THE TRAINING OF TEACHERS OF MATHEMATICS

CUPM's interest in the training of mathematics teachers has pervaded its activities throughout the Committee's existence.

The Panel on Teacher Training, one of the original four panels, began its work at a time when mathematics instruction in elementary and secondary schools was undergoing significant changes. Throughout the years since its original report was issued, the Panel's recommendations and ongoing activities have had a profound influence on the education of elementary and secondary school teachers.

The 1961 Recommendations for the Training of Teachers of Mathematics* identified five levels of mathematics teachers:

I. Teachers of elementary school mathematics--grades K through 6
II. Teachers of the elements of algebra and geometry
III. Teachers of high school mathematics
IV. Teachers of the elements of calculus, linear algebra, probability, etc.
V. Teachers of college mathematics

To complement the 1961 recommendations, CUPM also published Course Guides for the Training of Teachers of Elementary School Mathematics* and Course Guides for the Training of Teachers of Junior High School and High School Mathematics.* When it was proposed, the Level I curriculum received widespread attention and approval. It was approved formally by the Mathematical Association of America and it was endorsed by three conferences held by the National Association of State Directors of Teacher Education and Certification (NASDTEC) and the American Association for the Advancement of Science (AAAS). It formed a part of the Guidelines for Science and Mathematics in the Preparation Program of Elementary School Teachers, published by NASDTEC-AAAS in 1963.

In the years 1962-66 CUPM made an intensive effort to explain its proposed Level I program to that part of the educational community especially concerned with the mathematics preparation of elementary school teachers. Forty-one conferences were held for this purpose. Participants in these conferences, who came from all fifty states, represented college mathematics departments and education departments, state departments of education, and the school systems. The details of CUPM proposals were discussed and an effort was made to identify the realistic problems of implementation of the recommendations. As a result of these conferences and of other forces for change, there was a marked increase in the level of mathematics training required for the elementary teacher.

* Not included in this COMPENDIUM.

Level II and III conferences similar to those held for Level I were deemed unnecessary because the Level II and III guidelines had apparently been accepted by the teaching community through distribution of the recommendations and course guides. One indication of this acceptance has been the publication of numerous textbooks whose prefaces claim adherence to the CUPM guidelines.

Throughout the decade of the 1960's, CUPM continued to expend considerable effort on the problems associated with the preparation of teachers. Minor revisions of the original recommendations were produced in 1966, and the course guides for Level I were similarly revised in 1968.

In 1965 CUPM published A General Curriculum in Mathematics for Colleges* (GCMC) as a model for a mathematics curriculum in a small college. GCMC became a standard reference in other CUPM documents. The shortage of mathematicians, already severe by the late 1950's, had seriously impaired the ability of many colleges to implement CUPM recommendations, including GCMC. Qualified new faculty members were extremely difficult to obtain, and many established teachers were so overloaded with teaching responsibilities that they could not keep abreast of developments in their field. By 1965 the time was obviously ripe for CUPM to see what could be done to alleviate this problem. An *ad hoc* Committee on the Qualifications of College Teachers of Mathematics was appointed to study and report on the proper academic qualifications for teaching the GCMC courses. Simultaneously, CUPM established a Panel on College Teacher Preparation and instructed it to study a number of related topics: existing programs for the preservice and inservice training of college teachers, opportunities for support of college teacher programs by government and foundations, the supervision and training of teaching assistants, supply and demand data, etc.

In 1967 the Qualifications Committee issued its report, Qualifications for a College Faculty in Mathematics. The report identifies four possible components in the formal education of college teachers and describes teaching duties suitable for individuals with academic attainment equivalent to a given component. It also makes suggestions concerning the composition of a small undergraduate department.

Immediately upon publication of the qualifications report, the Panel on College Teacher Preparation fell heir to several tasks. One of these was the responsibility for a series of regional conferences designed to bring together mathematicians and college administrators to discuss some of the issues raised by the report. Another was the task of preparing a detailed description of a graduate program modeled after the "first graduate component" defined in the qualifications

* Not included in this COMPENDIUM. However, the 1972 Commentary on A General Curriculum in Mathematics for Colleges appears on page 33.

report. This latter project was undertaken by the Graduate Task Force, a group with membership drawn from the Panel and from CUPM. Its report, A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates, was issued in 1969. Meanwhile, other members of the Panel conducted a study on the supervision and training of teaching assistants in mathematics. Their findings were reported in a newsletter published in 1968.

The need for a "companion volume" for the qualifications report was established when the Panel on Mathematics in Two-Year Colleges issued its 1969 report A Transfer Curriculum in Mathematics for Two-Year Colleges. CUPM felt it was necessary to describe the qualifications for persons to teach the courses in the Transfer Curriculum, and for this purpose it appointed an ad hoc Committee on Qualifications for a Two-Year College Faculty in Mathematics. This group's recommendations are given in the document Qualifications for Teaching University-Parallel Mathematics Courses in Two-Year Colleges, published in 1969.

