics and individual college teachers. Reaction from readers indicates a need for several comments on the constructive use of this booklet.

The introductory pages should be read carefully before the course outlines which follow. These passages describe the underlying idealized objectives of the recommendations. Failure to understand these goals can result in misinterpretation of the spirit of the outlines.

The course outlines beginning on page 378 are illustrative samples which may be modified to fit local situations. Taken as a whole, they represent achievement of long-range goals; however, the reader is urged to use them as sources of mathematical ideas out of which to construct his own first steps toward these goals.

The booklet can be used as a guide to several levels of training in mathematics. The material in the outlines for Basic Undergraduate Mathematics together with roughly half of the algebra course on page 414 is generally considered adequate undergraduate preparation for current graduate programs. Departments and students are urged to use the booklet in planning curricula and individual programs of study.

BASIC ASSUMPTIONS

The recommendations presented in this booklet are idealized, as are most educational programs described in print. Hence, it is necessary to reveal some assumptions which were made by the Panel in constructing the suggested program and the course outlines. It is fully realized that the assumptions are somewhat unrealistic in the sense that few pregraduate programs can now include such courses in all detail. However, "honors" programs may well be able now to include some of these proposals. Such developments may lead to an absorption of the ideas into regular curricula, and the next task of the Panel is to provide indications as to how this may be done.

The program set forth here is designed for the first four years of a sequence of formal course study leading to the Ph.D. It is hoped that full-time students seeking a career in research mathematics will obtain the Ph.D. in a total of seven years, with the last year spent mainly in seminars and completion of the dissertation. Clearly, the number of beginning college students now possessing the knowledge and motivation necessary for entry into such a program is small indeed. If they realize their ambitions, these people can be expected to contribute to society as producers of mathematics--hence the title of the booklet.

We have thus assumed that the students will begin the program with some prior appreciation of mathematical proof, with a secondary school background conforming to the highest recommendations of such groups as the School Mathematics Study Group. More significantly, the program is directed toward all students who have profited fully from the mathematical opportunities afforded in their formal study, who hunger for deeper insight and more powerful techniques, who are intellectually curious and are capable of appreciating the elegance, scope, and excitement of mathematics.

Many persons may feel that no action need be taken to help this restricted group of students and, indeed, that it is impossible to prevent them from becoming creative, gifted mathematicians. Admittedly, this assertion is most often made by those who themselves have survived despite all adversity. The counterexamples, of course, do not survive to testify.

But the present document does much more than merely recommend a program for gifted students. It will also serve as a guide and source of ideas to persons interested in evaluating and modifying mathematical curricula. The Panel has spent more than two years in a critical examination of the basic structure of college mathematics, in relation to present mathematical research. It has attempted to discern important underlying patterns and to effect some unity, both of viewpoint and of technique, within a four-year curriculum. It has attempted to make use of the simplification of concept and technique resulting from recent discoveries, without sacrificing intelligibility.

The Panel has also agreed upon certain broad objectives for the college mathematics program. The student should be introduced to the language of mathematics, both in its rigorous and idiomatic forms. He should be able to give clear explanations of the meaning of certain fundamental concepts, statements, and notations. He should acquire a degree of facility with selected mathematical techniques, know proofs of a collection of basic theorems, and have experience with the construction of proofs. He should be ready to read appropriate mathematical literature with understanding and enjoyment. He should learn from illustration and experience to cultivate curiosity and the habit of experimentation, to look beyond immediate objectives, and to make and test conjectures.

The student should by these means be led to seek an understanding of the place of mathematics in our culture--in particular, to appreciate the interplay between mathematics and the sciences. The proposed program, in the portions dealing with analysis, exhibits the traditional role of the physical sciences as a source of mathematical ideas and techniques. The outlines offered for courses in probability and statistics also indicate the emergence of parts of mathematics from problems in the biological and social sciences.

A list of objectives such as these is largely independent of the content of courses and cannot be implemented completely by even the best collection of texts. This only emphasizes the obvious point that the quality of the mathematical education of the nation rests finally upon the caliber and initiative of teachers. We hope that the present document will stimulate widespread interest in a continuing examination and reformulation of mathematics programs in the college.

THE PROGRAM

The proposed program of pregraduate mathematical studies falls naturally into two parts: Introductory Undergraduate Mathematics occupies the first two years, Higher Undergraduate Mathematics the last two. As to subject matter, the former naturally is focused on the differential and integral calculus, and the latter is devoted mainly to basic material in the fields of analysis, algebra, and geometry. Each of these terms, however, is stretched beyond its traditional meaning to take account of contemporary mathematical development. Each of these principal areas is to be supplemented by related studies. And the two parts of the program are to be distinguished by their method of presentation as well as by their subject matter. In particular, the first two years must be shaped so as to lead gradually to an appreciation of the nature and role of definitions and proofs and an ability to employ mathematical language with precision. The last two years must be designed to merge smoothly with beginning graduate study, forming a period in which the most basic mathematical concepts, results, and methods are secured so as to provide a firm base for subsequent specialization and concentrated research.

We have recognized the pervasive character of linearity by recommending the early introduction of linear algebra and the recurrent use of linearity in the analysis courses. At the same time the analysis courses reflect the traditional role of the empirical sciences as a source of mathematical concepts and methods.

Geometry is construed so broadly as to contain differential geometry, differential topology, and algebraic topology, among others, but the spirit of classical geometry has been retained by emphasizing theorems having a geometric formulation.

Breadth, including a knowledge of fields of application, is of great value for the most significant mathematical research. Although difficult to achieve during the undergraduate years, at least a beginning is essential. For those students able to fit in a considerable variety of courses it is important to attempt to achieve as integrated a picture of mathematics as can be assimilated. In the present program the unity of mathematics is illustrated by emphasizing algebraic and topological ideas throughout.

It is important to emphasize that the ability to follow and formulate rigorous proofs must be balanced with the development of a free-ranging intuition in each mathematical field. Generally speaking, a rigorous treatment of some elements of the material should appear even in the earliest courses, and the rigorous segments should increase in length as the student advances to more recondite material. But failure to nurture intuition at any level can be stifling.

The idealized assumptions adopted by the Panel and described in the previous section, while simplifying the problem of formulating curriculum by abstracting from the host of practical problems which beset individual mathematics departments, are still far from sufficient to characterize a unique solution. Members of the Panel found themselves with continuing differences of opinion concerning such questions as the extent to which courses should deal with applications outside of mathematics, the order and relative emphasis to be given to certain groups of loosely associated topics, and the nature of subject development which is required by pedagogical considerations. These differences within the Panel are reflections, of course, of the varied attitudes, often strongly held, which prevail within the broader mathematical community.

In order to convey a fairly concrete idea of the nature of the program conceived by the Panel, course outlines are furnished in the Appendices which follow. To accommodate the variety of viewpoints which prevail, more than one outline is presented for certain basic courses. Each outline is prefaced by an indication of its scope and intended context.

INTRODUCTORY UNDERGRADUATE MATHEMATICS

Under the assumption of a student audience with strong mathematical training in high school and with excellent motivation, a unified two-year sequence of what might be called "vector space calculus" is recommended as a proper basis for the pregraduate program. Consistent with the historical development of calculus and with the flavor of modern mathematics, the program suggests that calculus be presented so as to introduce and utilize significant notions of linear algebra and geometry in the construction of analytic tools for the study of transformations of one Euclidean space into another. This demands that the material be arranged and presented in such a manner that students are ever mindful of mathematics as an inter-related whole rather than a collection of isolated disciplines. The presentation also needs a healthy balance of well-formulated mathematical arguments, of opportunity for discovery through independent work in solving problems and proving theorems, and of mathematical and physical motivation. The student must learn early that a highly significant aspect of mathematics is that of posing the right question.

This program of Introductory Undergraduate Mathematics comprises approximately 15 semester hours. Since the standard subjects are integrated, only a rough estimate of their proportions can be indicated: about nine semester hours of analytic geometry and calculus, with the remainder divided between linear algebra and differential equations.

Presented in Appendix A are three course outlines which display the embodiment of these ideas. The subdivisions in these outlines are in terms of topics and not in terms of days or weeks. The major differences between the outlines are explained at the outset in Appendix A.

HIGHER UNDERGRADUATE MATHEMATICS

This part of the program builds upon the foundation laid by the Introductory Undergraduate Mathematics curriculum. Assuming the ability to appreciate and handle rigor and abstraction, it is intended to broaden the areas of the student's mathematical knowledge with sufficient depth to provide a firm basis for later research and to allow for the formation of individual mathematical taste.

Courses appropriate for this part of the program should be considered as at the "undergraduate-graduate level," for the same type of course will be needed at the beginning of graduate study. Indeed, it is an historical "accident," certainly not related to intrinsic mathematical considerations, that the undergraduate degree is granted after four years of post-high school study. For this reason institutions with limited facilities should strive to provide courses with a full degree of depth and challenge even if this entails offering a narrower range of subjects. The student who comes to graduate school with one solid course behind him is ready to take a second in another field; but the one who comes with the equivalent of two half-courses is often forced either to repeat material or to proceed to more advanced work with a deficient background.

The advanced part of the program will reflect the interest of the faculty, as well as the needs of the student. Small institutions will concentrate on courses in the mathematical areas of primary interest to their professors. And the selection of materials and modes of presentation within each course will reflect the way in which the individual instructor looks at the subject.

Every college department undertaking this program should provide courses relevant to the central areas of mathematics:

Real analysis

Complex analysis

Abstract algebra

Geometry-topology

Probability or mathematical physics

Not every student will take courses in all of these areas; choices will depend on the student's intent. (See Appendix B for sample course outlines.)

In addition, to achieve a richer and more comprehensive program, a department should offer, as far as its resources will permit, a balanced selection of courses in:

Algebra

Analysis

Applied mathematics (in both the natural and social sciences)

Foundations and logic

Geometry (algebraic, differential, projective)

Mathematical statistics

Number theory

Topology

(See Appendix C for sample outlines of some of these courses.)

For the student, we recommend the following principles:

(a) For the upperclass years, at least three of the following four categories should be represented in the course program: (1) algebra, (2) analysis, (3) applied mathematics, (4) geometry-topology.

(b) Included in the program there should be, in order to achieve depth, at least two full-year courses--that is, courses in which the first semester is an essential prerequisite to the second.

(c) A major in mathematics should have at least seven semester-courses beyond our suggested Introductory Undergraduate Mathematics.

INITIATIVE AND INDEPENDENCE

So far in presenting the program, the greatest attention has been paid to describing the mathematical content of an idealized undergraduate curriculum in contemporary mathematics. In view of our assumptions, students participating in the program will not be satisfied to participate in a passive fashion, and so methods to engage the student as an active partner in scholarship should be devised. Indeed, independent intellectual activity of the student must be nurtured in preparation for the time when he will be independent of his professors and join them as a colleague. Thus, the student must increasingly take the initiative, not only to construct proofs by himself, but to develop his imaginative powers so that he can make conjectures for proof or disproof, perhaps even going on to contribute by creating new concepts or theories.

This process has its beginning in a small way when the student solves textbook problems. Another component is added when the student learns to read the textual material by himself, later making the passage to the reading of papers in the journals, which are more compactly written and therefore more difficult to read. There are other ways in which the undergraduate student can develop his initiative. Without attempting an exhaustive list, we mention a few common patterns.

There are seminars and colloquia wherein the student makes reports. There is the undergraduate thesis in which a student makes a contribution, original for him but not usually original in the larger sense. There is the developmental course, a version of the Socratic method, in which the student is led to develop a body of mathematical material under the guidance of the professor. The number of teachers who employ the developmental method completely is not large, but they are an enthusiastic band of people in their devotion to the procedure; a modified use of the developmental method is employed widely. Other devices have been developed to take advantage of special local conditions, or in line with experimental ideas reflecting special interests.

It would be desirable for all schools to give attention to the problem of enrolling the student actively in the study of mathematics, and so we urge that every pregraduate curriculum be designed to include some appropriate scheme.

Appendix A

INTRODUCTORY UNDERGRADUATE MATHEMATICS

Contained in this Appendix are three outlines for the proposed Introductory Undergraduate Mathematics program, each displaying an initial two-year sequence of idealized college mathematics. The three course outlines which follow have different points of view, and so they differ in emphasis and arrangement of material. The major mathematical differences are these:

(1) Outline I includes a self-contained section on linear algebra in the calculus but a separate course in differential equations. Outline II has a separate course in linear algebra but includes topics in differential equations in the main body. Outline III has no separate courses. It has a section on differential equations and develops linear algebra topics as they are needed to solve various problems in analysis.

(2) Outlines I and II introduce integration before differentiation while Outline III does the opposite.

(3) Outlines I and II treat integration via step functions while Outline III approaches the integral as a linear functional on the space of continuous functions.

(4) Outline II is more ambitious than Outline I. It includes, for example, a good deal of elementary point set topology in the second year.

Aside from such explicit differences, there are some subtler distinctions in attitude between Outline III and the other outlines. A prime motivation of Outlines I and II is concern for the internal structure of calculus and of linear algebra; applications are made when appropriate. In these outlines the generalized Stokes theorem is a fitting climax because of the merging of concepts in algebra, topology, and analysis needed in reaching it and because of its important applications in mathematics and physics. The approach in Outline III is to develop mathematical concepts directly as needed for the solution of important problems that arise in mathematics and physics. In particular, linear algebra is so treated. In addition, Outline III is oriented more in the direction of classical analysis: more emphasis on inequalities; Stokes' theorem is thought of as but one of a number of important theorems beyond the traditional calculus. Outlines I and II are more rigid than Outline III, the attitude being that these are the things juniors should know, and here is a reasonable order of doing it. Outline III is more flexible, the attitude being: it is more important to learn how mathematics is developed to solve problems than to insist that the students know a given amount of mathematics.

Outline I , First Year

Functions of One Variable and Linear Algebra

1. Review of function concept, the algebra of real numbers, order. Algebra of functions.

2. The historical background of the calculus: the problem of areas, the problem of tangents, the problem of instantaneous velocity. Heuristic discussion of area as an additive set function whose value is determined on rectangles. Transition to the integral of a function via negative areas. Definition of the integral of a step function. Uniqueness. The integral as a positive linear functional on the family of step functions on an interval.

3. Extension of the integral to other functions via upper and lower approximation by step functions. The family of integrable functions on a bounded closed interval. Show this family closed under addition and multiplication by scalars. The integral as a finitely additive interval function. Integrability of polynomials and the sine function (using summation formulae and trigonometric identities--the trigonometric functions are used here as learned in secondary school; precise definitions will come during the second semester). What other functions are integrable?

4. Definition of continuity in terms of neighborhoods (open intervals). Statement that a continuous function on a bounded closed interval is integrable (proof postponed). The continuous functions are closed under addition and multiplication by real numbers. Statement of uniform continuity of a continuous function on a bounded closed interval. Derivation of this from axiom that such intervals are compact (defined in terms of coverings by open intervals).

5. Proof that continuous functions are integrable (using uniform continuity). Some applications of the integral: moments, energy, work, etc.

6. Approximate integration (piecewise constant, linear, and quadratic approximation). Examples. The problem of a better method

of calculation awaits solution. Ways in which functions depart from continuity; kinds of discontinuity. This leads to definition of limit in terms of deleted neighborhoods.

7. Continuity phrased in terms of limit. Continuity of composites of continuous functions. Algebra of limits. Continuity of products and quotients of continuous functions.

8. Problem of tangents. Heuristic geometric definition of a tangent line to a curve. Calculation of the slope of a nonvertical tangent leads to the derivative of a function. Problem of instantaneous velocity does the same. Examples.

9. Rules for differentiation of sums, products, quotients, composites. Derivatives of identity function and of constant functions give derivatives of rational functions. Derivative of sine function gives derivatives of trigonometric functions.

10. Equations for derivatives from equations for functions give derivatives of algebraic functions (exact definition of fractional exponents next semester). Calculation of tangents to various second-degree curves. Implicit definition of functions.

11. Examples of maxima and minima problems. Attainment of maxima and minima by continuous functions on compact sets. Vanishing derivative test. Rolle's theorem and the Mean Value Theorem. Geometric interpretation.

12. Application of Mean Value Theorem to determine where a differentiable function is increasing, decreasing, constant. Higher derivatives and the second derivative test for maxima (minima). Intermediate Value Theorem on intervals. Applications to graphing. The Mean Value Theorem for integrals.

13. The indefinite integral. Continuity of the indefinite integral of integrable functions. Derivability of same at points of continuity of the integrand, and evaluation of the derivative. Geometric interpretation of this theorem.

14. Reduction of the problem of integration of piecewise continuous functions to that of finding primitives. Applications of this theorem: the practical solution of the problem of integration.

15. New functions. Log defined by indefinite integral. Properties of log. Its derivative. Inverse functions in general. Case

in point: exp function. Its derivative and integral.

16. The number $e = \exp(1)$. Arbitrary real powers of e in terms of exp. Arbitrary real powers of positive real numbers. Derivative and integral of x^{α}, α real. Definition of inverse trigonometric functions: the difficulty that trigonometric functions are not on a solid foundation. This leads to problem of arc length.

17. Analysis of arc length in terms of the integral. Definition of arcsine in terms of an indefinite integral. Other inverse trigonometric functions. Reprise of trigonometric functions, their derivatives and integrals, now solidly grounded.

18. Difficulty of integrating log and arcsine leads to integration by parts. Substitution. Certain trigonometric substitutions. Completion of square.

19. Integration of rational functions. Functions not integrable by elementary means leads to idea of uniform approximation. $|\int f - \int g| \leqq \int |f - g|$. Sequences of numbers and their limits. Maximum norm of a continuous function. Uniform convergence of sequences of functions. Pointwise convergence.

20. Taylor's theorem with integral remainder. Same with derivative remainder. Notion of a power series.

21. Series in general. Convergence of geometric series. Archimedean axiom introduced to prove L. U. B. theorem. Corollary: the Monotone Convergence Theorem. Comparison test.

22. Existence of radius of convergence of a power series. Uniform convergence on bounded closed intervals within the interval of convergence. Invariance of the radius of convergence under formal differentiation and integration. Justification of term-by-term differentiation and integration.