Publication of the several reports mentioned in the preceding paragraphs completed CUPM's original plan of providing course guides for each of the five teaching levels defined in 1961. By 1967, however, the pressure for further change was already beginning to be felt. A minor revision (1968) of the Level I course guides contained the statement, "The five years that have elapsed since the preparation of the Course Guides have seen widespread adoption of the ideas of the new elementary school curricula, not only of the work of such experimental or quasi-experimental groups as the School Mathematics Study Group (SMSG) or the University of Illinois Curriculum Study in Mathematics (UICSM), but also of many new commercial textbook series which incorporate such ideas. In addition, there have been attempts to influence the future direction of elementary school mathematics by such groups as the Cambridge Conference. In the near future, the Panel believes, it will be necessary to examine our courses to take account of these developments. We hope in the next couple of years to begin the sort of detailed, intellectual study of current trends in the curriculum and of predictions of the future which will be necessary in order to prepare teachers for the school mathematics of the next twenty years."

During the years 1968-72 the Panel on Teacher Training continued this promised study. It sought to understand current trends and future possibilities through a variety of means: in the spring of 1968 it sponsored a conference, "New Directions in Mathematics," to obtain the views and advice of a large number of mathematicians and educators; it followed the deliberations of the CUPM Panel on Computing; it followed with interest, and contributed to, continuing discussion on pedagogy, the changing attitudes toward experimentation in mathematics education, and the role of mathematics in society today; and, finally, the Panel met with representatives of the National Council of Teachers of Mathematics, the American Association for the Advancement of Science, and the National Association of State Directors of Teacher Education and Certification, and maintained contact with national curriculum planning groups. The Panel concluded from this study that a revision of the CUPM recommendations and course guides for Levels I, II, and III was indeed required. Its 1971 report, Recommendations on Course Content for the Training of Teachers of Mathematics, was a result of that decision.

During the early seventies the Panel on College Teacher Preparation continued its interest in the role and preparation of teaching assistants. A 1972 newsletter, "New Methods for Teaching Elementary Courses and for the Orientation of Teaching Assistants,"* contains a statement by the Panel on teaching experience as part of Ph.D. programs. In 1972 the Panel also issued a booklet entitled Suggestions on the Teaching of College Mathematics,* whose purpose was to disseminate some ideas about practices that are believed to have contributed to successful teaching of mathematics in colleges and universities.

* Not included in this COMPENDIUM.

QUALIFICATIONS FOR A COLLEGE FACULTY IN MATHEMATICS

A report of

The *ad hoc* Committee on the Qualifications of College Teachers of Mathematics

January 1967

BACKGROUND

[Editor's note: Although statements in this report which refer to the shortage of mathematics teachers are no longer valid, the main theme of these recommendations is still cogent.]

The curriculum reforms in mathematics at the elementary and secondary school levels during the past decade have necessitated immediate programs of support to provide the quality of teaching needed in those schools. Today we are beginning to notice many changes in college mathematics courses stemming from previous CUPM recommendations. Hence, it is now time to focus attention on the training and qualifications of teachers needed in our colleges and universities in order to effect the required changes in the undergraduate curriculum.

The recent dramatic growth of mathematical research activity, combined with the growing demands of industry and government for people with mathematical training, has created a severe shortage of mathematics teachers who have doctoral degrees. The rapidly increasing mathematics enrollments within a growing college population and the expansion of areas of application of mathematics have left many college mathematics departments seriously understaffed, greatly overworked, and quite unprepared to initiate urgently required modifications of their course offerings. It is imperative that decisions for curriculum changes, as well as for the other critical problems facing mathematics departments, be made and carried out by people with the highest possible mathematical qualifications.

The simple traditional requirement of many colleges, and of some junior colleges, that new appointments to the mathematics faculty be awarded only to people with a Ph.D. degree is, at the present time, quite unrealistic. Recipients of new Ph.D.s in mathematics are simply not available in the required numbers. For example, in 1964-1965 barely more than one quarter of the new full-time mathematics teachers employed by four-year colleges had Ph.D.s. The shortage is likely to continue, and junior colleges and four-year colleges will, of necessity, continue to use teachers whose academic preparation is intermediate between the bachelor's degree and the doctor's degree.

Our principal goal in this report is to set forth appropriate qualifications for teaching the courses recommended by CUPM in its report A General Curriculum in Mathematics for Colleges* (GCMC) in terms of a teacher's own academic background. As a further task, we consider the distribution of training within a mathematics

* The original GCMC report is not included in this COMPENDIUM. However, the 1972 Commentary on A General Curriculum in Mathematics for Colleges appears on page 33.

faculty today which makes it possible for the department to teach effectively the program recommended in the GCMC report.

It should be understood that no academic program or degree in itself qualifies an individual to teach effectively at any level unless this preparation is accompanied by a genuine interest in teaching and by professional activities reflecting continuing mathematical growth. These activities may assume the form of several of the following:

(a) taking additional course work

(b) reading and studying to keep aware of new developments and to explore new fields

(c) engaging in research for new mathematical results (even if unpublished)

(d) developing new courses and new ways of teaching

(e) publishing expository or research articles

(f) participating in the activities of professional mathematical organizations

The preceding list reflects our conviction that an effective teacher must maintain an active interest in the communication of ideas and have a dedication to studying, learning, and understanding mathematics at levels significantly beyond those at which he is teaching.

A college mathematics department, whose staff members are engaged in activies such as those described above and have the academic qualifications to be described below, should have confidence in its ability to provide the quality of teaching required of it.

THE TRADITIONAL ROLE OF THE Ph.D DEGREE

Colleges and universities have come to place considerable emphasis on the doctor's degree as a necessary requirement for college teaching. This emphasis is quite understandable, since the Ph.D. is the most advanced degree offered by American universities and is therefore a symbol of maximal academic achievement. Unfortunately, the relevance of the doctoral degree in the qualification of a college teacher is often misunderstood, and the resulting confusion has, in many cases, led to serious abuses. We have in mind such abuses as the preferential treatment frequently assured the holder of a doctoral degree over an otherwise well-qualified teacher who lacks a Ph.D.; or unrealistic emphasis at some institutions on the number of doctoral degrees, regardless of origin, held by members of the faculty. We shall examine the requirements for a Ph.D. in mathematics in order to analyze the relevance of each of them in evaluating the qualifications of a college teacher.

The Ph.D. in mathematics is by long tradition a research degree; research in mathematics has meant the creation of new mathematics and not, as in many fields, the scholarly analysis or synthesis of previous work. A mathematics student working toward a Ph.D. is expected to spend a considerable portion of his time, in the later undergraduate and early graduate years, acquiring a broad general background in mathematics. The breadth of his knowledge is usually tested by special examinations after one, two, or more years of graduate study. After these examinations the student's work becomes highly specialized with seminars, independent study, and thesis work penetrating in depth some area of particular interest.

The earlier years of graduate study provide a breadth of knowledge essential to a college teacher. The subsequent, very specialized, graduate study is equally essential for research work.

An institution that has, or that aspires to have, a legitimate graduate program must necessarily have a substantial number of research mathematicians on its faculty, to implement the program and to provide the necessary leadership. This is a condition which obviously should not be changed. Even in an undergraduate college which does not offer a graduate program in mathematics, there are good reasons for wanting faculty members to have the kind of preparation required for the Ph.D. Both the nature and the content of undergraduate courses in mathematics must undergo frequent revision to reflect the rapid developments in mathematics and in related fields. The competent teacher of undergraduate mathematics must be able to master new material independently and to prepare new courses involving material which he never studied in his own course work; frequently he must guide independent study by gifted undergraduates. These challenges call for a degree of mathematical maturity which comes only with extended effort. A confident approach to new material is made possible not alone by the amount of knowledge a teacher may have; it requires in addition a broad understanding and a deep appreciation of the nature of mathematics. A significant research experience such as that demanded for the Ph.D. dissertation is perhaps the best guarantee that a person actually has the kind of maturity we have in mind. The research work itself may not provide the prospective teacher with the necessary breadth of knowledge, but it provides him with maturity which should enable him to continue his mathematical education independently and indefinitely.

THE FORMAL EDUCATION OF COLLEGE TEACHERS OF MATHEMATICS

There are a number of levels of mathematical preparation which are appropriate for teaching the various courses described in A General Curriculum in Mathematics for Colleges. In discussing these levels we shall begin with a prospective college teacher's undergraduate program and then indicate the teaching responsibilities compatible with successive components of his additional mathematical education.

A. Strong Undergraduate Mathematics Major Program

Mathematics major programs differ widely from one institution to another. For present purposes we shall refer to a major program based on courses described in the GCMC report.* This report suggests that a mathematics major program for students preparing for graduate work in mathematics should include the lower-division analysis courses 1, 2, 4, 5, the lower-division probability course 2P, the lower-division linear algebra course 3, and the upper-division courses in algebra (6M and 6L), analysis (11, 12, 13), and applied mathematics (10). The report adds that, where possible, a stronger major is desirable, with options to be selected from the courses in probability and statistics (7), numerical analysis (8), and geometry (9). The CUPM Panel on Pregraduate Training, in reviewing these recommendations in their report Preparation for Graduate Study in Mathematics [page 447], observed that most graduate departments desire an incoming student to be especially well-grounded in algebra and analysis. Consequently they recommended, and described in outline, a year course in algebra to replace the GCMC course, as well as the content for the year course in real analysis (11-12) which they considered essential to preparation for graduate study.