23. Parametrized curves in R^2 and R^3. Reprise of function idea. Linearly parametrized lines in R^2, lines and planes in R^3 are functions. Addition of points in R^2 and R^3 and multiplication by real numbers introduced as a notational convenience. Definition of R^n, curves in R^n, vector operations in R^n. Linear and affine functions from R^n to R^m. Linear equations in terms of a single linear function equation.

24. Properties satisfied by the vector operations in R^n. Abstract notion of a vector space over R; linear and affine functions between vector spaces. Examples: function spaces, differentiation, definite and indefinite integral. Vector and affine subspaces: reprise of examples from Section 23, polynomial subspaces, solution space of a system of linear equations, direct and inverse images of affine subspaces under affine functions.

25. Dimension and linear independence, linear span. Basis. The standard basis of R^n. Representation of an n-dimensional vector space as R^n.

26. Representation of linear transformations with respect to bases. Matrix notation. The standard matrix of a linear transformation from R^n to R^m. The algebra of linear transformations and matrices.

27. Change of basis and similarity of matrices. Reprise of systems of linear equations as a single linear transformation equation. Rank and nullity theorem. Matrix notation for linear equations. (Column) rank and nullity of a matrix. General theorems on the dimension of the solution space of a linear system.

28. Elementary column operations and column equivalence of matrices. Echelon form of matrices. The use of this form to solve linear systems explicitly; comments on the numerical problem involved.

Outline I, Second Year

Functions of Several Variables and Linear Algebra

1. Review of cartesian products of sets. Cartesian products of n vector spaces. Multilinear functions with values in a vector space. Sums and real multiples of these.

2. Explicit solution of 2×2 and 3×3 systems of equations motivates notion of determinant. The determinant as a function of the columns of a matrix: multilinear, alternating, and unimodular on the standard ordered basis of R^n. The permutations of $\{1, \ldots, n\}$. Theorem: Let $X_1, \ldots, X_n$ be an ordered basis of V and let $Y \in W$; then there is a unique alternating multilinear function F

such that $F(X_1, \ldots, X_n) = Y$. Corollary: existence and uniqueness of the determinant, and an explicit formula for it. Corollary: the basis-free definition of the determinant of a linear transformation.

3. Multiplication of determinants; the group of nonsingular linear transformations. Theorem: An alternating m-linear function on an n-dimensional vector space, $m > n$, is necessarily zero. The solution of the equation $F(X) = A$ where F is a linear transformation from V into itself. Cramer's rule.

4. Invariant subspace of linear transformations; internal direct sums of subspaces. Consequences for matrix representation. One-dimensional invariant subspaces. The characteristic equation and eigenvalues. Cayley-Hamilton theorem.

5. Discussion of length and angle leads to notion of inner product. Length and norm. Schwarz inequality and definition of angle. Example of the integral inner product on the continuous functions on $[a, b]$. Trigonometric polynomials of order $\leqq m$. Orthonormal basis and the Gram-Schmidt process. The standard form of the inner product.

6. Symmetric and orthogonal linear transformations and matrices. Polar decomposition. Diagonalization theorem.

7. Application to conics and quadrics. Volume of a parallelepiped in terms of determinant. Orientation: defined by alternating function. Cross product in dimension 3, given an inner product and an orientation (use representation theorem for real-valued linear functions).

8. Neighborhoods of points as open spheres. Continuous functions between vector spaces with inner product, or more generally with a norm. Examples, including the continuity of the integral in the uniform norm. A linear transformation between two finite-dimensional inner product spaces is continuous.

9. Open and closed sets. Continuity in terms of open sets. Unions and intersections of open and closed sets. Interior, closure, and boundary of a set. Sequential limits. Characterization of previous notions using sequential limits.

10. Limit points in general. Theorem: An infinite subset of a compact (Heine-Borel) set S always has a limit point in S.

Corollary: compact sets are closed. Closed subsets of compact sets are compact. Continuous images of compact sets are compact; real continuous functions on a compact set attain their maxima.

11. Closed bounded sets in a finite-dimensional normed vector space are compact. Corollary: any two norms on a finite-dimensional normed vector space are equivalent. Cauchy criterion and completeness for a finite-dimensional normed vector space.

12. Uniform continuity of continuous functions on compact sets. Sequences of functions and uniform convergence of same. Completeness of space of continuous functions. Ascoli's theorem.

13. Connectedness; the Intermediate Value Theorem for real-valued functions.

14. Problem of volume in R^n. Heuristic discussion of volume as an additive set function whose value is determined on boxes. Problem of bad boundaries.

15. Volume of a domain with nice boundary. The integral defined in terms of step functions and its connection with signed volume under a hypersurface. The integral as a uniformly continuous positive linear function.

16. The integrability of continuous functions. The integral as a finitely additive set function. Differentiability of set functions.

17. Characterization of uniformly differentiable set functions. Reduction of multiple integration to iterated integration, calculation. Setting up iterated integrals.

18. Functions on open domains. The derivative

$$F(X,Y) = \lim_{t \to 0} \frac{F(X + tY) - F(X)}{t} .$$

Class C^1 and the linearity of $F'(X,\cdot)$. Geometric interpretation. Interpret $F(X_0) + F'(X_0, X - X_0)$ as "best" affine approximation to F at X_0. Matrix representation of $F'(X,\cdot)$ and partial derivatives.

19. Mean Value Theorem. Higher derivatives defined recursively:

$$F^{(n)}(X, Y_1, \ldots, Y_n)$$

$$= \lim_{t \to 0} \frac{F^{(n-1)}(X + tY_n, Y_1, \ldots, Y_{n-1}) - F^{(n-1)}(X, Y_1, \ldots, Y_{n-1})}{t}$$

The differentiability classes C^n. Symmetry and linearity of $F^{(n)}(X, \cdot, \ldots, \cdot)$ with $F \in C^n$. Taylor's formula.

20. Derivatives of composites and the ordinary chain rule (a matrix equation). Jacobians. Local one-one theorem for $J \neq 0$. Differentiability of invertible F with $J \neq 0$. $J(F^{-1}) = J(F)^{-1}$.

21. Implicit Function Theorem. Critical points. Lagrange multipliers.

22. Formula for volume change under C^1 function. $\int f = \int (f \circ F)|J(F)|$. This leads to notion of a differentiable n-form on R^n: $w(X, Y_1, \ldots, Y_n)$, alternating multilinear in the Y's.

23. Heuristic arguments concerning work and total flux integrals lead to notions of a differential p-form on an n-dimensional vector space and the definition of the integral of same over a parametrized p-cell. Singular simplices. Differential p-forms as functions from differential simplices to the reals.

24. Representation of 0-forms by real functions and (given a scalar product) of 1-forms by vector fields. Representation of $(n - 1)$-forms by vector fields and n-forms by real functions, given a scalar product and an orientation. The integral of a differential p-form over a totally nondegenerate singular p-simplex is independent of the parametrization.

25. d: 0-form $\rightarrow$ 1-form, by taking the derivative. The gradient. The extension of the notion to p-forms by differentiation and skew-symmetrization. $d^2 = 0$. Divergence and ($\dim V = 3$) curl of a vector field. Curl $\circ$ grad = 0; div $\circ$ curl = 0.

26. Multiplication of forms as ordinary multiplication of functions skew-symmetrized. Representation of forms with respect to coordinate functions; d in terms of a coordinate system.

27. Stokes' theorem over standard n-cell in R^n. General case of Stokes' theorem, classical Stokes' theorem, Gauss' theorem,

Green's theorem, and the Fundamental Theorem of the Calculus. The meaning of curl and div in fluid dynamics.

Outline I, <u>Differential Equations</u>

An adequate preparation for the course outlined here is the successful completion of the first one-and-a-half years' work of Outline I.

The course is designed for a semester

(i) to give the student the basic existence and uniqueness results for ordinary differential equations and systems of equations;

(ii) to develop in detail the properties of solutions of some important types of linear systems--constant coefficients, analytic coefficients, and systems with regular singular points--by exploiting the student's earlier preparation in linear algebra; and

(iii) to introduce the student to some topics of current research interest: stability of nonlinear systems, eigenvalue problems, elementary partial differential equations.

1. Complex numbers. Complex-valued functions. Polynomials. Complex series and the exponential function. Complex n-dimensional space and functions defined on it.

2. Examples of problems involving differential equations: Newton's laws of motion, heat flow, vibration problems. Initial value problems and boundary value problems.

3. Local existence of solutions to initial value problems for $y' = f(x,y)$, where x, y real and f real-valued. The method of successive approximations (fixed point theorem), using a Lipschitz condition. The polygon method, using the Ascoli lemma. Nonlocal existence, using a Lipschitz condition on f in a strip $|x - x_0| \leqq a$, $|y| < \infty$. Approximations to, and uniqueness of, solutions. Extension of results to case where x real, y complex, f complex-valued.

4. Existence and uniqueness for systems using vector, and vector-valued, functions. Extension of material in Section 2 to this case. Example: central forces and planetary motion.

Applications to equations of n^{th} order.

5. General results on homogeneous linear systems $y' = A(x,y)$ where A is linear in $y \in C^n$. The space of solutions as a vector space of dimension n. Nonlocal existence in this case. The solution of the nonhomogeneous system $y' = A(x,y) + b(x)$. Application to linear equations of n^{th} order.

6. Linear systems with constant coefficients $y' = A(y)$. Explicit structure of space of solutions using $\exp A$, and assuming Jordan canonical form for A. Explicit form that variation of constants takes in this case. Application to n^{th}-order equations. The case $n = 2$ in detail.

7. Linear systems with analytic coefficients (convergent power series as coefficients). Solutions as convergent power series. Application to n^{th}-order equations. Example: the Legendre equation.

8. Linear systems with regular singular points: $y' = x^{-1}A(x)y$, with A having convergent power series expansion. Structure of solution space using $x^A = \exp(A \log x)$. Application to second-order equations with regular singular points. Examples: the Euler equation and the Bessel equation.

9. Introduction to nonlinear theory. Perturbations of two-dimensional real autonomous systems. Classification of simple critical points. Phase portraits. Stability. Asymptotic stability. Relation of nonlinear case to linear approximation.

10. Poincaré-Bendixson theory (optional).

11. Self-adjoint eigenvalue problems for second-order linear equations--the regular case. The space of continuous functions $\mathcal{C}$ as a linear manifold in L^2. The existence of eigenvalues using complete continuity of the Green's operator in $\mathcal{C}$. Bessel's inequality and the Parseval equality. Expansion theorem.

12. Second-order linear partial differential equations. Classification: hyperbolic, elliptic, parabolic. Equations with constant coefficients. Typical initial and boundary value problems in each case. Application of results in Section 11.

Functions of One Variable

1. Functions and the real numbers. Review of the general concept of function. Characterization of the real numbers, the Archimedean axiom, and suprema. The algebra of real functions defined on a set, polynomial functions from the reals to the reals.

2. The problems of area. Historical background, including the work of Archimedes. Heuristic discussion of the problem of defining area as an additive set function whose value is determined on rectangles. Transition to the integral of a fuction via negative areas. Definition of the integral of a step function, uniqueness. The integral as a positive linear functional on the family of step functions on an interval.

3. Extension of the integral to more general functions. Upper and lower approximations. The family of integrable functions on a bounded closed interval. The observation that this family is closed under addition and scalar multiplication. Discussion of some functions which are integrable, monotone functions, sums of monotone functions. Integration of some explicit functions such as polynomial functions and some of the trigonometric functions, assuming that at least an intuitive definition of these functions together with their principal algebraic and geometric properties has been learned in an earlier study of mathematics. Approximations to the integral and estimates of error.

4. Continuous functions. Definition of continuity in terms of open intervals. Observation that the continuous functions are closed under both addition and multiplication. Continuity of the polynomial functions. Derivation of uniform continuity of a continuous function on a closed bounded interval, assuming such intervals are compact (defined in terms of coverings by open intervals).

5. Integrability of continuous functions. Applications of integrals to problems such as calculating areas, moments, work, and energy.

6. Approximations to the integral. Piecewise constant, linear, and quadratic approximations. Estimates of error. Ways in which

functions depart from continuity; discontinuities. Definition of approximations.

7. The algebra of continuous functions. Continuity in terms of limits. Continuity of composites, products, and (under appropriate circumstances) quotients. The algebra of limits.

8. Historical backgound of the problem of tangents. Heuristic geometric definitions of the tangent to a curve at a point. The problem of velocity. Definition of the derivative of a function. Derived function and its geometric interpretation as the function which to every point assigns the slope of the tangent of the original function.

9. Formal differentiation. Derivation of rules for calculating derivatives of sums, products, quotients, and composites. Numerous calculations to develop techniques. Algebraic functions. Calculation of tangents to various second-degree curves. Implicitly defined functions.

10. Maxima and minima. Proof that a continuous function on a closed bounded interval attains its maximum. Criteria for determination of maxima. Vanishing derivative test. Introduction of the second derivative. Sufficient conditions for local maxima. Graphs, geometric ideas of convexity, and maxima. Interpretation of the second derivative as acceleration. Rolle's theorem and the Mean Value Theorem. Intermediate Value Theorem on intervals. Application of the preceding to problems involving graphing, velocity, and acceleration. The Mean Value Theorem for integrals.

11. Relation between integration and differentiation. The indefinite integral, and continuity of functions defined by integration of integrable functions. Differentiability of such functions at points of continuity of the integral. Geometric interpretation of the preceding. Piecewise continuous functions and reduction of the problem of integrating such functions to the problem of finding primitives. Various calculations via this last result.

12. Functions defined by integrals. The logarithm function and its properties. Inverse functions in general. The exponential function and its properties. The number $e = \exp(1)$. Arbitrary real power of e and hence of any positive real number.

13. Methods of integration. The difficulty of integrating the logarithm function. Integration by parts. Substitutions, including certain trigonometric substitutions. Completion of the square. Integration of rational functions.

14. Uniform approximation. Functions not integrable by elementary methods. Approximations of the integral of such functions by approximating the function itself uniformly by functions with elementary integrals. Sequences of numbers and their limits. The sup norm of a continuous function. Uniform limits of continuous functions. Space of continuous functions closed under uniform limits. Pointwise limits.

15. Taylor's theorem with remainder. Various forms of the remainder. Use of Taylor's theorem to approximate functions by polynomials; estimates of the error of approximation in concrete examples. The idea of a power series.

16. Series in general. Infinite series of real numbers. Various tests for convergence, including the comparison test, n^{th} root test, and the ratio test. Power series. Radii of convergence of power series and their determinations. Uniform convergence of the partial sums on bounded closed intervals within the interval of convergence. Proof of the invariance of the radius of convergence under formal differentiation and integration. Justification of term-by-term integration and differentiation.

17. Further properties of power series. The algebra of power series converging in a fixed radius. Analytic functions, Taylor's theorem, and power series. The possibility of defining functions by means of power series. The power series for certain classical functions, particularly the exponential. Possibility of defining sine and cosine functions by power series and observation that this would eliminate the difficulty that they have not been well-defined until this time.

18. Definition of R^n as n-tuples of real numbers. Distances and limits in R^n, the norm, and perpendicularity. R^2 and R^3 discussed explicitly, together with the physical intuition concerning them. Addition and scalar multiplication in R^n. Linear functions mapping intervals in R^1 into R^n, with particular attention to lines

in R^2 and R^3.

19. Integration of functions from intervals to R^n. Observation that in the definition of the integral of real functions, it was important that the domain be a subset of the line, but that only certain properties of the range entered. Introduction of some examples. Differentiation of functions from intervals to R^n. The derivatives of such functions are defined directly, and then it is observed that they could have been obtained from the coordinate functions.

20. Curves in R^n and their tangents. Newton's laws of motion. Two curves f, g meet at a point t if $f(t) = g(t)$. Their order of contact at such a point is the largest integer n such that

$$\lim_{s\to t} \frac{|f(s) - g(s)|}{|s - t|^n} = 0,$$

or ∞ if no such integer n exists. Tangent lines and the order of contact of a tangent line with a curve.

21. Taylor's theorem for functions from an interval to R^n. Geometric relation of Taylor's theorem with the order of contact of a curve and a "polynomial" curve. Approximation of a curve by "polynomial" curves. Arc length in R^n studied carefully. Inverse trigonometric functions and arc length in R^2. Principal normal to a curve. Curves in R^2 and R^3 and their curvature. The osculating circle as an approximation to such a curve.

22. Velocity, acceleration, Newton's laws of motion. Other physical problems involving differential equations (e.g., vibrating string). Families of curves and differential equations. Solutions of certain simple differential equations. Initial and boundary value problems.

23. First-order differential equations. Approximation to solutions. Lipschitz condition and the existence and uniqueness of solution, both local and nonlocal, by the Picard method. Cauchy's proof of existence. Examples of differential equations with distinct solutions passing through a point. Application to central forces and planetary motion.

24. Power series and linear differential equations with

analytic coefficients. Existence of solutions. The possibility of defining functions as the solution of certain differential equations, illustrated by examples such as sine and cosine. Derivation of properties of the sine and cosine from their defining differential equations.

Outline II, Second Year

<u>Topology and Functions of Several Variables</u>

1. Definition of topological space; continuous mappings in terms of open sets. Definition of metric space, the associated topological space of a metric space, continuity at a point for functions mapping one metric space into another, and the equivalence of continuity with continuity at all points. The examples of Euclidean spaces, spheres, and real projective spaces (defined as quotients of spheres).

2. Subspaces, quotient spaces, and product spaces of topological spaces; restriction of continuous mappings. Subspaces and product spaces for metric spaces, and relation with the same operations on topological spaces. The examples furnished by Euclidean spaces and tori.

3. The notion of Hausdorff space, and proof of its stability under the operations of taking subspaces or products. Observation that metric spaces are Hausdorff. Examples to show that quotient spaces of Hausdorff spaces are not necessarily Hausdorff.

4. The notion of compactness defined for a Hausdorff space by the finite covering property. Closed bounded subsets of Euclidean spaces are compact. Proof that a metric space is compact if and only if it is complete and totally bounded.