We do not favor special undergraduate curricula for prospective college teachers. Instead, we recommend a strong mathematics major program which begins with the mathematics major as described in GCMC and includes the analysis courses 11-12 outlined by the Pregraduate Panel, the additional work in algebra recommended by the Pregraduate Panel, and two additional courses selected from probability and statistics (7), numerical analysis (8), and geometry (9). We firmly believe that applications should be presented in all mathematics courses and that, where possible, students also should have some courses in fields where mathematics is applied (for example, theoretical physics or mathematical economics).

* These courses will be cited below using the numbers given them in the report A General Curriculum in Mathematics for Colleges; all of them are semester courses.

While the strong mathematics major program which we have described is certainly desirable for a college teacher, one must expect and encourage wide variation in the undergraduate programs which students actually encounter. Indeed, it is to be expected that strongly motivated research-oriented students will be advised to proceed to graduate work without some of the undergraduate courses we have listed. There are institutions where this strong major will be completed by many students at the time they receive the bachelor's degree. On the other hand, some students, including many in other disciplines or in training programs for secondary school teachers, will not encounter some of the more advanced upper-division courses until they reach graduate school.

Graduate students who have completed a strong undergraduate mathematics major program with distinction and who have a definite interest in teaching are qualified to assist more mature teachers in teaching elementary courses at the college level. Completion of this strong mathematics major should not be considered permanent qualification for a teacher of even the most elementary college courses. As we pointed out, continued intellectual growth is an essential qualification for sustained competence as a teacher. (In junior colleges, or at other institutions where remedial mathematics courses are offered, there could be some justification for having outstanding teachers with training equivalent to that of a strong mathematics major as members of the faculty, responsible for these courses.)

B. First Graduate Component

In this section we describe the additional graduate work which a prospective college teacher, who has completed the strong major program, will need in order to acquire the mathematical background necessary to teach the lower-division curriculum of GCMC (and hence the mathematics courses for junior college students who plan to transfer to a college or university). Those who complete both the strong major program and this first graduate component will also have the technical qualifications needed to teach some of the upper-division courses of the GCMC program.

We must emphasize that the courses to be described are not meant to be minimal introductions to their subject matter. The courses demand a serious involvement with graduate mathematics. Where questions of substance arise, mathematics departments should tend in the direction of the recommendations of the CUPM Pregraduate Panel's report Pregraduate Preparation of Research Mathematicians [page 369].

The time required to complete the first graduate component will vary considerably; obviously, a student who achieves only minimal success in his course work or whose undergraduate training has fallen short of the strong mathematics major will require more than the usual amount of time to reach the necessary level of mathematical maturity. We have found, incidentally, that the programs of many

Academic Year Institutes bring the student to a level only slightly beyond that of a strong mathematics major.

Hopefully, a student who completes the first graduate component will have developed a mathematical maturity that will enable him to bring to his classes an awareness of the fact that the mathematics taught in lower-division courses is a part of the basic fabric of applied mathematics. He should be able to present illustrations from outside of mathematics including both the physical and the behavioral sciences, where appropriate. It would be desirable, but it is not necessary, that he have made a serious study of some field of application (as represented, for example, by a year's course work), but it would also be possible for him to broaden his appreciation for the applications of mathematics by supplementary reading outside of his regular course assignments.

The first graduate component, which is an essential part of the preparation of a college mathematics teacher, and for which a master's degree would be suitable recognition, includes:

1. The completion of the strong mathematics major, if it has not been completed by the time the student begins graduate work.

2. At least two of the following three items:

    a. A substantial year's work in modern algebraic theory building on the earlier courses which presented the fundamental concepts of algebra.

    b. A year's work in analysis designed to follow the undergraduate analysis courses 11, 12, 13 of GCMC.

    c. A full year of "geometry" from a topological point of view following an undergraduate geometry course such as 9 of GCMC. This should include a semester of general topology and at least an introduction to algebraic topology.

3. At least one semester, preferably two, of teaching a class of undergraduate mathematics under the close supervision of an experienced teacher. Serious special attention should be devoted to the pedagogical problems involved in developing mathematical material for an immature audience. If possible, this teaching experience should also be accompanied by a proseminar designed to give students experience in articulating mathematical concepts before a critical audience.

We have repeatedly stated that a college teacher must continue his mathematical growth throughout his career. While the early graduate years are themselves a period of growth, it is also desirable that the student review college mathematics from the more advanced point of view of his graduate courses. There are many books by distinguished mathematicians which can help in this review and which provide a wealth of illustrations to enrich his teaching.

C. Advanced Graduate Component

In this section we describe a program of study which, when offered in a graduate department having an established Ph.D. program in mathematics, should provide the prospective college mathematics teacher with the mathematical background and with the maturity he will need to be prepared to teach all of the courses in the four-year GCMC program. Successful completion of both the first graduate component and the advanced graduate component should also provide a sound basis for the continued professional and intellectual growth which a college teacher requires in order to qualify, in due course, for promotion, tenure, and administrative responsibility in his department--whether or not he subsequently earns an advanced degree. Some of the work which we include in the advanced graduate component is intended specifically for prospective college teachers and to this extent it complements regular graduate programs designed to prepare research mathematicians.

The work of the advanced component builds on that of the first component and consists of the satisfactory completion of the following:

1. A year course in any of the three fields--algebra, analysis, topology-geometry--not included in satisfying Recommendation 2 of the first graduate component.

2. A second year of graduate study in at least one of the three fields mentioned above, as well as additional graduate courses in mathematics representing areas of special interest to the faculty.

3. A graduate research seminar designed to bring the student into active contact with the creative efforts of a member of the research faculty.

4. A seminar or reading course designed to provide a critical review of the relationship of the student's graduate courses to the undergraduate courses he might be called upon to teach: briefly, a form of "Elementary Mathematics from an Advanced Viewpoint."

5. A general examination designed to test the breadth of knowledge essential to a college mathematics teacher. It would cover each of the major areas of mathematics in which the student has taken courses at the graduate level.

6. A lecture project designed to test the student's ability to prepare and deliver a seminar talk and to provide him

an opportunity to develop his expository ability. We suggest that the topic assigned for the lecture be one outside of the student's field of specialization, in order that he may also demonstrate his competence to pursue mathematics on his own initiative.

While the time and the course work required to complete the advanced graduate component will vary a great deal among individuals and institutions, it should be clear that a candidate who reaches this level must have a strong personal commitment to mathematics and that he will have successfully completed at least two or three years of serious full-time graduate study beyond the strong major program.

The depth of understanding, the breadth of knowledge, and the mathematical maturity attested to by the successful achievement of the advanced graduate component are essential for the effective teaching of the various courses in mathematics offered at the college level. We believe that such achievement should be recognized by appropriate certification. Recent action of the faculties at Michigan, at Yale, and on the Berkeley Campus of the University of California seems to indicate a growing sentiment in favor of some such formal recognition.

D. The Doctorate

Although we have asserted that a Ph.D. degree in mathematics should not be regarded as an absolute necessity for the academic qualification of a college teacher of mathematics, we certainly would not suggest that the work and the study required to earn a Ph.D. are not important or that they would not enhance the effectiveness of any college teacher. The significant difference between the advanced graduate component and the Ph.D. degree consists of research seminars and independent reading in the candidate's field of specialization, leading to an original contribution to mathematical knowledge reported in the thesis. Making an original contribution to mathematical knowledge is extremely valuable for the college teacher, for by engaging in research he becomes a participating member of the mathematical profession and thus is able to transmit to his students, both in the classroom and outside of it, the knowledge and the stimulation that comes from the experience of creating new ideas.

It is our intention that the successful completion of the advanced graduate component when followed by an appropriate thesis should be worthy of a doctor's degree. Thus we believe that it should be offered only in those departments which already have established Ph.D. programs in mathematics; only in the vital research atmosphere of such a department can the required quality be attained. We also believe that graduate schools should be encouraged to seek ways of increasing the opportunities for qualified college teachers of mathematics to earn the Ph.D. after some years of teaching.

THE COMPOSITION OF AN UNDERGRADUATE DEPARTMENT

It is clear from the preceding discussion that we consider it neither necessary nor desirable to specify a single standard to be applied to all college teachers of mathematics. A very effective department can be composed of staff members with different levels of preparation and experience. Of course, there is no such thing as being too highly qualified to teach any course: higher qualifications can always be translated into more effective teaching, the design of an improved course, the preparation of better materials, and so on. However, the critical shortage of mathematics teachers requires that the available staff be used as effectively as possible, both in the individual college and in the country as a whole.

Let us consider reasonable academic qualifications for the mathematics faculty of a small college, one with a mathematics staff of six. We assume that the college has no graduate students and hence no graduate teaching assistants. At least two thirds of the teaching load is likely to be in lower-division courses. We believe that if three or four of the six staff members are at or near the level of the advanced graduate component or have Ph.D.s in mathematics, the department will have the technical qualifications needed to do an excellent job. Care must be exercised in the selection of staff members to assure that advanced study is not concentrated in only one area of mathematics. For example, the GCMC courses in applied mathematics, numerical analysis, and probability and statistics require special attention; there should be members of the staff who have graduate work in these areas.