5. Tychonoff's theorem for finite products. The notion of local compactness, and proof that finite products of locally compact spaces are locally compact. Euclidean spaces and tori.

6. Introduction of complex numbers. Real and complex topological vector spaces. Uniqueness of the topology on finite-dimensional vector spaces.

7. Real and complex projective spaces as the lines in real or

complex n-space. Equivalence of this definition of real projective space with the earlier definition. Complex projective spaces as quotient spaces of spheres. Properties of quotient spaces when the set of points equivalent to any point is compact. Conditions insuring that the quotient of a metric space is metric.

8. Inner products and norms on topological vector spaces. Equivalence of norms of finite-dimensional vector spaces. Isometry of inner product spaces having the same dimension. Cauchy criterion and completeness.

9. Uniform continuity of continuous functions on compact sets. Sequences of functions; the proof that the set of continuous functions is complete. Ascoli's theorem.

10. Connected spaces and components. Continuous images of connected spaces are connected. The Intermediate Value Theorem for real-valued functions on connected topological spaces.

11. Contraction maps in metric spaces and the fixed point theorem for contractions in complete metric spaces. Relation of this theorem to Picard's method for the existence of solutions of ordinary differential equations in open domains in Euclidean space.

12. General results on homogeneous linear systems of differential equations. The solution space. Nonhomogeneous systems. Applications to linear equations of the n^{th} order.

13. Linear system with constant coefficients. Explicit structure of the solution space using Jordan canonical form and exponential. Application to n^{th}-order equations.

14. Integration and volume in Euclidean spaces. Volume of boxes. Domain with smooth boundaries; difficulties involved with bad boundaries. Integral defined using step function and shown to be a uniformly continuous positive linear function. Integrability of continuous functions. The integral as a finitely additive set function. Reduction of multiple integration to iterated integration. Examples and calculations using iterated integrals.

15. The idea of two functions, defined on a domain in Euclidean space with values in a Euclidean space, touching at a point. Explicitly, two continuous functions f and g have order of contact n at x if

$$\lim_{y \to x} \frac{|f(y) - g(y)|}{|x - y|^n} = 0,$$

and touching means order of contact 1. Proof that if an affine function touches f at x, then it is unique; f is defined to be differentiable at x if such an affine function exists. Continuously differentiable functions.

16. Geometric implications of order of contact with particular emphasis on touching. Connection of differentiability with differentiability along lines and with affine approximations. Examples, formulae, and matrix representation using the standard coordinates in Euclidean space.

17. Mean Value Theorem. Recursive definition of higher derivatives. The class of n-times continuously differentiable functions. Taylor's formula, and proof that an n-times differentiable function f has order of contact n at x with the standard approximation to f obtained using the first n derivatives of f at x.

18. Derivatives of composite functions, the chain rule using linear transformations. The Jacobian matrix; examples involving the Jacobian matrix.

19. The Inverse and Implicit Function Theorems in geometric form. Their formulation using coordinates. Invariance of domain under diffeomorphism.

20. Changes of volume induced by a continuously differentiable function. Calculations for a range of examples. The notion of an n-form on a domain in n-space; connection with volume and volume change.

21. Intuitive discussion of differential forms on Euclidean n-space, their use in Newtonian mechanics. Intuitive description of the integral of a q-form over a differentiable singular q-simplex.

22. Exterior algebras for finite-dimensional vector spaces. Morphisms of same induced by linear transformations. Orientations of real vector spaces via the exterior algebra. Duality between q-forms and (n-q)-vectors in an oriented vector space.

23. Introduction of differential forms and vector fields on domains in Euclidean space. Duality between q-forms and fields of

(n-q)-vectors. Integration of differentials over differentiable singular chains.

24. The star isomorphism in the exterior algebra of a Euclidean vector space and its extension to forms and vector fields on a domain in Euclidean space. Morphisms of forms and vector fields induced by differentiable functions.

25. The connection of integration of n-forms over singular n-chains in domains in n-space with the integration of functions defined earlier. Subdivision of domains with smooth boundary and singular chains. Volume, exterior algebras, determinants, and the idea of a Riemannian metric.

26. The exterior derivative and its properties. Connection with gradients; further geometric ideas. Poincaré lemma for convex regions. Interpretation of Poincaré lemma in terms of existence of solutions of differential equations. Exact equations, integrating factors, and calculations in low dimensions. Special properties of 3-space. Curl, divergence, and the exterior derivative.

27. The general Stokes theorem integrating q-forms over singular q-chains. Classical form of Stokes' theorem, including Gauss' theorem, Green's theorem, and the Fundamental Theorem of Calculus. Physical interpretations: study of flows, charges, etc.

28. Further applications of the calculus of differential forms to physical problems, including Maxwell's equations in both Newtonian and relativistic form. Hamilton's equations in dynamics.

Outline II, Linear Algebra

This one-semester course, as part of Outline II, presents separately the fundamental notions of linear algebra which are recommended as suitable for Introductory Undergraduate Mathematics. The course can be presented during any part of the first two years that is deemed appropriate for the students involved.

It is noted here that Chapters 1 through 13 of this course are designated as prerequisite to the algebra courses outlined in Appendix B. This is consistent with the amount of linear algebra in Outlines I and II of Introductory Undergraduate Mathematics.

1. The complex numbers and subfields of the complex numbers. The integers modulo p.

2. Vector spaces and linear transformations. Examples of vector spaces, particularly spaces of n-tuples and spaces of functions. Subspaces, quotient vector spaces. Linear independence, generating sets, and the notion of basis.

3. Dimension for finite-dimensional vector spaces, invariance of dimension, finite-dimensional subspaces of general vector spaces. Behavior of dimension with respect to subspaces and quotient vector spaces.

4. Inner products for real vector spaces; length and volume. Euclidean vector spaces defined as finite-dimensional vector spaces with inner product. Orthogonal bases. Gram-Schmidt process and its relation to volumes. Subspaces, complementary subspaces, and their relation with quotient vector spaces. Lines, planes, hyperplanes, and distances.

5. Hermitian vector spaces defined as complex vector spaces with a complex (Hermitian) inner product. Length of vector, volume of boxes, the associated Euclidean vector space of a Hermitian vector space. Orthogonal bases. Gram-Schmidt process and its relation to volumes. Subspaces, complementary subspaces; lines, planes, hyperplanes, distances.

6. Recollection of definition of linear transformations. Definition of matrix; representation of linear transformations by matrices. Composition of linear transformations. Change of basis.

7. Orthogonal, symmetric, and skew-symmetric transformations of Euclidean vector spaces; their relation with matrices and bases.

8. Unitary, Hermitian symmetric, and skew-Hermitian transformations of Hermitian vector spaces; their relation with matrices and bases.

9. Inductive definition of the determinant of a matrix. Relation of determinants to volumes of boxes.

10. Permutations. New definition of determinant and equivalence with the old. Multiplicative properties of the determinant. The determinant of orthogonal and unitary matrices, and of the transposed matrix.

11. Inverses, adjoints, elementary matrices, and reduction to diagonal form. Applications to systems of linear equations.

12. Proof that in a Euclidean space any nonsingular linear transformation is the product of a positive-definite symmetric transformation and an orthogonal transformation. The idea of Euclidean geometry. Invariance under rotations and translations.

13. Proof that in a Hermitian space any nonsingular linear transformation is the product of a positive-definite symmetric transformation and a unitary transformation. Hermitian geometry.

14. Decompositions of a vector space into irreducible cyclic subspaces relative to a linear transformation. Jordan canonical form.

15. Definition of minimal polynomial. The characteristic polynomial as a product of certain minimal polynomials of irreducible subspaces of a cyclic decomposition.

16. Characteristic vectors, characteristic values, relations with the characteristic polynomial. Special cases involving orthogonal, symmetric, unitary, and Hermitian symmetric transformations.

Outline III, First Year

Functions of One Variable

The questions of what to teach in calculus and how are notoriously difficult to answer, and the answer has to be reargued by each generation. The main difficulty is that calculus has to be both problem-oriented and theory-oriented. The former means that the student must be made aware of how theories arise to deal with concrete problems, that these concrete problems often originate in the external world, and that the external world is an important source of our intuition (and of our aesthetic criteria). The latter means that the basic concepts should be introduced in the same spirit in which they are used by working mathematicians, and that proofs ought to have the same clarity and elegance which distinguishes all first-rate mathematics.

Fortunately, the two views do not conflict but complement each other: to demonstrate how an abstract theory is developed to deal with a concrete problem and unify what is common in various problems is one of the most valuable lessons for budding young mathematicians, far more valuable than merely presenting the postulates for real numbers or the axioms of linear algebra.

The choice of subjects and their arrangement is not entirely rigid, e.g., the construction of the real numbers, which is put at the beginning, can just as reasonably be done later. The same goes for the uniform continuity of continuous functions over compact intervals.

1. Real numbers. The intuitive notion of the continuum of real numbers. The gaps in the rational numbers (Pythagorean theorem); construction of the real number system, either by nested intervals, Dedekind cuts, or infinite decimals. The topology of real numbers; the algebra of limits.

Three basic theorems: the real numbers are complete; closed bounded intervals are compact; a bounded set of real numbers has a supremum.

Nondenumerability of real numbers, denumerability of rational and algebraic numbers.

Mathematical induction.

2. Analytic geometry. Points of 2-, 3-, and n-dimensional space as ordered n-tuples of real numbers. Addition, multiplication by scalars. Straight lines, convex sets, hyperplanes, linear subspaces. Dimension of linear subspaces.

Euclidean distance, scalar product, Schwarz inequality. Orthogonality. Gram-Schmidt process.

Complex numbers.

3. Differentiation. The concept of a function; illustration, graphical representation. Intuitive notion and rigorous definition of a continuous function. The algebra of continuous functions.

Intuitive notion and rigorous definition of the derivative as slope and instantaneous velocity. Derivatives of polynomials. Algebraic rules for differentiating sums, differences, constant multiples, products, and quotients of functions.

Differentiation of trigonometric functions, based on geometric definition.

Linear approximation to functions; derivation of the chain rule.

Local existence and differentiability of the inverse of a function with nonzero derivative. Newton's method.

4. Integration. The intuitive notion of integral as signed area, work; examples of integrals which can be calculated by a direct passage to the limit.

Existence of the integral of a uniformly continuous function over a finite interval.

Basic properties of the integral: linearity, positivity. The Mean Value Theorem.

The integral as function of its upper limit. Differentiation as the inverse of integration. The log function and its inverse. Statement of the theorem that a function with zero derivative is constant. Integration as antidifferentiation. The inverse trigonometric functions. Techniques of integration, partial fractions, integration by parts, change of variables.

Estimation of integrals, Stirling's formula.

Arc length, surface area, and volume of bodies of revolution.

5. More about continuous and differentiable functions. Three theorems about continuous functions: existence of maximum and minimum over a finite closed interval, existence of intermediate values, and uniform continuity of continuous function in compact intervals.

Proof of the Mean Value Theorem. Proof that if $f' = 0$, then f is constant.

Calculation of maxima and minima.

Higher derivatives; their geometric and physical significance.

Taylor's theorem with remainder (both derivative and integral form). Taylor series for the exponential and trigonometric functions, the logarithm, the binomial series. Examples of functions [e.g., $\exp(-1/x^2)$] which are not represented by Taylor series.

The notion of the maximum norm; uniform convergence. The completeness of the continuous functions under the maximum norm. Continuity of the integral with respect to the maximum norm.

Termwise differentiation of series.

The interval of convergence of a power series; calculus of convergent power series.

Improper integrals.

Outline III, Second Year

Linear Algebra and Functions of Several Variables

Again, the choice of subjects and their arrangement is not entirely rigid. Sections 6 and 7 contain more advanced topics; some fraction of these may be covered if there is time.

1. Vector- and matrix-valued functions and their applications in geometry and mechanics. Linear transformations of the plane and 3-space into themselves; their description with the aid of matrices. Definition of a matrix as a linear transformation of R^n into R^m. The multiplication of matrices via the composition of transformations.

Definition of symmetric, antisymmetric, and orthogonal matrices. Orthogonal matrices form a noncommutative group. Description of orthogonal matrices in two and three dimensions in terms of rotation and reflection.

Curves in n-dimensional space as vector-valued functions. The notion of continuity and differentiability of vector-valued and matrix-valued functions. Algebraic rules for differentiating scalar and matrix products of functions.

Arc length and curvature in 2- and 3-dimensional space.

Orthogonal transformations depending on a parameter; their derivative expressed in terms of antisymmetric transformations. Geometric interpretation as infinitesimal rotation. Introduction of vector product in 3-dimensional space.

Mechanics: Newton's laws for particles. Systems of particles acting on each other by central forces. Center of mass, the moment of forces. Rate of change of momentum, angular momentum, and energy.

Motion of rigid systems of particles. Moment of inertia.

2. Ordinary differential equations; application of some notions from linear algebra. Examples of differential equations from physics, chemistry, and geometry, and their explicit solution in terms of elementary functions.

Radioactive decay, vibrating spring, law of mass action, two-body problem, oscillation of electric circuits, trajectories of simple vector fields, etc. Examples where physical intuition suggests the qualitative behavior of solution: under- and over-damp, etc.

Examples of differential equations which cannot be solved explicitly: the three-body problem, etc. The need for a theory, i.e., existence and uniqueness theorems, qualitative estimates, and methods for finding approximate solutions.

Statement and motivation of the existence and uniqueness theorem for the initial value problem. Proof of uniqueness by conservation of energy in special situations (e.g., vibrating spring). Solution of analytic initial value problems by power series; recovery of the exponential and trigonometric functions.

Difference methods for solving the initial value problem. Comparison of exact and approximate solutions in simple cases which can be handled explicitly.

The fixed point theorem for contracting transformations of a complete metric space. Proof of the existence and uniqueness theorem (do it in the special but typical case of a single first-order equation).

The abstract notion of a linear space over the complex numbers. Dimension, coordinates. Linear transformations.

The notion of an operator mapping a certain class of functions into another. Linear operators, linear differential operators.

The set of solutions of a homogeneous linear differential equation forms a linear space. Calculation of the dimension of this space by the existence and uniqueness theorem.

First-order matrix equations $y' = A(t)y$. The solution operator $U(t)$ defined by $y(t) = U(t)y$. Solution of the inhomogeneous equation $y' = A(t)y + f$ given by $y(t) = U(t) \int_0^t U^{-1}(s)f(s)\, ds$.

The algebra of scalar differential operators with constant coefficients: factorization, commutation. Main theorem: if $L_1, \ldots, L_k$ are pairwise relatively prime, then the nullspace of their product is the direct sum of their nullspaces. Proof based on main lemma about relatively prime polynomials. Solution of $(D - \lambda)^n u = 0$ in terms of exponentials and polynomials.

Existence of a complete set of generalized eigenvectors of a linear transformation of a linear space into itself, based on main lemma about polynomials. Triangular form of a matrix.

Differentiation transforms any solution of a linear differential equation with constant coefficients into another solution. The eigenvectors of this transformation are exponentials times polynomials.

The signature of a quadratic form.

The spectral theory of symmetric matrices. Extremal property of eigenvalues. Positive-definiteness. Spectral theory of unitary matrices.

Small vibrations of mechanical systems. Monotonic dependence of characteristic frequencies on the potential energy.

Sturm separation theorem. Simple two-point boundary value problems. Characteristic frequencies, resonance.

3. Differentiation of functions of several variables. Determinants as alternating multilinear functionals.

Open and closed subsets of n-dimensional space. Compactness of bounded, closed subsets.

Functions defined on subsets of n-space. Continuity. Existence of maxima and minima on compact sets. Uniform continuity on compact sets.

Differentiability at interior points in terms of approximation by linear functions. Chain rule. Partial derivatives of first order. Maxima and minima, stationary points. Geometric interpretation of grad f in Euclidean space as normal to surface $f = \text{const}$. Examples.

Functions k times differentiable; approximation by polynomials of k^{th} order. Higher partial derivatives; commutation of partial differentiation. Classification of stationary points. Examples.

Extreme values under side conditions; Lagrange multiplier.

Vector fields; fields of force, gradient fields, conservation of energy, Newtonian potential.

Mapping of n-space into m-space; Jacobian. Composition of mappings. Implicit and Inverse Function Theorem.

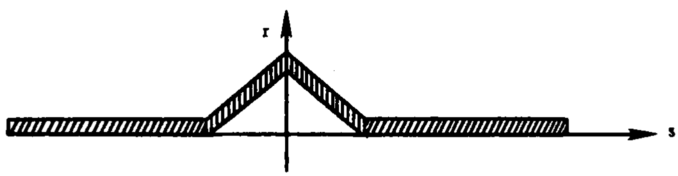
Conformal mapping.

The degree of a mapping, following the method of E. Heinz. Journal of Mathematics and Mechanics, 8 (1959), pp. 231-247.

4. Integration of functions of several variables. The intuitive notion of the integral in Euclidean space: volume, mass, momentum, moment of inertia, potential of mass, etc.

Rigorous definition, starting with the integral of continuous functions with compact support defined by using rectangles in a fixed orthogonal frame. Properties of the integral: linearity, positivity, translation invariance.

Theorem. These properties characterize the integral up to a positive multiple. Proof: Let $I(f)$ be a linear, translation-invariant, positive functional defined for all continuous functions with compact support. Denote by $r(s)$ the "roof" function graphed below:



Denote $r(ms)$ by $r_m(s)$. Every piecewise linear function whose derivatives are discontinuous only at the points i/m, i an integer, can be expressed as a linear combination of $r_m(s)$ and its translates. In particular,

$$r(s) = \sum_{-m}^{m} \left|1 - \frac{k}{m}\right| r_m\left(s + \frac{k}{m}\right).$$

Define now

$$h(x) = \prod_{j=1}^{n} r(x_j)$$

$$h_m(x) = h(mx).$$

Putting $s = x_j$ and multiplying we get

$$h(x) = \sum a_{k,m} h_m\left(x + \frac{k}{m}\right),$$

k a multi-index,

$$\sum_k a_{k,m} = m^n.$$

So, using the first two properties of I, we get

$$I(h) = m^n I(h_m).$$

Every piecewise linear function $\ell(x)$ in R^n whose derivatives are discontinuous only on the hyperplanes $x = i/m$, i an integer, can be expressed as a linear combination of h_m and its translates. This shows that $I(\ell)$ can be expressed in terms of $I(h)$. Since every continuous function f with compact support can be approximated by such piecewise linear functions, it follows that $I(f)$ can be expressed in terms of $I(h)$.