We do not suggest that all the lower-division courses ought to be taught by teachers in the first group and all the advanced courses by the others. On the contrary, we consider it essential that some of the most highly qualified teachers be involved in the elementary courses, just as we believe that many of the less well-prepared teachers can be expected to do excellent work in some of the more advanced courses. Indeed, one very effective way for any teacher to increase his knowledge is for him to give an advanced course in which he may learn as he teaches. The level of qualification of any staff member cannot be regarded as permanent or fixed. Since continued intellectual growth is required for good teaching, every staff member at whatever level must be considered as on his way to higher qualifications. This applies just as much to a man with a Ph.D. in mathematics as it does to any other teacher in the department.

Finally, we do not wish to imply that rank or salary should depend entirely on the levels of academic preparation we have described. In general, rank should correspond to professional competence and achievement, as indicated by all professional activities and by teaching effectiveness, as well as by earned degrees.

Our suggestions are, of course, subject to modification to fit the needs of individual institutions. We predict, for example, that for the foreseeable future the first graduate component should represent adequate preparation for teaching transfer students in junior colleges, provided the teacher continues to remain "intellectually alive." At universities, and at colleges near universities, it is certainly appropriate to make use of teaching assistants who have reached only the level of a strong mathematics major, or who have not yet completed the first graduate component, provided that the teaching is adequately supervised and that there is clear evidence of progress toward the next level.

FINAL REMARKS

We have repeatedly stated our conviction that continued intellectual and professional growth is essential to continued competence as a teacher. One needs to move forward in order not to fall behind. A significant reason for recognizing the Ph.D. as a meaningful and desirable level of qualification for college teachers is that it is both evidence of an individual's ability to continue his mathematical growth by himself and an indication of momentum in that direction. However, for reasons of isolation or inadequate training, many college teachers are unable to provide for their own professional growth. For them, and for college teachers who do not have even minimal academic qualifications for the responsibilities they are asked to assume, there is an urgent need for expanded programs of external stimuli for improvement: guidance, financial assistance, and easily accessible and attractive study programs. Institutes, internships, and new forms of retraining need to be explored and developed. We must recognize, however, that the intellectual growth of college teachers depends primarily not on opportunities of this kind but on the conditions of their daily work. If their teaching and administrative duties leave them no time or energy for study and reflection, then it cannot be expected that their scientific qualifications will improve from year to year, or even that they will be maintained.

We have no definite advice to offer for solving these problems. We can only call attention to them and suggest that the difficulties involved in upgrading many of our present teachers and in stimulating continued growth in others provide some of the most important and pressing problems faced by the mathematical community.

A BEGINNING GRADUATE PROGRAM
IN MATHEMATICS
FOR PROSPECTIVE TEACHERS OF UNDERGRADUATES

A Report of

The Graduate Task Force

February 1969

TABLE OF CONTENTS

INTRODUCTION

More than half of the college and university mathematics teachers in the United States do not hold a Ph.D. in one of the mathematical sciences. Thus, the bulk of undergraduate teaching of mathematics in the country is being done by men and women whose graduate training, for one reason or another, has broken off short of the doctorate. There is no convincing evidence that this situation will soon change: the rising output of Ph.D.s in mathematics is probably more than offset by rapidly rising college enrollments and by increasing demands on mathematics as a service discipline.*

Those concerned with the preparation of college teachers are therefore faced with a basic problem: What is the best way to arrange the early part of the graduate program in mathematics to provide background for effective college teaching? This problem is further complicated because the research potential of most students is still untested when they begin graduate work; thus, it is not possible to separate those who will complete a Ph.D. from those who will not. Accordingly, the choice of topics for the first year or two of graduate study must permit students to progress unretarded toward the Ph.D. This booklet explores one solution to this problem.

We contend that all graduate students of mathematics should be treated as future teachers--first, because most of them do in fact go into teaching, and second, because virtually all professional mathematicians are engaged to some extent in the communication of mathematics. Hence, graduate programs aimed at producing better teachers may be expected to benefit everyone.

The CUPM *ad hoc* Committee on the Qualifications of College Teachers of Mathematics, in its report *Qualifications for a College Faculty in Mathematics* [page 102], outlined a graduate program ("first graduate component") which provides both a reasonable first segment of a Ph.D. program and adequate background for teaching the lower-division courses described in *Commentary on A General Curriculum in Mathematics for Colleges* (GCMC) [page 33]. In 1967 the Graduate Task Force, a group with membership drawn from CUPM and its Panel on College Teacher Preparation, was given the assignment of preparing a more detailed description of the first graduate component. A description appears in the pages that follow.

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* The situation has, in fact, changed dramatically since this report was written. According to the 1972 document *Undergraduate Education in the Mathematical Sciences, 1970-71* (Report of the Survey Committee of the Conference Board of the Mathematical Sciences), university departments of mathematical sciences had, in the fall of 1970, 6,304 doctorates in mathematical sciences, 348 other doctorates, and 971 nondoctorates. In four-year college departments there were 3,508 mathematical science doctorates, 758 other doctorates, and 5,158 nondoctorates.

Clearly, this set of recommendations is not the only possible solution to the problem stated above, but we believe that it forms a sound basic program which each university can adapt to local conditions. The time for its completion will depend upon the student's ability and preparation, but in most cases one to two years beyond the bachelor's degree should be adequate. Satisfactory completion will insure that the student has sound academic qualifications for teaching lower-division courses in calculus, linear algebra, probability, and advanced multivariable calculus.

Besides serving the purposes already described, appropriate parts of the course of study we recommend would constitute an excellent sabbatical program for established teachers who wish to improve their acquaintance with modern approaches to mathematics.

The recommended program is discussed in detail in the following section. Here we mention some of its characteristic features and reasons for them. Two considerations figure prominently in the selection of topics for courses: first, the relative importance of the topic in all of mathematics, and second, its relevance to teaching the lower-division courses described in the GCMC report.

In analysis, to follow a year of undergraduate real analysis and a semester of undergraduate complex analysis (like the GCMC courses 11, 12, 13), the program includes a semester of measure and integration followed by a semester of functional analysis. The course in measure and integration is obviously relevant to the GCMC courses in calculus and probability. We believe that the course in functional analysis is more important at this stage than a second course in complex analysis, since functional analysis will further develop the methods of linear algebra, the concept of uniform convergence, and various other topics in analysis. Moreover, functional analysis provides an immediate application of the course in measure and integration.

In topology we recommend a sequence which, in addition to the usual material in basic topology, includes an introduction to manifold theory and differential forms, to provide the prospective teacher with a deeper understanding of multivariable calculus.

Since lower-division mathematics needs to be illustrated liberally with uses of the subject, college teachers must command a broad knowledge of the applications of mathematics. Also, many will be called upon to teach elementary probability and statistics. Hence, we recommend that a course of study for college teachers include two or three semesters of work at the advanced undergraduate or beginning graduate level chosen from courses in probability, statistics, differential equations, numerical analysis, or subjects in applied mathematics. Moreover, all courses in the program should give attention to the relevance of their subjects to undergraduate mathematics and related disciplines.

In algebra we believe that a year-long advanced undergraduate course such as that described in the CUPM publication Preparation for Graduate Study in Mathematics [page 453] is essential, but that further study of algebra is less important in the preparation of teachers of lower-division mathematics than the suggested work in analysis, topology, and applied mathematics; therefore, graduate-level algebra has been treated as an elective. Likewise, we do not advocate a special requirement in geometry, partly because a considerable amount of geometry in various forms is distributed throughout other recommended courses, and partly because advanced training in geometry does not seem essential either as background for lower-division teaching or as general preparation for further graduate study. Nevertheless, geometrical points of view should be stressed in courses whenever they are appropriate.

It is certain that many teachers of lower-division mathematics will, in the very near future, be called upon to use the computer to some extent in their courses. However, in view of the rapid developments in computer science and the many nonmathematical factors involved, any explicit recommendations on the role of computing in this program must be regarded as tentative at this time. We do expect that students completing this program will have acquired at least a basic knowledge of computers [for example, the content of the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics, page 563].

Apart from formal course work, we feel that a meaningful apprenticeship in teaching is an essential aspect of the student's preparation. Activities to provide such an apprenticeship should form an integral part of beginning graduate work.

A master's degree would suitably recognize completion of the first graduate component. However, in place of a master's thesis we strongly recommend the substitution of a comprehensive examination. Foreign language requirements are not discussed here because we believe that they are irrelevant for a student whose graduate training stops at the first graduate component; however, the student who hopes to earn a Ph.D. should be advised that a reading knowledge of foreign languages is likely to be essential in his subsequent work.

It must be understood that the student who has completed this or any other program will not be, by that reason alone, a complete teacher or mathematician for the rest of his career; sustained intellectual and professional growth is essential to continued competence as a teacher and as a mathematician. For this reason, we urge a graduate faculty to make vigorous efforts to involve the students seriously, as participants rather than observers, in the mathematics they are studying. It is important for the student who stops short of the Ph.D., even more than for the one who will complete it, that course work of the first two years of graduate study emphasize fundamentals and basic understanding. This applies especially to the prospective college teacher who must be able to relate his graduate work to the material he will be teaching later. Courses which involve the student in doing mathematics as distinct from hearing about mathematics would seem to be particularly valuable. Depth of understanding on the part of the student is to be preferred to superficial exposure to mathematical terms. Our course outlines should be understood in this context.