Corollary 1. Volume integral = repeated integral.

Corollary 2. Integral is independent of orthogonal frame chosen (consider functions which depend only on $|x|$).

Corollary 3. Under a linear change of variables, the integral is multiplied by a factor, which is a multiplicative functional of the matrix of the transformation.

Corollary 4. This factor is the absolute value of the determinant of the matrix of the transformation. (Proof by writing the matrix as a product of orthogonal and diagonal transformations and using the first three corollaries.)

The formula for integration by parts:

$$\int f_{x_j} g \, dx = -\int f g_{x_j} \, dx$$

follows from Corollary 1.

Integration over open sets. Intuitive notion of volume of an open set in terms of filling it up with cubes of unequal size. Rigorous definition:

$$V(D) = \operatorname{Sup} \int f \, dx \begin{cases} f \leqq 1 & \text{in } D \\ f \leqq 0 & \text{outside } D \end{cases}$$

Corollary. Volume is unchanged under rotation and translation. Interpretation of determinant as volume.

Intuitive notion of the integral of a function over an open set in terms of approximating sums over cubes of unequal size.

Rigorous definition:

$$\int_D f\,dx = \operatorname{Sup} \int g\,dx \left\{ \begin{array}{ll} g \leqq f & \text{in } D \\ g \leqq 0 & \text{outside } D \end{array} \right.$$

Corollary.

$$\int_D f\,dx \leqq V(D)|f|_{max}.$$

Evaluation of integral over D by repeated integration when f is continuous up to the boundary of D and the boundary of D is nice, i.e., each line cuts it only in a finite number of places (convex domains and unions of convex domains). Examples.

Change of variables in one-to-one multiple integrals: Let $x \to y$ be a mapping of an open set D in x-space onto a set G in y-space with continuous first derivatives and nonzero Jacobian. Let $f(y)$ be a continuous function with support in G. Then

$$(*) \qquad \int_D f(y(x))\left|\frac{\partial y}{\partial x}\right| dx = \int_G f(y)\,dy.$$

Proof: Let $\sum_j p_j(y) \equiv 1$ be a smooth partition of unity in y-space, $g_j(y) = p_j(ny)$ a refinement of it. Write the left side of (*) as

$$\sum_j \int g_j(y(x))f(y(x))\left|\frac{\partial y}{\partial x}\right| dx.$$

For j fixed, replace the Jacobian by its value at x_j, $y(x)$ by a linear approximation to it. By Corollary 4 the resulting integral equals $\int g_j(y)f(y)\,dy$; the total error committed is easily estimated and tends to zero with increasing n.

Examples: Change to polar coordinates. Evaluation of various integrals, such as the error integral.

Area-preserving maps. Canonical transformations.

Domain with smooth boundary defined by possibility of smooth local parametrization. Proof that $f(x) < 0$ has a smooth boundary if $\operatorname{grad} f \neq 0$. Intuitive notion of surface area. Definition by integral; independence of parametrization. Surface integrals.

Integration by parts over domains with smooth boundaries.

Continuous functions form a Euclidean space under scalar product $(f,g) = \int fg\,dx$. Notion of a linear operator. Symmetry and positivity of the Laplace operator under boundary condition $u = 0$

or $du/dn = 0$. Analogy to symmetric positive matrices. Uniqueness of boundary value problem for the Laplace equation and the mixed initial and boundary value problem for the wave equation.

5. Exterior forms. The Gauss and Stokes theorems in special cases. Their interpretation for flows and in the theory of electricity and magnetism.

Definition of exterior form, Grassmann algebra, differential of a function.

Integration of forms over singular chains.

The exterior derivative; gradient, curl, and divergence as special cases. The Poincaré lemma.

The general Stokes theorem. Applications to Cauchy's integral theorem.

6. Introduction to the calculus of variations. Examples of problems in the calculus of variations for functions of one variable. The general problem of finding extrema for

$$\int f(x,y,y')\,dx.$$

The Euler equation; examples where Euler equation can be solved explicitly.

Quadratic variational problems. Proof that the integral is definite if the underlying interval is short enough.

The second variation; examples where the second variation is not positive (catenoid); conjugate points. Geodesics; example of the Poincaré half-plane.

Variational problems for functions of several variables. The Dirichlet integral. Plateau's problem.

7. Harmonic analysis. Fourier transform, Parseval's formula. Convolution.

Appendix B

HIGHER UNDERGRADUATE MATHEMATICS

In this section we have placed one or more outlines for each of the courses which the Panel suggests should be provided by every college department undertaking the program and which are relevant to the central areas of mathematics:

Real analysis

Complex analysis

Abstract algebra

Geometry-topology

Probability or mathematical physics

It is not intended that these outlines shall be construed as completely determining the content of these courses; alternatives will be welcomed by the Panel.

Real Analysis (One Year)

This outline deals with the following three major topics in real analysis:

(1) Various classes of generalized functions such as L_1, L_2, L_∞, distributions, etc.

(2) Measure theory

(3) Nondiscrete decomposition

These topics are basic in a wide variety of fields in analysis, such as the theory of differential equations, the calculus of variations, harmonic analysis, complex variables, probability theory, topological dynamics, spectral theory, and many others.

We advocate presenting this material, notably that listed under (1), within the framework of general topology and functional analysis. The necessary background is developed in Sections 3, 4, and 5; to a certain extent this constitutes a review of material already covered in the Introductory Undergraduate Mathematics. As the outline shows, we believe strongly, as did the founding fathers, in dealing first with special cases and in presenting applications along with the general theory.

Perhaps our most radical departure from tradition is advocating the presentation of the notions of strong derivatives in the sense of Friedrichs and Sobolev, and distributions in the sense of

Schwartz, rather than the Lebesgue theory of differentiation. We feel that this is justified by the simplicity and general usefulness of the newer theories.

There is more material presented here than would fit into a year's course. Sections 8-12 are offered as a variety to choose from. The subjects in the first seven Sections are basic, but the material outlined is a little more than what is strictly necessary for a self-contained treatment.

Fortunately, most of the subjects discussed in the outline are available in textbooks, although not all within the covers of one text.

1. Set theory. Review of the terminology of set theory. One-to-one correspondence, countable and uncountable sets; the uncountability of the real numbers and of other interesting sets. Equivalence relations, order. The Schroeder-Bernstein theorem. The axiom of choice and Zorn's lemma.

2. Real numbers. The construction of real numbers by completion (equivalence classes of Cauchy sequences). The compactness of closed, finite intervals. Hamel basis.

3. Metric spaces. Definition, examples: the continuous functions, L_1, L_2, L_p; the Schwarz and Hölder inequalities. Open and closed sets, dense and nowhere dense sets, separability. Bernstein polynomials and the Weierstrass approximation theorem; Chebyshev's theorem on best approximation.

Completeness and the process of completion. Fixed point theorem and its application. Baire category theorem and its applications. Continuous functions; Tietze's extension theorem.

Compactness and local compactness, Arzelà-Ascoli and Rellich compactness theorems and applications.

4. Topological spaces. Definition, examples. Open and closed sets. Hausdorff spaces. Separability. Compactness; Stone-Weierstrass theorem, Tychonoff's theorem. Topological groups.

5. Normed linear spaces. Hilbert space: definition, orthonormal base, projection theorem, representation of linear functionals. Bounded operators, adjoints, symmetric and unitary operators.

Banach spaces: definition, linear functionals, dual space, bounded linear operators.

Banach-Steinhaus theorem, applications. Hahn-Banach theorem, applications (moment problems).

6. Integrals and measures. There are two competing approaches, neither of which should be slighted. (A) is functional analysis-oriented with applications in classical analysis generally (e.g., orthogonal series, differential and integral equations, classical probability theory). (B) is measure-theoretical, with applications in stochastic processes, ergodic theory, and statistics.

(A) The space C_0 of continuous functions on a complete, locally compact metric space. Signed and complex measures. Relation of measures: absolute continuity, Radon-Nikodým theorem. Convergence theorems. Fubini's theorem. Riesz representation theorem.

(B) Classical general measure-theoretic methods: outer measure, extension of a measure through outer measure, Kolmogorov consistency criterion, conditional measures. Product measure. Mention of finitely additive measures.

7. Differentiation. Functions in n-dimensional Cartesian space. Strong derivatives in the sense of Friedrichs and Sobolev. Sobolev's theorem. Applications to differential equations. Schwartz theory of distributions; applications.

Vitali covering theorem and differentiation almost everywhere.

8. Applications to classical analysis. Orthonormal series, Fourier and other transforms, Riesz-Fischer theorem, Fourier transforms of L_2 and of tempered distributions. Convolution. Classical inequalities based on convexity. Theory of approximation. Applications.

9. Integration on groups. Construction of the Haar measure. Examples.

10. Measure spaces. Definition of an abstract measure space. Measure on the Cartesian product of a countable number of circles. Application: the convergence of random series.

11. Banach algebras. Definition, the Gelfand theorem on the existence of multiplicative linear functionals. Applications to Fourier series and function theory.

12. Spectral resolution of self-adjoint operators. The spectral resolution of bounded, symmetric operators. The discrete,

singular, and absolutely continuous parts of the spectrum.

Complex Analysis, Outline I (One Year)

The first semester of this course, covering Sections 1-8, includes the standard elementary (but basic) topics from the theory of functions of one complex variable. The content of the second semester centers about the conformal mapping theorems for regions of finite connectivity, including the necessary tools for their proof. Other suitable topics for the second semester may be found in Selected Topics in the Classical Theory of Functions of a Complex Variable by Maurice Heins (New York, Holt, Rinehart and Winston, Inc., 1962) and in Banach Spaces of Analytic Functions by Kenneth Hoffman (Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962).

1. Complex numbers as ordered pairs of reals, field properties. Conjugate and absolute value, geometric properties. Polar representation. Stereographic projection and the extended plane.

2. Elementary functions. The derivative. Analytic functions on open connected sets. Detailed treatment of examples: polynomials, rational functions, the group of linear fractional functions, exponential and trigonometric functions.

3. Conformal mapping by elementary functions. Proof that an analytic function is conformal at points where its derivative does not vanish. Specific conformal mappings.

4. Integration along piecewise continuously differentiable curves. Cauchy's theorem for rectangle and circular disk. Integral representation of the derivative. Morera's theorem, Liouville's theorem, and the Fundamental Theorem of Algebra.

5. Taylor series development. Proof that the uniform limit of analytic functions is analytic. Classification of isolated singularities--removable, poles, essential singularities. Zeros of nontrivial analytic functions are isolated. Laurent series.

6. Nonconstant analytic functions are open. The Maximum Modulus Theorem. Schwarz' lemma. The one-to-one analytic maps of the unit disk onto itself.

7. Cauchy's theorem and homology. Simply and multiply connected regions. The Residue Theorem. Argument principle. Rouché's theorem. Evaluation of definite integrals using the Residue Theorem.

Implicit Function Theorem.

8. Analytic continuation.

9. Meromorphic functions. Infinite products; the Weierstrass factorization theorem. Mittag-Leffler theorem.

10. Compact families of analytic functions. Montel's theorem. Conformal equivalence of simply-connected regions. The Riemann mapping theorem.

11. Harmonic functions. Elementary properties: Mean Value Theorem, maximum principle, isolated singularities.

12. The Dirichlet problem for the disk, with continuous boundary values. The Poisson integral.

13. Applications of the Poisson integral: a continuous function having the mean-value property is harmonic, uniform limit of harmonic functions is harmonic. Harnack inequalities and convergence theorem.

14. Subharmonic functions. Elementary properties. Perron's theorem.

15. The Dirichlet problem for a region. Sufficient conditions for existence of a solution. Barriers.

16. Green's function for a region. Relation with conformal mapping of the region. Regions of finite convexity. Harmonic measures.

17. Conformal mappings of regions of finite connectivity onto standard regions.

18. The Hardy H^p-spaces of analytic functions on the unit disk. (This Section assumes some knowledge of Lebesgue integration.) Fatou's theorem; Herglotz's theorem.

Complex Analysis, Outline II (One Year)

Complex analysis offers a unique opportunity to convince the young student who has only a minimal knowledge of algebra and topology that these subjects can interact with analysis in a useful way. The aim in this outline is to present the few key concepts which are remembered by mathematicians of all fields. Simplicity and a hope to excite the student with continuing ideas are emphasized at the expense of an occasional time-honored result or point of view. It is assumed that the student will be familiar with topological concepts appropriate to the plane, including properties of continuous mappings and of arcwise connectivity.

1. The complex field. Characterization of the real field, complex numbers as pairs of real numbers, the complex field C as a field with valuation and conjugation as its automorphism, geometric interpretations, intuitive and rigorous adjunction of a point at infinity as a compactification of the plane.

2. Power series. The ring of formal power series $K[[x]]$ over a field K, operations with series (formal derivative, reciprocal, inverse), convergent series ($K = R$ or C), uniform convergence of a series of functions, radius of convergence of a formal series, operations with convergent power series (differentiation, reciprocal, inverse), exponential function and logarithm functions.

3. Analytic functions. Real and complex analytic functions defined as functions, on open sets, which are locally power series, and the algebra of functions analytic on a region D; principle of analytic continuation (uniqueness of continuation), zeros of an analytic function (discreteness), rational function, poles, the field of meromorphic functions on D.

4. Integration. Differential forms $P\,dx + Q\,dy = \omega$, differential chains γ, integration $\int_\gamma \omega$, exact chains, closed forms, complex forms, homotopy, winding number of closed chain, generalized principle of argument, and Rouché theorem for mappings.

5. Holomorphic functions. f is holomorphic at z_0 if $f'(z_0)$ exists. Cauchy-Riemann equations, Cauchy theorem (if f is holomorphic in D, then $f(z)\,dz$ is a closed form); existence of local primitives that are holomorphic, Cauchy integral representation, holomorphic functions are analytic, Morera's theorem.

6. Applications of integration. Liouville's theorem, algebraic closure of C, Maximum Modulus Theorem, Open Mapping Theorem, Schwarz' lemma, Laurent representation, isolated singular points, residues, calculation of contour integrals, counting of zeros and poles of a meromorphic function, Schwarz reflection, doubly-periodic functions.

7. Functions of several variables. Formal power series in several variables, domain of convergence, operation with series,

analytic functions in several variables, principle of analytic continuation; harmonic functions, holomorphic functions are (complex variable) harmonic functions, every real harmonic function is (locally) the real part of a holomorphic function, harmonic functions are analytic.

8. Global problems. The Riemann sphere, functions holomorphic in regions on the sphere, the fundamental group $\pi(D)$ of a region D, integration as a homomorphism of $\pi(D)$ into the additive group of C, generators of $\pi(D)$, the covering space of D and general solution of the problem $\mathrm{Re}(f) = u$, where u is harmonic in D, subcovering spaces and normal subgroups of $\pi(D)$.

9. Holomorphic function of several variables. Cauchy's theorem, Taylor's theorem, composition of functions and the Implicit Function Theorem, statement of Hartogs' theorem.

10. Spaces of holomorphic functions. The spaces $C(D)$ and $H(D)$ of functions continuous in D and holomorphic in D, fundamental theorems on convergence in compact sets in D, continuity of differentiation in $H(D)$, the univalent functions as a subset of $H(D)$, series of meromorphic functions, the Weierstrass periodic function; infinite products of holomorphic functions, representation of $\sin(\pi z)$ and $1/\zeta(z)$; closed bounded sets in $H(D)$ are compact.

11. Holomorphic mappings. Local properties of a mapping $\omega = f(z)$, special mappings, the conformal automorphisms of a disc, of the plane, and of the Riemann sphere; the Riemann mapping theorem.

12. Analytic spaces. The general notion of an analytic space, holomorphic mappings of analytic spaces, meromorphic functions on an analytic space, fundamental theorem on conformal equivalence of simply connected analytic spaces, differential forms on an analytic space; Riemann surfaces, analytic continuation.

13. Application to differential equations. Existence theorem and uniqueness theorem, dependence upon initial conditions, higher-order equations.

Abstract Algebra (One Year)

The purpose of this course is to introduce the student to basic structures of abstract algebra and to provide an introduction to applications to various branches of mathematics. The prerequisite for this course consists of the material in Chapters 1-13 of the semester course in linear algebra appearing in Outline II of Appendix A.

The main body of the course is divided into nine Sections outlined below. This initial segment should cover approximately 5/3 semesters; for the remaining 1/3 semester, five options are presented, each starting with Section 10 and each representing an introduction to further specialized study.

1. Groups. Definition of groups and morphisms of groups. Notion of subgroup; quotient group. The permutation groups, representation of any group as a group of permutations; groups of some regular solids. Groups of linear transformations. Orthogonal groups, unitary groups, etc. Abelian groups; abelianization of an arbitrary group.

2. Commutative rings. The definition of commutative ring and discussion of examples including the integers, Gaussian integers, and the integers modulo n. Definition of ideal and quotient ring; further examples. The definition of field, integral domains, Euclidean domains, and principal ideal domains. The Euclidean algorithm. Maximal ideals. The problem of unique factorization; proof that principal ideal domains are unique factorization domains. Examples showing that not all integral domains are unique factorization domains.

3. Commutative rings. Definition of a commutative algebra. Polynomial algebras in a finite number of indeterminates, including both existence and universal properties. The polynomial algebra in one indeterminate over a field as a Euclidean ring. Proof that polynomial algebras in a finite number of indeterminates over a unique factorization domain are again unique factorization domains.

4. Modules over commutative rings. Definition of module, morphism, epimorphism, monomorphism, and isomorphism; examples including vector spaces and abelian groups. Sums and products of modules including explicit constructions and universal properties. Observations that finite sums and products coincide. Exact sequences

of modules, submodule, quotient module, modules of morphisms. The fundamental decomposition theorem for modules over principal ideal domains; application such as a review of cyclic decompositions of vector spaces and Jordan canonical form.

5. Graded and exterior algebra. Representation of morphisms of finitely generated free modules by matrices; dual modules, dual bases, duals of morphisms; relation with matrices, transposes, etc. Graded algebras, commutative graded algebras; examples including free associative algebras, polynomial algebras. Exterior algebras, rank, invariance of dimension of free modules, traces, determinants via the exterior algebra.

6. Polynomial algebras and finite-dimensional vector spaces. Given a vector space over a field and a linear transformation, the vector space becomes a module over the polynomial ring in one indeterminate over a field. Cyclic decompositions. Minimal polynomial and characteristic polynomials of linear transformations; the Cayley-Hamilton theorem. The Jordan and other canonical forms of matrices. Trace and determinant via the characteristic polynomial. Eigenvectors and eigenvalues.

7. Field theory. Splitting field of a polynomial, prime factors, finite fields, and fields of fractions. Algebraic extensions, separability, inseparability, norms and traces. Roots of unity, algebraic number field, the theorem of the primitive element. Algebraically closed fields; existence and uniqueness of the algebraic closure.

8. Group theory. Isomorphism theorems for group theory. Composition series; Jordan-Hölder-Schreier theorem. Product of groups. The Remak decomposition for finite groups. Solvable groups, the Sylow theorems; examples. Further study of the permutation groups. Simplicity of the alternating group for $n > 4$.

9. Galois theory. Automorphism of fields, fixed fields of groups of automorphisms of the splitting field of a polynomial as a permutation group. Galois extensions defined using finite automorphism groups; criteria for an extension to be Galois. Fundamental Theorem of Galois Theory from the Artin point of view; discussion of other proofs. Fields of fractions of polynomial rings. Galois

extensions with a symmetric group or Galois group, solvable extensions and relations with solvability of equations by radicals.

Option I: Algebraic Number Theory

10. Rings of integers. Finite extensions of the rational number field; calculation of sample Galois groups. Definitions of rings of integers, calculations of the primes and prime ideal in various examples; examples of rings of integers in number fields which are not principal ideal domains, and examples which are principal ideal domains but not Euclidean rings. Definition of Dedekind ring; proof that the rings of integers in finite separable extensions of the field of fractions of a Dedekind ring are again Dedekind. Various characterizations of Dedekind rings. Study of the rings of integers in quadratic extensions of the rationals.

11. Dedekind rings and modules. Fractional ideals, classical ideal theory for Dedekind rings; examples. Modules over Dedekind rings; fundamental theorem for finitely generated modules. The ideal class group. Finiteness via Minkowski's lemma. Class numbers. Finitely generated torsion-free modules are characterized up to isomorphism by their rank and ideal class, using the exterior algebra to determine the ideal class of the module. Calculations of ideal class groups for a few simple examples.

12. Introductory algebraic number theory. Integral bases, examples, and proof of existence in general. Units, the cyclotomic fields, units in quadratic extensions. The Dirichlet-Minkowski theorem on units. Calculation of various examples.

13. Further introductory number theory. Ramified and unramified primes; examples. Decomposition groups, ramification groups, etc.; examples. Abelian extensions. Cyclotomic and quadratic fields; quadratic reciprocity law.

Option II: Noetherian Rings and Modules

10. Rings with minimum condition. Definition and fundamental properties, equivalence with descending chain condition. Correspondence to rings with no nonzero nilpotent (left) ideals. Reduction to semisimple rings, matrix characterization of simple rings. Modules and their structure as sums of a minimal ideal, fields; quadratic specialization to vector spaces.

11. Noetherian rings. Definition and fundamental properties, equivalence with ascending chain condition. Hilbert basis theorem. Normal decompositions. Correspondence with local rings, decompositions again.

12. Dedekind domains. Definition and some equivalent notions. Characterization of ideals. Finitely generated modules, torsion-free characterization. Torsion modules, connections with matrices. Hilbert zero theorem. Special case: ideal theory for quadratic number fields.

13. Representation theory of groups. Representation of degree n (over an algebraically closed field F whose character does not divide order of the group G), equivalent representations, connections with finitely generated left F(G)-module. Characters, equivalence, direct sum decomposition, irreducible characters. Computation for the symmetric group.

Option III: Geometry of Classical Groups

10. Affine and projective geometry. Affine geometry; synthetic approach and construction of a field. Desargues' theorem, Pappus' theorem and commutativity. Projective geometry, introduction and fundamental theorems. Examples of projective geometries; the projective plane.

11. Quadratic forms. Definition of quadratic forms and their elementary geometry. Orthogonal quadratic forms, orthogonal sum of subspaces. Orthogonal geometry (especially over finite fields). Symplectic forms, symplectic geometry (especially over finite fields).

12. Orthogonal and symplectic groups. Euclidean orthogonal groups. General orthogonal groups, Clifford algebras, spinor norms. Structure of the orthogonal group; structure of the symplectic group.

Option IV: Equations of Fifth Degree

10. Representations of finite groups. Definitions and general properties of representations. Characters; complete reducibility (under the appropriate assumption concerning the characteristic of the field). Schur's lemma; relations on characters (over the complex numbers). Some computations for the symmetric group. Invariants of finite groups.

11. Equations of fifth degree. Lüroth's theorem; group of automorphisms of a rational function field of one variable. Determination of all finite subgroups (and their invariants for character zero); peculiarities of the modular case. The icosahedral equation. Bring's equation. The icosahedral equation as resolvent of the general equation of the fifth degree. Kronecker's theorem on the nonexistence of rational resolvents for general equations of degree greater than or equal to five.

Option V: Elliptic Function Fields

10. Algebraic function fields of one variable. Places and valuations; completion of a field with respect to a valuation. Existence of places; order functions. Divisors and divisor classes. Differentials. Special cases: partial fractions for rational function fields; Riemann-Roch theorem for elliptic function fields. Riemann-Roch theorem for hyperelliptic function fields. Analogies with quadratic number fields.

11. Algebra of elliptic functions. Algebraic group structure of a nonsingular plane cubic curve; the absolute invariant. Addition theorem for elliptic functions, multiplication and division of elliptic functions. Divisor classes of finite order in an elliptic function field. Modular equations (e.g., their Galois groups). Euler's theory of elliptic functions. Gauss' theory of the lemniscate.

Function-theoretic viewpoint.

Geometry-Topology

Aspects of point set topology are prerequisite to most beginning graduate programs--e.g., metric spaces, compactness, connectedness, effects of continuous mappings on such properties, uniformly continuous mappings, Tychonoff's theorem. A substantial part of this material is discussed in the Introductory Undergraduate Mathematics, and it is not regarded as necessary that a separate course in point set topology be in the undergraduate curriculum.

It is, however, highly desirable that every undergraduate take part in some sustained, deep geometric development. Such a variety of significant geometric developments is possible, differing in method and aim from the very start, that the Panel is reluctant to suggest any one or two of them as belonging to every undergraduate's program. Instead, we propose a larger number of courses and recommend that each student take one or two of them. The specific ones of these courses that may be offered in a given college depend on the interests and training of its staff. Hopefully, this will lead to a wide divergence in the types of geometers eventually produced.

We have also felt that the hybrid title "Geometry-Topology" is more descriptive of this area than either title alone.

Outline I. Set-Theoretic Topology (One Year)

The set-theoretic topology included in the calculus course is limited to that needed for the multidimensional calculus. In this course it is developed in a more abstract setting. These techniques and results are then applied to study topological groups, covering spaces, the fundamental group, and 2-dimensional manifolds.

The course outlined is a one-year course. However, Sections 10 through 17 and 18 through 26 are independent. This permits various choices as determined by the interests of the group concerned.

1. Hausdorff spaces. Compactness, local compactness, one-point compactification, sequential compactness. Continuous, open, closed mappings. Uniform continuity.

2. Connectedness, local connectedness, components. Preservation under mappings. Nonlocal connectedness.

3. Product spaces, quotient spaces. The Hilbert cube. Hausdorff maximality principle or the axiom of choice. Product of compact spaces is compact.

4. Separability, 2nd countability. Countability arguments (Brouwer reduction theorem; i.e., irreducibility). Baire category theorem and general method of argument.

5. Metric spaces, equivalent metrics, completeness, topological completeness. G_δ-sets. Baire category theorem in complete metric spaces.

6. Urysohn's lemma, Tietze's extension theorem, metrizability for locally compact 2nd-countable Hausdorff spaces. Paracompactness and Smirnov metrization theorem.

7. Upper semi-continuous and continuous decompositions of compact metric spaces. Hausdorff metric. Relationship of decompositions to mappings.

8. Hahn-Mazurkiewicz theorem, arcwise connectivity.

9. Characterizations of arcs and 1-manifolds.

10. Topological groups, nuclei, quotient spaces.

11. Projection mapping $G \to G/H$ is closed mapping if H is compact. G is compact (locally compact) if H is compact and G/H is compact (locally compact). Examples: orthogonal and unitary groups, Stiefel manifolds.

12. Local isomorphism of topological groups; $G \to G/N$ is a local isomorphism if N is a discrete normal subgroup. If G and G' are locally isomorphic, there exists H with discrete normal subgroups N and N' so that G is isomorphic to H/N and G' is isomorphic to H/N'.

13. Paths, homotopies of paths, fundamental group. Pathspace PX of a topological space X with base point e; continuity of projection map π: $PX \to X$.

14. Pathspace of topological group is topological group and π is a homomorphism; π is open, onto if X is a pathwise connected, locally pathwise connected topological group.

15. $\Omega X = \pi^{-1}(e)$, e the unit of X, and $\Omega_0 X$, the identity component of X, are closed normal subgroups of PX; $\tilde{X} = PX/\Omega_0 X$ is the universal covering group of X and φ: $\tilde{X} \to X$ induced by π is the covering map. The kernel of φ is $\Omega X/\Omega_0 X$ and φ is open, onto, continuous homomorphism if X is pathwise connected and locally pathwise connected.

16. If X is a pathwise connected, locally pathwise connected, semi-locally simply connected topological group, then $\varphi: \tilde{X} \to X$ is a local isomorphism with kernel $N = \Omega X/\Omega_0 X$ = fundamental group of X; $\tilde{X}$ is simply connected.

17. In the class of all pathwise connected, locally pathwise connected, semi-locally simply connected topological groups which are locally isomorphic to one such group, there is uniquely, up to isomorphism, a simply connected group C^* in the class, and for any C in the class, $C^*/N \cong C$, where N is the fundamental group of C and is in the center of C^*.

18. Fundamental group of the unit circle (as topological group). Winding number of closed path in the plane relative to a point (as element of fundamental group of punctured plane). Fundamental Theorem of Algebra. Simple arc does not disconnect the plane.

19. Jordan curve theorem.

20. Arcwise accessibility of points of arcs and simple closed curves in the plane from their complements. Schoenflies theorem.

21. Simplicial complex, abstract complex, geometric realizations and polyhedra. Imbedding theorem for n-dimensional complexes. Simplicial approximation theorem. Fixed point theorem for n-cells (Hirsch's proof).

22. Manifolds. Triangulability of compact 2-manifolds. Haupt-vermutung for compact 2-manifolds.

23. Cuts and handles. Orientability.

24. Invariance of Euler characteristic. Connectivity of 2-manifolds.

25. Classification of 2-manifolds.

26. Bicollaring, Brown-Mazur theorem.

Outline II. Algebraic Topology (One Year)

This course introduces the student to the tools and techniques of homology theory through a continuation of the study of differential forms as in Outline II of the Introductory Undergraduate Mathematics. For those with a background from Outlines I or III, several topics in Outline II must be studied first.

1. Differentiable manifolds of various classes, charts, atlases. Differentiable mappings. Orientation.

2. Differential forms in coordinate neighborhoods, morphisms, coordinate transformations. Differential forms on manifolds.

3. Exterior derivative, effect on products and transformations and of iteration. The differential forms $F^*(M)$ on a differential manifold M and exterior derivative viewed as a cochain complex, with product. Contravariant homomorphism induced by differentiable mappings of manifolds; compositions.

4. The standard and affine simplices in Euclidean space, face operation. Singular and differential chain group. Boundary of affine and singular chains. Induced mappings and commutation with boundary, $\partial\partial = 0$. Stokes' theorem on compact manifolds.

5. Closed and exact differential forms; cycles and boundaries; cohomology of forms and homology of singular (differentiable) chains. Stokes' theorem establishing dualities between the various classes of forms and chains. Closed forms as linear functionals on homology classes.

6. Definition of singular homology groups and the de Rham groups (i.e., graded quotients of closed by exact forms). Statement of de Rham's theorem in form that the de Rham groups are dual vector spaces to the singular homology groups.

7. Local triviality of the singular and de Rham groups for contractible (or differentially contractible) spaces; in the case of de Rham groups, by integration by parts; introduction of chain and cochain homotopies. Cone construction for singular groups.

8. Singular cohomology groups. Singular cochains with coefficients in an abelian group G as the group of homomorphisms of the group of singular chains into G. Coboundary operator as $\mathrm{Hom}\ (\partial)$. Cocycles, coboundaries, cohomology. Properties under mappings. Isomorphism $H^p(X;\ G) \cong \mathrm{Hom}\ (H_p(X);\ G)$ for divisible groups. Restatement of de Rham's theorem as saying the de Rham groups $R^p(M) \cong H^p(M;R)$, where R is the real numbers.

9. Sub-cochain complexes, quotients; homomorphisms of cochain complexes; Bockstein exact sequence for cohomology for a short exact sequence of coefficient groups.

10. System of local coefficients for the singular chain complex; cochains with local coefficients; cohomology. Homomorphisms of local coefficient systems; short exact sequences and the Bockstein sequence.

11. Simplicial complexes, abstract simplicial complexes, polyhedra, geometric realization; simplicial mappings. Oriented simplicial chain complex; alternating simplicial cochain groups. Natural mapping of simplicial homology of an abstract simplicial complex into the singular homology of its geometric realization; same for cohomology; proof later of its isomorphism. Local coefficients for simplicial complexes.

12. Nerve N of a covering; presheaves; cohomology of the nerve with coefficients in the local system of the presheaf. Examples; significance of $H^0(N;G)$.

13. Proof that for a contractible (differentially) covering $\{U\}$ of the connected manifold M, if $\mathcal{C}^p(\mathfrak{F}^p)$ is the presheaf of singular p-cochains (differential p-forms), as a system of local coefficients on the nerve of $\{U\}$, we have

$$0 \to \overset{0}{\mathcal{C}}{}^{p-1} \to \mathcal{C}^{p-1} \overset{\delta}{\to} \overset{0}{\mathcal{C}}{}^{p} \to 0 \text{ is exact}, \quad p > 0, \quad \overset{0}{\mathcal{C}}{}^{0} = G,$$

constant

$$0 \to \overset{0}{\mathfrak{F}}{}^{p-1} \to \mathfrak{F}^{p-1} \overset{d}{\to} \overset{0}{\mathfrak{F}}{}^{p} \to 0 \text{ is exact}, \quad p > 0, \quad \overset{0}{\mathfrak{F}}{}^{0} = R,$$

where $\overset{0}{\mathcal{C}}{}^{p-1}$ is the local system of (p-1)-cocycles, etc. (local triviality). Proof that for a finite open covering of M with nerve N, $H^p(N;\mathcal{C}^q) \cong H^p(N;\mathfrak{F}^q) = 0$ for $p > 0$, $q \geqq 0$. (Use partition [differentiable] of unity subordinate to the covering.)

14. Existence of a differentiably contractible finite open covering on a compact differentiable contractible finite open covering on a compact differentiable manifold (assume some elementary Riemannian geometry). For such a covering, observe by use of Bockstein sequence that

$$R^p(M) \cong H^1(N;\overset{0}{\mathfrak{F}}{}^{p-1}) \cong H^2(N;\overset{0}{\mathfrak{F}}{}^{p-2}) \cong \ldots \cong H^p(N;\overset{0}{\mathfrak{F}}{}^{0}) \cong H^p(N;R)$$

$$H^p(M;G) \cong H^1(N;\overset{0}{\mathcal{C}}{}^{p-1}) \cong H^2(N;\overset{0}{\mathcal{C}}{}^{p-2}) \cong \cdots \cong H^p(N;\overset{0}{\mathcal{C}}{}^{0}) \cong H^p(N;G)$$

and conclude de Rham's theorem. Similarly, obtain isomorphism of simplicial and singular cohomology.

15. Eilenberg-Steenrod axioms for singular homology and cohomology.

16. Cell-complexes; cellular-homology; isomorphism with singular theory; isomorphism with simplicial theory, when defined.

17. Computations; suspensions; complex projective space, real projective space, homology homomorphism induced by the double covering of real projective n-space by the n-sphere.

18. Tensor products of modules; right exactness; homology with coefficients. Bockstein sequence for homology.

19. The functor Tor; universal coefficient theorem for homology.

20. The Eilenberg-Zilber theorem; Künneth sequence for the singular homology of a direct product.

21. Exterior cross-product in cohomology; cup product, properties. Chain approximation to diagonal map for regular cell-complexes, uniqueness.

22. Computation of chain approximation to diagonal for n-sphere; for mod 2 chains on real projective space; for integral chains on complex projective space. Ring structure of $H^*(P_n(R);Z_2)$ and $H^*(P_n(C);Z)$.

23. Borsuk-Ulam theorem; Flores nonembedding examples, invariance of domain.

24. Euler-Poincaré formula; Lefschetz fixed point theorem; application to existence of vector fields on manifolds.

Outline III. Surface Theory (One Year)

This course consists of a year's study of surfaces, their topological, differential geometric, conformal, and algebraic structure. Much modern mathematics consists of partial generalizations of what happens on surfaces. The student should find this material a good source of concrete examples in depth of subjects he will meet as a graduate student. It should develop his geometric insight and show him how analysis and algebra implement geometric intuition. It should solidify his previous mathematical training because it draws heavily on his knowledge of advanced calculus, complex variables, and algebra. Finally, it will display the interplay and overlap of various fields: the genus occurring topologically, geometrically via Gauss-Bonnet, and analytically via holomorphic differentials; or the surfaces of constant curvature, the simply connected complex 1-manifolds, and the non-Euclidean and Euclidean geometries.

1. Combinatorial topology. Homotopy of curves, the fundamental group, covering spaces, deck transformations.

Simplicial complexes, barycentric subdivisions, simplicial approximation theorem.

Simplicial homology. Betti numbers, Euler characteristic, genus.

Classification of triangulable compact 2-manifolds; the only simply connected triangulated 2-manifolds are S^2 and R^2.

The de Rham theorem for triangulated 2-manifolds.

2. Differential geometry. Definition of Riemannian 2-manifold. Bundle of frames. Riemannian connection. Parallel translation--motivation via surface in R^3.

Geodesics, minimizing property of geodesics. Structural equations. Curvature. Exponential map. Gauss' lemma; Gauss-Bonnet theorem for simply connected region bounded by broken curve (as an application of Stokes' theorem); global Gauss-Bonnet theorem for triangulated compact 2-manifold.

Surfaces of constant curvature; Poincaré model for negative curvature; uniqueness theorem for simply connected complete 2-manifolds of constant curvature, constant curvature manifolds as models of hyperbolic and elliptic non-Euclidean geometries.

Surfaces in R^3. 2nd fundamental form and the spherical map. Curvature again. Gauss-Codazzi equation. Uniqueness of the imbedding, given the 2nd fundamental form.

The spherical map for compact surfaces with positive curvature. Rigidity theorem.

Flat surfaces in R^3. The tangent developable. Geometric interpretation of parallel translation via the tangent developable. The only complete flat surface in R^3 is a cylinder.

Minimal surfaces; spherical map for minimal surfaces.

3. Complex manifolds. Definition of a complex 1-manifold. Complex tangent space. Conformal mapping. Reinterpretation of Cauchy-Riemann equations. Review of analytic continuation and examples of complex 1-manifolds as Riemann surfaces of an analytic function element.

Existence of isothermal coordinates in a Riemann 2-manifold;

every Riemannian 2-manifold carries a complex structure.

Riemann mapping theorem; the three different simply connected complex 1-manifolds. Relation with 2-manifolds of constant curvature.

The spherical map of a minimal surface is conjugate conformal. Complete minimal surfaces in R^3.

Potential theory. Hodge's theorem. The dimension of the space of holomorphic differentials is the topological genus.

4. Algebraic geometry. Algebraic function fields of one variable over the complex numbers. Places. The Riemann surface of a function field, the meromorphic functions of this Riemann surface, the meromorphic functions on the Riemann surface of an analytic function element as an algebraic function field.

Algebraic curves in the complex projective plane. Pictures of singularities. The Riemann surface of a nonsingular curve. Birational equivalence of nonsingular curves is the same as conformal equivalence of their Riemann surfaces and the same as algebraic isomorphism of their function fields.

Application of the Hodge theorem to show that any compact complex 1-manifold is the Riemann surface of an algebraic function field.

Divisors as 0-chains. The divisor of meromorphic functions. Bilinear relations. The Riemann-Roch theorem via potential theory. Abel's theorem, and other applications of Riemann-Roch.

Genus zero and the rational functions in the plane. Genus one and the study of the complex structures on the torus. Elliptic functions.

<u>Probability</u> (One Semester)

The development of classical mathematics was principally inspired by problems of physics and engineering. In the usual classical engineering problem the variables of the system are assumed to satisfy a set of well-defined and deterministic relations. These are analyzed, by and large, by the methods of ordinary and partial differential equations and related mathematical techniques. Such deterministic concepts and methods are no longer entirely suitable for treating mathematical problems in the biological and social sciences; furthermore, even in the physical sciences there arise problems which involve uncertainties and variability. Probability theory and stochastic processes provide language and tools by which to analyze such problems.

The course outlined here is designed to develop facility in the language, concepts, motivation, and techniques of probability theory. Stress is put on those stochastic models which are of mathematical importance as well as of interest in other disciplines. The course should aim at rigor in its treatment of both theory and applications.

The subject of probability and stochastic processes combines intuitive and analytical aspects. It draws upon and interacts with much of real analysis, functional analysis, linear algebra, complex variables, etc. It is also a basic subject for many applications. Many of these areas of application signal new directions for pure mathematical research.

1. This Section introduces the basic concepts and terminology, suggesting both an axiomatic and intuitive formulation of the mathematical model underlying probability structure.

Sample space and probability distributions, empirical background, frequency concept, relations amongst events, axiomatic foundations (Kolmogorov formulation).

2. Occupancy problems, random walks, realization of m among N events, coin tossing, run theory.

3. Random variables, conditional probabilities. Stochastic independence, Bayes' theorem, repeated trials, joint and marginal probabilities.

4. This Section seeks to develop certain analytical methods and classical distribution examples.

Expectations, variance, moments of distributions, characteristic functions, generating functions, convolutions, compounding, Chebyshev's inequality, Kolmogorov inequality, three-series theorem, correlation coefficients, classical examples: binomial, Poisson, normal, gamma, t, and F distributions, multivariate distributions, etc.

5. The classical limit theorems of probability theory are the content of the material.

Borel-Cantelli theorem, Law of Large Numbers, Central Limit Theorem, Law of Iterated Logarithm.

6. Introduction to stochastic processes. The structure of stochastic processes is delimited and its classification is outlined.

Time parameter, state space, dependence relations, introduction to Markov processes, independent increments processes, stationary processes, martingales, diffusion.

7. This Section introduces the principal concepts of stochastic processes.

Recurrence and absorption, renewal theorems, first passage probabilities, transient states, arcsine laws, occupation time of a given state.

8. Important categories of stochastic processes.

Random walk, Poisson process, birth and death, Brownian motion, branching processes.

Formulation and analysis of some simple stochastic processes occurring in physics, engineering, biology, and the social sciences (e.g., Ornstein-Uhlenbeck process, gene frequency and population growth models of Wright and Feller, learning models, etc.).

Mathematical Physics (One Semester)*

A large part of analysis originates in problems of the physical sciences; our intuition and our sense of what is important is partly based on experience in dealing with problems of the physical world. This has been so in the past and is likely to remain so in the future, although mathematicians will increasingly look for inspiration to the biological and social sciences and to computing.

It is of greatest importance for the continued vigor of mathematics to keep open the channels of communication with other sciences; colleges should offer courses on a high intellectual plane to accomplish this. The courses must then deal with fundamental ideas as well as techniques, modern analytical concepts and methods should be employed, and subjects of current research interest need to be introduced. Unfortunately, in most American colleges there is no tradition for teaching such courses, there is not a wide enough variety of suitable texts, nor are there enough people inclined or able to teach them. The Panel presents the present outline as a step toward filling the gap.

* Though we are recommending a one-semester course, we include enough material for two or more semesters, to allow for individual variations.

1. Equilibrium problems. Derivation of the Laplace and Poisson equations for: the equilibrium position of a stretch membrane, electrostatic and gravitational potential, steady state incompressible, irrotational flow.

Statement and physical motivation of boundary value problems.

Uniqueness theorems (a) via the maximum principle (proved by Mean Value Theorem) and (b) via the Dirichlet integral.

Invariance of harmonic functions under various groups of transformations: translation, rotation, contraction. Special solutions which are eigenfunctions under these transformations (generalization of the principle that the matrices in a finite set of commuting matrices have common eigenfunctions). Application to the representation of the orthogonal group.

The fundamental solution of Laplace's equation and its physical significance. Green's function. Explicit construction of Green's function for the half-plane, circle, and sphere by the principle of reflection.

Construction of the Poisson kernel for a half-plane by similarity.

Invariance of harmonic functions under conformal map. Conjugate harmonic functions; relation of harmonic and analytic functions. Role of conjugate harmonic functions for flows. The relation of Green's function of a domain to the conformal mapping function.

Dirichlet's principle and the basic principles of the calculus of variations. The Euler equation. The equations of elasticity and of minimal surfaces.

The Laplace difference equation; its relation to random walk.

Free boundary value problems of hydrodynamics.

2. Conservative time-dependent problems (wave propagation). Newton's laws of motion. Derivation from physical principles of the equations governing the motion of a vibrating string and membrane, and the equations of acoustics. The wave equation.

Statement and physical motivation of initial and of mixed initial boundary value problems.

Uniqueness theorems based on the Haar Maximum Principle and on the energy method. The notion of domain of dependence and speed of propagation signals.

Invariance of the wave equation under translation; exponential solutions. Plane waves and the D'Alembert solution. Hamilton's principle. The equations of time-dependent compressible flow; shock waves. The equations governing the flow of traffic. Finite difference approximations to the wave equation.

3. Dissipative time-dependent equations. Derivation of the equations governing heat conduction, diffusion, and viscous flow.

Statement and physical motivation of initial and mixed problems for the heat equation.

Uniqueness theorems based on the maximum principle and on the energy method.

Translation and rotation invariance, exponential solutions, radial separation. The fundamental solution derived by similarity. Uniqueness and existence of solutions to the initial value problem in the entire space.

Relation of the heat equation to probability theory. Finite difference approximation to the heat equation.

4. Introduction to Hilbert space and operator theory. A brief review of linear algebra; Hilbert space. Orthonormal sets, completeness. Bessel inequality, Parseval relation. Projection theorem.

Examples of orthogonal systems: Fourier series and classical orthogonal polynomials. Weierstrass approximation theorem. Gram-Schmidt procedure.

Notion of symmetric operator. Orthogonality of eigenvectors. Completeness of eigenvectors of compact operators. Compactness of integral operators; discussion of the inverse of a differential operator.

Positive-definite operators.

Operational calculus for symmetric operators. Definition of exp A through 1) operational calculus, 2) contour integral, 3) eigenvector expansion, 4) Yosida formula (semigroups).

Theory of Fourier transform; the classes L_2 and $\mathcal{L}$. Application of Fourier series and integral to solve anew the boundary value problem for the Laplace equation in the circle and half-plane, and the initial value problem for the wave and heat equations on the real axis, with and without periodicity.

Determination of eigenfunctions and eigenvalues of the Laplace operator in simple geometries.

5. Existence theorems. Solution of various boundary value problems for the Laplace equation by the method of orthogonal projection, the Hahn-Banach theorem, or one of the many other methods.

Using the existence theory for the Laplace operator and the operational calculus developed in Section 4 to treat the initial value problem for the wave equation $u_{tt} = \Delta u$ and the heat equation $u_t = \Delta u$.

6. Quantum theory and statistical mechanics.

1) The harmonic oscillator in classical and quantum mechanics:

Let ψ_n be the normalized eigenfunction of the system with energy E_n. Then the probability that its position is between a and b is

$$\int_a^b |\psi_n(x)|^2 \, dx.$$

This tends in the classical limit $(h \to 0,\ E_n \to E)$ to the proportion of time which the classical harmonic oscillator spends between a and b (proof based on asymptotic properties of Hermite polynomials).

2) The motion of electrons in crystals:

Consider the Schrödinger equation

$$\psi'' - V(x)\psi = -E\psi$$

with a periodic potential V. Show that there exist solutions bounded for all x only when E lies in certain intervals, and identify these with conduction bands.

3) The classical and quantum-mechanical partition functions; limiting behavior as h tends to zero. For an ideal gas, we are led to the problem of asymptotic distribution of the eigenvalues of the Laplacian under the boundary condition $u = 0$ on the boundary of the container. Since the thermodynamical properties do not depend on the shape of the container, this suggests that the asymptotic distribution of the eigenvalues depends only on the volume

(Weyl's theorem). Explicit determination for a cube.

Further suggested subjects: the time-dependent Schrödinger equation, the application of group representations in quantum theory.

Appendix C

HIGHER UNDERGRADUATE MATHEMATICS

This Appendix contains some sample outlines of courses in Higher Undergraduate Mathematics which might be considered for inclusion in a program already containing basic courses which cover the fields of Appendix B. Future reports of the Panel will contain additional outlines for this section.

Mathematical Methods in the Social Sciences: Game Theory, Programming, and Mathematical Economics (One Semester)

The desire to formulate quantitative methods for analyzing phenomena in the social, management, and behavioral sciences has led to new types of mathematical problems. The tools needed in dealing with such problems combine principally probabilistic, statistical, and decision-theoretic concepts and techniques. Three specific developments of this kind, *inter alia*, are exemplified by the areas of mathematical research known as game theory, programming, and mathematical economics. The structure of game theory seems suitable for describing some monopolistic practices in addition to providing a norm for certain patterns of rational behavior. The methods of mathematical programming are particularly appropriate for determining optimal policies in a variety of management problems. The formulation of mathematical economics is useful in explaining the workings of some economic systems.

These disciplines appeal to devices from topology (e.g., fixed point theorems), the stability theory of nonlinear differential equations, methods of the calculus of variations, inequalities, linear algebra, convexity, and similar subjects. The intuitive content of the underlying economic interpretation frequently suggests new mathematical theorems. The influence of these disciplines on developments in statistics and probability has also been substantial.

1. Game theory. Classification of games (number of players, zero-sum versus nonzero-sum, personal and chance moves, information structure, utility concepts).

Zero-sum matrix games, minimax theorem, dominance concepts, Snow-Shapley characterization of extremal solution, completely mixed games, dimension relations of solutions, examples.

Infinite zero-sum games (optional material). Separable games (polynomial kernels), convex games, games of timing, bell-shaped games, games over function space, recursive games, games of survival.

n-person games, cooperative and noncooperative games, coalitions, von Neumann solution, simple games, Shapley value, Nash equilibrium point, examples.

2. Linear programming. Formulation of linear and dual linear programming problems. Examples, optimal assignment problem, transportation model, network flow models, etc.

Two principal theorems of linear programming: (i) Existence theorem of solution, (ii) Duality theorem.

Interpretation of dual problem in terms of shadow prices.

Computing algorithms for solutions. Simplex method, primal and dual algorithm, special methods for the transportation problem, application to minimal-cut and maximal-flow theorem.

Equivalence of linear programming and game theory.

3. Nonlinear programming. Equivalence to saddle point theorem. Kuhn-Tucker theorem, Arrow-Hurwitz gradient method. Fenchel formulation of nonlinear programming problem.

4. Methods of mathematical economics and management science. Production, consumption, and competitive equilibrium models.

Frobenius theory of positive matrices. Application to linear production model (Leontief model), Samuelson substitution theorem, formulation of theories of consumer preference. Axiomatic approach. Principle of revealed preference. Derivation of consumer preference relation as a utility maximization. Relation of production theory and nonlinear programming problem. Existence of competitive equilibrium. Formulations of Arrow-Debreu, Wald, McKenzie, and others.

5. Welfare economics, stability theory, and balanced growth. Relation of welfare economics and the vector nonlinear programming problems. Characterization of Pareto optimum solutions. Local and global stability properties of competitive equilibrium. Gross substitutibility, models of balanced growth, von Neumann model of expanding economy, turnpike theorem.

6. Control problems in management sciences and economics. Hohn-Modigliani model of smoothed production, models of optimal inventory analysis, application of Pontryagin maximal principle to two sector growth models, introduction to replacement programs, repairmen problems, queueing theory, reliability models.

Mathematical Logic (One Year)

Many mathematicians think of logic as having for its principal purpose the laying of a firm "foundation" on which the rest of mathematics can be built securely. This can be understood historically, because it was the discovery of paradoxes in set theory which first led a broad segment of mathematicians to take up the study of logic in a quest for consistency proofs.

To make headway toward the twin aims of developing a foundational logic and providing guarantees of consistency, it was necessary to restrict sharply the mathematical methods employed. In particular, early workers laid great stress on the "constructive" character of their work.

As with other branches of mathematics, so with logic: the original aims were partly realized, partly found unrealizable, and partly altered to conform to the broadened perspective arising out of new discoveries. Some logicians began to notice the mathematical structures arising in the earlier work and became interested in these for their own sake. Through the study of these structures, contact has been made with other parts of mathematics at points far removed from the "foundational level" which was the starting point.

As a result of this development, it seems fair to say that the idea ascribed above to "many mathematicians," that the principal purpose of logic is to lay "foundations," does not accurately reflect the spectrum of current activities in the field. Roughly, logicians are now concerned with two large areas of work. One, based on the notion of recursive function, deals with such things as abstract computing machines, nonexistence of decision methods, hierarchical classification of sets of numbers and functions on numbers, and recursive analogues of portions of set theory and analysis. The other, combining Boole's original impulse to algebraize with Tarski's mathematical analysis of semantical notions, includes portions of the field which have come to be known as "algebraic logic" and "theory of models." In both of these principal areas the bulk of the work is carried on without restriction to "elementary" or "constructive" methods. The basic attitude is that any method may be used if it answers a question--and any question may be raised which is interesting! Particularly in model theory, there is generally a heavy use of set theory.

Despite this turn of events, almost all textbook treatments of logic lay great emphasis on the restriction to constructive methods and seem to concern themselves principally with demonstrating how logic can be developed so as to provide a foundation--i.e., to be a beginning--of mathematics. It seems time to attempt a presentation of the subject more closely related to current events. The foundational role of logic is explained as one aspect of the subject, but this is not allowed to restrict and distort the methodology.

In formulating a first course, one might either attempt to give introductions to the concepts in both of the principal areas mentioned above or to go more deeply in one of these directions. Very likely both schemes have merit, but we have preferred to follow the latter. Our judgment has been that for students proceeding toward a Ph.D. in mathematics, serious acquaintance with the ideas of algebraic logic and theory of models is of greatest value.

What do we presuppose of the student entering our course? Competent books on logic are now available for use in the 6th grade; indeed, grade school seems the proper place to compute with truth-functions and thereby learn the mathematical meaning of sentential connectives. High school seems to be the proper place to learn how to formalize sentences employing quantifiers and to learn (in an informal way) some of the elementary rules for handling quantifiers. The early college years will begin to develop the student's ability to apply effectively the basic apparatus of set theory. The proposed course carries on from there.

It is customary to approach mathematical logic by considering first sentential logic and then (first-order) quantifier logic. The course outline given below follows this pattern, except that we interpolate between these parts of the course a substantial section on quantifier-free predicate logic. If we were concerned solely with formal deductive systems for logically valid formulas, this would be ridiculous, since the axiom schemes and rules of inference of (q.f) predicate logic are indistinguishable from those of sentential logic. However, when we deal with model-theoretic aspects of the subject, the situation is quite otherwise. And the section on predicate logic forms a valuable bridge, both from the mathematical and the pedagogical viewpoints, between sentential logic and quantifier logic.

To understand properly the role of logic in mathematics, it is necessary to deal with (i) systems of symbols, (ii) the use of these systems in languages interpreted as referring to mathematical structures, and (iii) the manipulation of symbolic expressions according to formal deductive rules and the relation of such rules to the semantical concepts of (ii). Each section of logical material--sentential, predicate, and quantifier--is subdivided according to the classifications (i), (ii), (iii).

It is possible to treat the high points of sentential and quantifier logic in a single semester. However, to explore the subject in the depth desirable for achieving both a full understanding of the relation of logic to other parts of mathematics and a firm basis for future graduate work, two semesters is not excessive. During 1962-63 an experimental course patterned after the following outline is actually being given, and, despite the encouragement of excellent students, there is difficulty in fitting all of the material into two semesters. But it is felt that after accumulating experience in teaching material so organized, and if a suitable text becomes available, it should be possible to incorporate substantially all of the material in the indicated time.

1. Historical background. Intuitive account of principle concepts such as consequence, deduction; role of sentential connectives in natural languages.

2. Systems of formulas (absolutely free algebraic systems). Axiomatic treatment; various examples and their interrelations; fundamental existence theorem (justifying definition by recursion over formulas); definition of substitution, part, occurrence, and derivation of their fundamental properties from axioms.

3. Truth functions. Relation to connectives; projections, composition of functions; closed sets (examples); generating bases (mention of Post's theorem); proofs of definability and nondefinability (of a given function in terms of a given set of functions); lattice of closed sets; Boolean algebra B_n of n-placed truth-functions, $n = 1, 2, \ldots, \omega$; isomporphisms $B_n \rightarrow B_{n+1}$, and the direct limit of $B_1, B_2, \ldots$, as subalgebra of B_ω; infinite sums and products in B_ω; topological aspects of B_ω; compactness.

4. Semantical concepts of sentential logic. (Classical) models (truth-value assignments), associated homomorphism of system of formulas into algebra of truth values, validity, consequence, satisfiability, equivalence, independence; their interrelations; fundamental laws for consequence-relation; connection with substitution; equivalence as congruence relation (replacement law); positive and negative parts of formulas (partial replacement); natural mapping of formulas into B_n; definability by formulas; significance of the consequence relation in B_n; finitary character of consequence from compactness; Boolean algebras as models; the consequence relation determined by a Boolean ideal; normal forms; interpolation theorem; nonclassical interpretations (n-valued, intuitionistic, modal).

5. Deductive aspects of sentential logic. Axioms for derivations;* consequence satisfies these; basic laws obtained from axioms; effective proof of weak completeness (every valid formula derivable from empty set, for any derivation); proof by Zorn's lemma of strong completeness (semantical consequence is the minimal derivation); connection with compactness; characterization of finitary derivations; derivations defined by formal axioms and rules of inference; discussion of deductive logic as foundation for mathematics. Fragments of sentential logic; their deductive interconnections.

6. Systems of open predicate formulas (individual symbols, relation symbols, operation symbols). Terms and formulas; fundamental existence theorem; substitution, part.

7. Semantical concepts of predicate logic. Relational systems; models and variable-assignments; values of terms and formulas; validity, satisfiability, implication, definability, equivalence--with respect to a model and to a class of models; examples; properties of the class of definable relations, characterization of such classes; concept of a Boolean substitution algebra; predicate implication = propositional implication; compactness; decision procedure; Skolem-Löwenheim; implication relative to class of equality-models and its relation to predicate implication; compactness, decision-procedure, and S-L for predicate-equality logic; simple applications of compactness (e.g., condition for abelian semigroup to be imbeddable in group); subsystems, homomorphisms, direct products, direct limits, etc., for relational systems; invariance of validity for sets of equations, and of more general formulas, under these operations; characterization of equational classes and universal classes.

8. Deductive aspects of predicate logic. Formal axioms and rules of inference for predicate logic reduce to those for sentential logic (strong completeness); treatment of predicate-equality logic; complications of formalization for these systems if variables in some of the hypotheses of an implication are treated as universalized;

* The word "derivation" is not in common use. It is employed to indicate any relation (between sets of formulas and formulas) which satisfies certain laws for the consequence relation.

detailed consideration of a mathematical example, such as natural numbers under addition, obtaining complete axiomatization, detailed description of definable relations, decision procedure, strong incompleteness, analysis of nonstandard models.

9. Systems of quantifier-formulas (first order). Free and bound occurrences of variables, complications with substitution; sentences and formulas. Semantical concepts of quantifier-logic: same notion of model as in predicate logic; modified notions of variable-assignment and value-of-formula; same definitions of validity, satisfiability, implication, definability, and equivalence; class of definable relations, characterization of such classes; concept of polyadic algebra; prenex normal forms; reduction of validity and implication for quantifier logic to that of predicate logic via added individual constants or operation symbols; semantical versions of Herbrand's theorem and Skolem normal forms; Skolem-Löwenheim theorem, compactness, applications to algebra; treatment of quantifier-equality logic; equivalence of any formula with one having variables in standard order; simplified description of definable relations; concept of cylindric algebra; reduced products and ultraproducts of relational systems; invariance of validity for quantifier-formulas under latter; characterization of elementary classes; use of ultraproducts to replace compactness arguments in algebraic constructions.

10. Deductive aspects of quantifier logic. Formal axioms and rules of inference (with and without equality); notions of formal proof and formal theorem, formal deduction and formal implication, consistency for sets of sentences; derivation of basic laws of logic (i.e., properties of formal implication); strong completeness (alternative proof of compactness); Craig's interpolation theorem, Beth's theorem on definability, version of A. Robinson; Lyndon's characterization of sentences invariant under homomorphisms; detailed consideration of quantifier theory of natural numbers under addition, axiomatization, method of elimination of quantifiers applied to obtain complete description of definable relations, decision procedure, strong incompleteness, analysis of nonstandard models.

Differential Geometry (One Semester)

This outline of a one-semester course in differential geometry differs from the classical course in these respects: elementary theory of manifolds is presented; some of the classical matrix groups are studied; the Frenet formulas are given in n dimensions; intrinsic Riemannian geometry is studied before imbedded hypersurfaces; and some global theorems are included.

1. Basic facts about smooth manifolds and mappings between them.

2. The rotation group $R(n)$, Euclidean group, and affine group as examples of manifolds. Invariant 1-forms on these groups. The Lie algebras as matrix Lie algebras. Fundamental uniqueness theorem: two maps of a manifold into G differ by a left translation if and only if the left invariant 1-forms of G pulled back by the two maps are equal.

3. Parameterized curves in R^n. Canonical parameterization via arc length. Adapted frames and mapping of curve into Euclidean group. Curvature and higher torsions. Frenet formulas. Application of uniqueness theorem to give determination of curve up to Euclidean motion.

4. Introduction of Riemannian metric. Bundle of frames. Riemannian connection. Parallel translation. Structure equations. Curvature. Geodesics and minimizing property. Exponential mapping. Gauss' lemma. Specialization to 2-manifolds. Gauss-Bonnet theorem for 2-manifolds.

5. Manifolds of constant curvature. Uniqueness for simply connected ones.

6. Hypersurfaces in R^{n+1}. Induced Riemannian metric. 2nd fundamental form and spherical map. Mapping of bundle of frames into Euclidean group. Curvature in terms of 2nd fundamental form. Gauss-Codazzi equation. Application of fundamental uniqueness theorem to give determination of surface up to a Euclidean motion. Principal curvatures, mean curvature, umbilical and parabolic points, curvature, and asymptotic lines, for 2-manifolds in R^3.

7. Rigidity theorem for compact hypersurfaces of positive curvature.

8. Flat hypersurfaces. Tangent developable. Geometric

interpretation of parallel translation using tangent developable. A complete flat 2-surface in 3-space is a cylinder.

9. Isothermal coordinates. A Riemannian 2-manifold as a complex 1-manifold. Minimal surfaces and their spherical maps.

Statistics (One Semester)

Probability theory and stochastic processes provide mathematical tools for a descriptive analysis of certain mathematical models. Statistics gives a means of testing the adequacy of the model. The statistics course outlined below is designed to cover the methodology and basic theory of statistical analysis. The approach is a combination of the classical and the modern, emphasizing at the start the standard procedures of statistics, while the latter part contains the general theory of statistical decisions.

Statistical techniques lean heavily on probability theory, real analysis, and linear algebra. Its content motivates and inspires problems in convexity, inequalities, real analysis, and probability theory. The outline presumes a course in probability theory as prerequisite.

1. Review of probability. Emphasis on theoretical distributions including the important examples of chi-square, t, F distributions of order statistics and functions of order statistics. Multivariate distributions and similarity.

2. Sampling. Description of sample data-means, standard deviation, frequency histogram, etc. Distribution theory of various statistics arising in sampling from normal populations, asymptotic distribution theory of various statistics.

3. Estimation. Formulation of the problem. Discussion of criteria for estimators (unbiasedness, consistency, efficiency, minimizing mean square error, absolute error, etc.).

The concept of sufficiency. Fisher-Neyman characterization, applications to the exponential family of distributions, extremal range distributions, principle of completeness, Rao-Blackwell inequality for improving estimates using sufficient statistics, Cramer-Rao inequality.

Confidence interval estimation. Maximum likelihood estimator and its properties.

4. Testing hypotheses. Formulation of the general problem.

Analysis of the case of a simple hypothesis versus simple alternate hypothesis including the celebrated Neyman-Pearson theorem.

Discussion of the composite hypothesis problem, concept of uniformly most powerful test. Tests derived from likelihood ratio criteria.

5. Statistical decision theory. The preceding approach was classical. Formulation of the Wald statistical decision theory. The concepts of utility, loss, and risk should be discussed. The derivation of the simplest complete class and "admissibility" theorems should be given.

Principles to be explored: Bayes' criteria, minimax, invariance, etc. Comparisons to classical statistical procedures.

Introduction to sequential analysis.

6. Regression theory and design of experiments. The formulation of the general linear hypothesis, linear regression. The Markov principle and the method of least squares. Analysis of variance.

7. Nonparametric statistics. Order statistics and derivation of confidence intervals for percentiles. Tolerance limits, goodness of fit, two sample problem. Kolmogorov-Smirnov statistics, rank procedures.

Number Theory (One Semester)

The theory of numbers has some attractive features which make it a very appropriate topic in the undergraduate curriculum. Having flourished over a very extensive period of time, number theory can be classified among those mathematical topics having a steady appeal. Perhaps this is due in part to its close relationship to algebra and analysis, an aspect of number theory that has been given greater emphasis in recent textbooks on the subject.

Number theory is not related to analysis in quite the same way as it is related to algebra, however. Analysis is used in number theory primarily as a matter of powerful analytic techniques of proof; for example, in the prime number theorem and in various proofs of transcendency of numbers. On the other hand, algebra has found in number theory a rich source of examples for the study of algebraic structures. It is not surprising to find in many books on algebra, therefore, an introductory chapter on the theory of numbers. Indeed, there are mathematical topics that are not easily classified as algebra proper or as number theory; the theory of algebraic numbers is an example of this.

The theory of numbers is an excellent vehicle for clarifying in the mind of the student the nature of proof. It may be that by the junior year in college a student should have no doubt about what is and what is not a proof. Nevertheless, since even some graduate students have occasional difficulty with this, the wide variety of proof techniques used in number theory can serve as excellent models for the student's attention. Furthermore, there is in the theory of numbers much for the student to do over and above examining the basic results. There is an almost unparalleled wealth of problems, including not only applications and examples of the theory, but also extensions and alternative foundations of the theory. Thus, the student has an opportunity both to develop his ingenuity and to discover results for himself through a program of exploration, conjecture, and attempts at proof. This aspect of number theory, often a source of frustration for the average student of mathematics, provides the superior student with much satisfaction and pleasure.

The following topics are presumed to be known by the student at the start of the course: unique factorization of integers, greatest common divisor, least common multiple, simple observations on the distribution of prime and composite numbers.

1. Congruences and residue classes. Congruences as equivalence relations, basic properties of congruences, changes of moduli; residue classes as groups, rings, and fields; theorems of Fermat and Euler on powers of residue; the language of algebra (order of an element of a group, generator, etc.) and the language of number theory (belonging to an exponent, primitive root, etc.); general theorems on solutions of congruences of degree n.

2. Quadratic residues. The Legendre and Jacobi symbols and their properties; the Gaussian reciprocity law.

3. Diophantine equations. The linear case and its relationship to linear congruences and greatest common divisor; $x^2 + y^2 = z^2$, $x^2 + y^2 = n$, $\sum_{i=1}^{4} x_i^2 = n$; impossibility of $x^4 + y^4 = z^4$; $ax^2 + by^2 + cz^2 = 0$.

4. Number-theoretic functions. Euler ϕ-function, divisor function, sum-of-divisors function; multiplicative and totally multiplicative functions; the Möbius function and the inversion formula; estimates of the order of magnitude of various number-

theoretic functions, including lattice points in various configurations; recurrence functions, Fibonacci sequences; the partition function.

5. The approximation of irrationals by rationals. Farey sequences, continued fractions; the best possible theorem (Hurwitz) on approximations; the uniform distribution of the fractional parts of the multiples of an irrational number.

6. Quadratic forms. Definite and indefinite forms; equivalence classes and the class number; questions of representation.

7. Prime numbers. Bertrand's theorem (a prime between n and $2n$); the Prime Number Theorem (either by an analytic proof or the "elementary" proof, or, if there is not the time for either of these, the weaker form of the theorem due to Chebychev).

8. Algebraic and transcendental numbers. Algebraic numbers form a field, algebraic integers and integral domains, quadratic fields, the Euclidean algorithm, unique factorization; the irrationality of π and e, the transcendence of e.

9. Optional topics. Infinitude of primes in an arithmetic progression; arithmetic properties of roots of unity, cyclotomic polynomials.

Geometry: Convex Sets (One Semester)

The study of convex figures is one of the oldest branches of mathematics; indeed, most figures studied by the classical geometers were either convex or stars. On the other hand, many branches of modern mathematics, e.g., functional analysis, game theory, numerical analysis, etc., have found that many problems in their scope are related to problems of convexity. Neither of these points of view of convexity, either as a branch of classical geometry or as a tool for other subjects, catches the essence of the theory. The geometry of convexity is very much alive today. A course in convexity should try to preserve the geometry as much as possible, even though in higher dimensions rigor often demands analytical technique. With this in mind, the course outlined below attempts to develop the subject starting from the simple intuitive notions and ending on the borders of the unknown in such a fashion that the geometric relationships are always in sight.

The course outlined below is intended for one semester. The material listed is more than enough for that length of time. Sections 8, 9, and 10 are all independent of one another and may be considerably shortened and used in any order without serious harm to the course.

1. Elementary properties of E_n. Coordinate systems and vectors. Scalar products, norm, and distance. Limits. Topological notions. Equivalence of topological definition of limit. n-dimensional analogs of the Bolzano-Weierstrass theorem, Heine-Borel theorem, and the theorem that a filter of closed bounded sets with finite intersection property has nonempty intersection. k-flats in parametric form and as solution space of inhomogeneous linear equations. Incidence properties and affine invariance.

2. Properties of individual convex sets. Dimension of a convex set. Interior and boundary, relative interior and boundary. Intersection properties of k-flats with boundary and interior. Preservation of convexity under affine transformations, interior and closure operations, projections, intersection, and lim inf operations. The existence of a support plane at every boundary point and in every direction. Regular points. Separation properties. Hull operator. Equivalence of intersection definition with constructive definition. The existence of exposed points for a closed bounded convex set (CBCS). Every exposed point is extreme. Every point of a CBCS is in the hull of $2n$ extreme points; every interior point of a CBCS is in the interior of the hull of $n + 1$ extreme points. A CBCS is characterized by its extreme points. The hull of a closed bounded set S is a CBCS, namely the intersection of all closed half-spaces containing S.

3. Convex cones and polyhedra; polarity. Support planes, extreme points and rays, and hull formation. Projecting cones and asymptotic cones. Polarity or duality theory for cones and polyhedra, and equivalence of various definitions of polyhedra. Applications to dual systems of linear inequalities, and to game theory and linear programming.

4. The algebra of convex sets. The sum of convex sets is a convex set. Addition is associative, commutative, and satisfies the usual rules for positive scalar multiplication. The sum of closed (open) sets is closed (open). Essential invariance under choice of origin. The same for cartesian products. Relations between sums,

products, and scalar multiplication. Faces, support planes, and diameters of sum in terms of those of summands.

5. Symmetrization operation. Steiner symmetrization in E_2; its relation to area and perimeter, diameter, and width. The symmetral of the sum of two sets. Similar results for Steiner symmetrization about a hyperplane in E_n. Steiner symmetrization about a k-flat in E_n. Central symmetrization

$$K \to \frac{K - K}{2}.$$

Its relation with volume, area, diameter, etc.

6. Helly's theorem for a finite family of convex sets. Extension to infinite families of CBCS. Further generalizations and relatives of Helly's theorem. Applications: Chebychev's approximation theorem, Jung's theorem, Krasnoselskii's theorem, solutions of convex inequalities.

7. The space of CBCSs, K_n. The inner and outer parallel sets of a CBCS. The distance $\Delta(K_1,K_2)$ between two CBCSs. K_n is a complete metric space. The polyhedra are a countable dense subset. Bolzano-Weierstrass, Heine-Borel, and filter theorems for K_n. Continuity of volume, area, diameter, sum, symmetrization, etc. Applications: surface area problems, isoperimetric problem, etc.

8. Brünn-Minkowski theorem. Linear arrays.

9. Convex functions. Distance and support functions of CBCS, and polar reciprocals. Continuity of convex functions. $f(Z)$ is convex with convex domain in E_n if and only if

$$\{(Z_1, Z_2, \ldots, Z_n, r) | r \geqq f(Z_1, Z_2, \ldots, Z_n)\}$$

is convex in E_{n+1}. Preservation of convexity under transformation of domain, sup, composition with monotone increasing convex functions, etc. Differential conditions which imply convexity. The a.e. differentiability of a convex function, i.e., almost all points of the boundary of a convex set are regular (possibly only the case $n = 2$). Extrema of convex functions with convex domain. Helly's theorem for convex functions. The convexity of certain special functions; e.g., Hadamard's 3-circle theorem.

10. Constant width sets. A set has constant width if and only

if it is equivalent in breadth to a sphere. Projection properties. Properties involving area, perimeter, etc. (n = 2).

11. Further refinements and generalizations. Some comments on separation theorems (infinite-dimensional spaces, for open and closed sets). Introduction to convex topologies. Hahn-Banach theorem--applications. Applications to constructive function theory (summability of series, etc.). Derivation of classical inequalities (e.g., Hölder, Minkowski, etc.). Non-Euclidean spaces.

PREPARATION FOR GRADUATE STUDY

IN MATHEMATICS

A Report of

The Panel on Pregraduate Training

November 1965

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FOREWORD: THE GOAL

The task of the Panel on Pregraduate Training is to recommend programs for undergraduates who intend to go on to graduate work in mathematics. The first stage of the Panel's work, to consider an ideal undergraduate program for the future research mathematician, was completed in 1963, and the conclusions have been published in the report Pregraduate Preparation of Research Mathematicians [page 369].

The recommendations in the 1963 booklet are for the first four years of a seven-year program leading to a solid Ph.D. and a career in mathematical research. The booklet contains guidelines and several detailed sample course outlines. Note, however, that in forming these recommendations the Panel made no allowance for the student's possible deficiencies in preparation or tardiness in selecting a goal, for inadequacies of staff, for lack of suitable textbooks, etc. Furthermore, the program was designed for the very gifted undergraduate working under ideal circumstances. Some schools already offer programs compatible with the 1963 recommendations. At many others, however, the recommendations cannot be put into effect very soon. For them, the 1963 booklet provides a goal. The present booklet provides interim guidance.

THE PRESENT RECOMMENDATIONS

The Panel on Pregraduate Training, in considering the CUPM reports Pregraduate Preparation of Research Mathematicians [page 369] and A General Curriculum in Mathematics for Colleges*, makes the following observations:

1. The first two years of the GCMC program include a thoughtful study of calculus through partial derivatives and multiple integrals, and of linear algebra through the elementary theory of vector spaces and linear transformations.

2. If all mathematics students follow the first two years of the GCMC program, those who decide relatively late on graduate work in mathematics will not lose very much, even though early work in the spirit of the 1963 recommendations (Pregraduate Preparation of Research Mathematicians) is very desirable for the future professional mathematician.

* The original GCMC report is not included in this COMPENDIUM. However, the 1972 Commentary on a General Curriculum in Mathematics for Colleges appears on page 33.

3. The first two years of the GCMC program are within the reach of all schools, even if they have to offer a precalculus course as preparation.

The lower division of the GCMC program and its relation to pregraduate training are discussed in more detail later on.

The upper division (last two years). Because of the heterogeneity of the mathematical world, the Panel recognizes that no single curriculum will work for all schools. We therefore recommend the following priorities.

1. A minimal upper-division program for mathematics majors who intend to continue the study of mathematics in graduate school appears in the 1965 GCMC report and is reprinted below:

> A mathematics major program for students bound toward graduate mathematics: Mathematics 1, 2, 2P, 3, 4, 5, 6, 10, 11, 12, 13. A stronger major would be desirable, but this is adequate to enter good graduate schools at the present time [1965]. With two semesters' advanced placement the student still has Mathematics 7, 8, 9 from which to complete a better major.

Any college not already offering a comparable program should take immediate steps to do so.

2. If, however, the college can also supply courses designed especially for the pregraduate student, then it should provide one, or if possible both, of the one-year sequences (analysis and algebra) described below. The choice, if only one can be given, should be determined by the staff's capabilities. The one-semester analysis course (Mathematics 11) of the GCMC recommendations is itself aimed primarily at the pregraduate student; it could be replaced by the one-year course (Mathematics 11-12). On the other hand, the one-semester algebra course (Mathematics 6M) of GCMC serves many purposes; it should be retained for these purposes and the one-year algebra course described below should be added for the pregraduate student. [See page 93 for outlines of Mathematics 11-12 and of Mathematics 11. See page 68 for an outline of Mathematics 6M.]

The Panel has consulted many outstanding graduate mathematics departments; a solid grounding in algebra and analysis is what they most want from incoming students. If time and resources permit, it is, of course, desirable to introduce the pregraduate student to a broader range of material, but not at the expense of depth in algebra and analysis.

Introducing the program. Many departments of mathematics can adopt the present recommendations now. Many departments, indeed, are already offering even more substantial programs; they are referred to the 1963 recommendations and urged to proceed as far and as fast as they can in the directions suggested there. We have no universal advice for departments that feel unable to go as far as the present recommendations now.

Objectives of the program. Our concern is with all prospective graduate mathematics students regardless of their destination in today's diversified mathematical profession. The mathematical involvement of the professional mathematician, heavily influenced by the rapid developments in computer science, continues to broaden and deepen in education, in industry, and in government.

The subject matter recommended for the pregraduate program is discussed briefly above and in more detail below. This subject matter is, of course, very important; but equally important are the spirit and tone of the teaching, not only because they are reflected in the ultimate quality of the student's performance but also because they can influence the student to decide for or against a career in mathematics.

The student should be introduced to the language of mathematics in both its rigorous and idiomatic forms. He should learn to give clear explanations of some fundamental concepts, statements, and notations. He should develop facility with selected mathematical techniques, know proofs of a collection of basic theorems, and acquire experience in constructing proofs. He should appreciate the power of abstraction and of the axiomatic method. He should be aware of the applicability of mathematics and of the constructive interplay between mathematics and other disciplines. He should begin to read mathematical literature with understanding and enjoyment. He should learn from illustration and experience to cultivate curiosity and the habit of experimentation, to look beyond immediate objectives, and to make and test conjectures. In short, the student must be helped to mature mathematically as well as to acquire mathematical information.

There are many ways in which the student can be helped to mature mathematically. He can be taught in special "honors" classes for superior students. In the earlier stages he can be given independent reading assignments in textbooks, and later he can be assigned the more difficult task of reading papers in journals. He can be taught through "reading courses." He can make reports in seminars and colloquia. He can prepare an undergraduate thesis containing work original for him although not necessarily original in the stricter sense. He can be taught through the "developmental course," in which he is led to develop a body of mathematical material under the guidance of the professor. In general, the Panel feels very strongly that every pregraduate curriculum should include work to develop mathematical self-reliance, initiative, and confidence.

Identifying students. Far too few college students successfully complete a graduate program in mathematics. The Panel recommends strongly that every effort should be made to identify pregraduate mathematicians as early as possible, preferably when they enter college (a task often complicated by the students' own incorrect notions about their mathematical capabilities).

Lower-division courses. As we have already said, the Panel regards the basic sequence Mathematics 1, 2, 3, 4, 5 of the GCMC recommendations as essential for the pregraduate student. We comment briefly on this program.

For the first semester of college-level mathematics, the GCMC presents a course dealing with the integral and differential calculus of the elementary functions, together with the associated analytic geometry. To meet the needs of the future graduate student most effectively, this course should be designed with three specific objectives in mind. First, the course should build a strong intuitive concept of limits based on concrete examples. These examples can be drawn from geometry, physics, biology, etc. This may be followed by setting down a strong enough axiom system about limits to encompass their elementary properties obtained intuitively. The student should be told which of his current axioms will be future theorems. The formal definition of a limit is too difficult to be swallowed whole by the student at this point; our greatest service to the student would be to give him a firm intuitive grasp of the concept.

The second objective of this course should be to improve the student's ability to handle mathematical rigor. This can be done, for example, by using the axioms about limits in a rigorous development of the calculus.

The third and final purpose of this course should be to teach the student to calculate--for, certainly, it is the ability to calculate with the calculus that makes the calculus the powerful tool that it is.

In the next calculus courses, Mathematics 2 and 4, the GCMC is concerned, in part, with the possible introduction of a fair amount of multivariable calculus much earlier than is customary. This is less important for pregraduate mathematics students than for some other students, since pregraduate students will take all the courses.

The attitude in presenting the material is more important. After courses 1 or 3 there should be a gradual but considerable increase of mathematical maturity. Course 11-12 treats continuity, differentiation, and integration at the level of sophistication required in the theory of "real variables." Consequently, the study of these concepts in courses 2 and 4 should bring the student to an insight which makes the transition easier. There will be little in Mathematics 5 to help in this direction. Thus, the student must be led in Mathematics 2 and 4 to a considerable appreciation of rigor and to the effective personal use of mathematical language.

The 1965 GCMC recommendations included an introductory course in probability, Mathematics 2P, for all students in their first two years. The present Panel feels that the student preparing for graduate work in mathematics might better be making faster progress toward upper-division courses, deferring his work in probability.

Mathematics 3 is a short course in linear algebra. The Pregraduate Panel concurs with the GCMC committee in recommending that this course should come no later than the beginning of the second year.

Upper-division courses. Mathematics 5 contains material frequently presented as the second half of a course called "advanced calculus." Certainly the pregraduate student, whatever his branch of mathematical study, needs to acquire skill in the techniques and understanding of the concepts of mappings between Euclidean spaces of dimension at least 2 (i.e., systems of several functions of several variables).

After Mathematics 1, 2, 3, 4, 5 the Panel recommends for pregraduate students, for reasons discussed earlier, a year course in abstract algebra instead of GCMC Mathematics 6M and a year course in real analysis instead of GCMC Mathematics 11. Possible outlines for such an algebra course are presented below to indicate the flavor and scope that the Panel considers desirable for the pregraduate student; an outline for the analysis course, which is the same as GCMC Mathematics 11-12, can be found on page 93.

We repeat that any department which can offer more than these two courses should turn to the 1963 recommendations (Pregraduate Preparation of Research Mathematicians, page 369) for further suggestions.

COURSE OUTLINES

Abstract Algebra

The purpose of this year course is to introduce the student to the basic structures of abstract algebra and also to deepen and strengthen his knowledge of linear algebra. It provides an introduction to the applications of these concepts to various branches of mathematics. [Prerequisite: Mathematics 3]

OUTLINE A

1. Groups. (10 lessons) Definition. Examples: vector spaces, linear groups, additive group of reals, symmetric groups, cyclic groups, etc. Subgroups. Order of an element. Theorem: Every subgroup of a cyclic group is cyclic. Coset decomposition.

Lagrange theorem on the order of a subgroup. Normal subgroups. Homomorphism and isomorphism. Linear transformations as examples. Determinant as homomorphism of $GL(n)$ to the nonzero reals. Quotient groups. The first two isomorphism theorems. Linear algebra provides examples throughout this unit.

2. Further group theory. (10 lessons) The third isomorphism theorem. Definition of simple groups and composition series for finite groups. The Jordan-Hölder theorem. Definition of solvable groups. Simplicity of the alternating group for $n > 4$. Elements of theory of p-groups. Theorems: A p-group has nontrivial center; a p-group is solvable. Sylow theory. Sylow theorem on the existence of p-Sylow subgroups. Theorems: Every p-subgroup is contained in a p-Sylow subgroup; all p-Sylow subgroups are conjugate and their number is congruent to 1 modulo p.

3. Rings. (10 lessons) Definition. Examples: the integers, polynomials over the reals, the rationals, the Gaussian integers, all linear transformations of a vector space, continuous functions on spaces. Zero divisors and inverses. Division rings and fields. Domains and their quotient fields. Examples: construction of field of four elements, embedding of complex numbers in 2×2 real matrices, quaternions. Homomorphism and isomorphism of rings. Ideals. Congruences in the ring of integers. Tests for divisibility by 3, 11, etc., leading up to Fermat's little theorem, $a^{p-1} \equiv 1 \pmod{p}$, and such problems as showing that $2^{32} + 1 \equiv 0 \pmod{641}$. Residue class rings. The homomorphism theorems for rings.

4. Further linear algebra (continuing Mathematics 3). (12 lessons) Definition of vector space over an arbitrary field. (Point out that the first part of Mathematics 3 carries over verbatim and use the opportunity for some review of Mathematics 3.) Review of spectral theorem from Mathematics 3 stated in a more sophisticated form. Dual-space adjoint of a linear transformation, dual bases, transpose of a matrix. Theorem: Finite-dimensional vector spaces are reflexive. Equivalence of bilinear forms and homomorphism of a space into its dual. General theory of quadratic and skew-symmetric forms over fields of characteristic different from 2. The canonical forms. (Emphasize the connections with corresponding material in

Mathematics 3.) The exterior algebra defined in terms of a basis--2- and 3-dimensional cases first. The transformation of the p-vectors induced by a linear transformation of the vector space. Determinants redone this way.

5. Unique factorization domains. (12 lessons) Primes in a commutative ring. Examples where unique factorization fails, e.g., in $Z[\sqrt{-5}]$. Definition of Euclidean ring, regarded as a device to unify the discussion for Z and $F[x]$, F a field. Division algorithm and Euclidean algorithm in a Euclidean ring; greatest common divisor. Theorem: If a prime divides a product, then it divides at least one factor; unique factorization in a Euclidean ring. Theorem: A Euclidean ring is a principal ideal domain. Theorem: A principal ideal domain is a unique factorization domain. Gauss' lemma on the product of two primitive polynomials over a unique factorization domain. Theorem: If R is a unique factorization domain, then $R[x]$ is a unique factorization domain.

6. Modules over Euclidean rings. (14 lessons) Definition of module over an arbitrary ring viewed as a generalization of vector space. Example: vector space as a module over $F[x]$ with x acting like a linear transformation. Module homomorphism. Cyclic and free modules. Theorem: Any module is a homomorphic image of a free module. Theorem: If R is Euclidean, A an $n \times n$ matrix over R, then by elementary row and column transformations A can be diagonalized so that diagonal elements divide properly. Theorem: Every finitely generated module over a Euclidean ring is the direct sum of cyclic modules. Uniqueness of this decomposition, decomposition into primary components, invariant factors, and elementary divisors. Application to the module of a linear transformation, leading to the rational and Jordan canonical forms of the matrix. Several examples worked in detail. Similarity invariants of matrices. Characteristic and minimal polynomials. Hamilton-Cayley theorem: A square matrix satisfies its characteristic equation. Application of module theorem to the integers to obtain the fundamental theorem of finitely generated abelian groups.

7. Fields. (10 lessons) Prime fields and characteristic. Extension fields. Algebraic extensions. Structure of $F(a)$, F a

field, a an algebraic element of some extension field. Direct proof that if a has degree n, then the set of polynomials of degree $n-1$ in a is a field; demonstration that $F(a) \cong F[x]/(f(x))$, where f is the minimum polynomial of a. Definition of $(K:F)$, where K is an extension field of F. Theorem: If $F \subset K \subset L$ and $(L:F)$ is finite, then $(L:F) = (L:K)(K:F)$. Ruler-and-compass constructions. Impossibility of trisecting the angle, duplicating the cube, squaring the circle (assuming π transcendental). Existence and uniqueness of splitting fields for equations. Theory of finite fields.

OUTLINE B (including Galois theory)

A course culminating in and climaxed by Galois theory can be constructed by compressing the topics in Outline A into somewhat less time and adding material at the end. Outline B suggests such a course. The appeal of Galois theory as a part of a year course in algebra is obvious. The material ties together practically all the algebraic concepts studied earlier and establishes a clear connection between modern abstraction and a very concrete classical problem. The cost is equally obvious; the depth of much of the earlier material must be reduced, or several topics eliminated. Whether the advantages justify the cost is debatable.

Recognizing that there is merit on both sides, the Panel offers Outline B as an alternative to Outline A with the following words of caution and explanation:

(a) For most pregraduate students today, Outline A probably represents the better balance between coverage and pace.

(b) Outline B contains all the material in Outline A plus Galois theory. Thus Outline B should be attempted only when more time is available or the students are <u>clearly</u> capable of an accelerated pace. Accordingly, each unit in this outline is assigned a range of suggested times, the extremes representing these two alternatives.

(c) In order to break up the rather substantial concentration on group theory at the beginning of Outline A, some of this material has been moved to a position near the end of Outline B, where it fits naturally with Galois theory. This same shift may be made in Outline A by taking the units in the order 1, 3, 4, 5, 6, 7, 2.

1. <u>Groups</u>. (6-10 lessons) Outline A, unit 1.
2. <u>Rings</u>. (7-10 lessons) Outline A, unit 3.

3. Further linear algebra. (10-12 lessons) Outline A, unit 4.

4. Unique factorization domains. (10-12 lessons) Outline A, unit 5.

5. Modules over Euclidean rings. (12-14 lessons) Outline A, unit 6.

6. Fields. (7-10 lessons) Outline A, unit 7.

7. Galois theory. (8-10 lessons) Automorphisms of fields. Fixed fields. Definition of Galois group. Definition of Galois extension. Fundamental Theorem of Galois Theory. Separability. Equivalence of Galois extension and normal separable extension. Computation of Galois groups of equations. Existence of Galois extensions with the symmetric group as Galois group. Theorem on the primitive element: If $(L:F)$ is finite and there exist only a finite number of intermediate fields, then $L = f(a)$ for some $a \in F$.

8. Further group theory. (9-10 lessons) Outline A, unit 2.

9. Galois theory continued. (8-10 lessons) Hilbert's theorem 90: If L is finite and cyclic over F, g is a generator of the Galois group, and x is an element of L of norm 1 over F, then $x = y(yg)^{-1}$ for some $y \in L$. Also the additive form of Hilbert's theorem 90. Galois groups of $x^n - a$. Definition of solvability by radicals. Theorem: An equation is solvable by radicals if and only if its Galois group is solvable. The unsolvability of the general equation of degree n, $n \geqq 5$. Other examples. Roots of unity and cyclotomic fields.