Finally, we emphasize that it has not been our objective to design a separate track in graduate mathematics. This program is intended to prepare the student as an effective and well-informed teacher of lower-division mathematics; but, at the same time, we believe it moves him toward the Ph.D. at a satisfactory rate.

PROGRAM DESCRIPTION

The "first graduate component," as described in Qualifications for a College Faculty in Mathematics [page 107], is a program of graduate study built upon strong undergraduate preparation in mathematics. Because the undergraduate preparation of graduate students varies widely, it is useful to describe the first graduate component in terms of the combined undergraduate and graduate preparation of the candidate. It is likely that many students will have to complete in graduate school some undergraduate-level work; for such students, up to two years of post-baccalaureate study may be required to complete this program.

We assume that every student has already completed lower-division courses equivalent to the GCMC courses Mathematics 1, 2, 2P, 3, 4, 5, including a basic course in computer science, like C1 [page 563].

In addition, he will have studied some, but probably not all, of the following upper-division courses: Mathematics 7 (Probability and Statistics)*, 8 (Introduction to Numerical Analysis), 9 (Geometry), 10 (Applied Mathematics) [see page 79].

The following five courses form the core of preparation for graduate study: Mathematics 11-12 (Introductory Real Variable Theory), 13 (Complex Analysis), D-E (Abstract Algebra). For outlines of Mathematics 11-12 and 13, see page 93. For outlines of D-E, see page 453.

Graduate courses which are especially appropriate for the first graduate component, and for which suggested course descriptions are given starting on page 121 of this COMPENDIUM, are:

* We use 7A and 7B to refer to the probability component and statistics component, respectively, of Mathematics 7.

P Measure and Integration

Q Functional Analysis

R Complex Analysis

S Topology

T Homology and Multivariable Integration

U Topology and Geometry of Manifolds

V Galois and Field Theory

W Ring Theory and Multilinear Algebra

X Advanced Ordinary Differential Equations with Applications

Y Problem-oriented Numerical Analysis

Z Seminar in Applications

Courses 11-12, 13, D-E, P, Q, S, and T should be in every student's program. Because of the great importance of applied mathematics, every program of study should include at least one year of applied work, of which the following four sequences are examples: 7A-7B, X-10, X-Y, X-Z.

Each student should, if possible, include a third course from among the courses 7A, 7B, 10, X, Y, Z in his program. Students who plan to continue into the advanced graduate component and to specialize in some area of pure mathematics are advised to take as many as possible of the courses R, U, V, W. Other students may substitute electives in geometry, logic, foundations, number theory, or other subjects.

For the sake of convenience, we have stated our recommendations in terms of semester courses. However, we believe that courses at the graduate level are best thought of as year courses. The material outlined for pairs of related courses can, of course, be rearranged within the year to suit local conditions.

Since beginning graduate programs ordinarily include year courses in analysis, topology, and algebra, our recommendations depart from the norm only in ways intended to enhance the ability of the student to teach lower-division mathematics.

Effective exposition is a skill of major importance to any prospective mathematician, whether he expects his principal professional emphasis to be research or teaching. However, it is unrealistic to assume that a beginning graduate student is qualified to teach well, even in introductory undergraduate courses. Therefore, we propose that he be required to complete an apprenticeship in teaching under the thoughtful direction of experienced members of the faculty. Suggestions for such a program are discussed later in this report.

To complete the program, we suggest that every student be required to take a comprehensive examination designed specifically to test the breadth and depth of the candidate's understanding of mathematics relevant to the undergraduate curriculum. Whenever feasible, the examination should be scheduled so that students have several weeks devoted exclusively to preparation for it. We believe that a truly comprehensive examination is a more appropriate requirement than the traditional master's thesis, principally because preparation for such an examination demands that the student regard his subject as a whole rather than a collection of parts.

In summary, the first graduate component, as described here, consists of the following work:

(1) Completion (if necessary) of a strong undergraduate major program which includes these upper-division courses: three semesters of real and complex analysis, a full year of abstract algebra (the equivalent of Mathematics 11-12, 13, D-E).

(2a) A year of graduate topology, including differential forms.

(2b) A year of graduate analysis: measure and integration and functional analysis.

(2c) A year of work at the advanced undergraduate or beginning graduate level, emphasizing the applications of mathematics: e.g., a year of probability and statistics; or a semester of differential equations followed by a semester of numerical analysis, a seminar in applications, or a "model building" course.

(3) A year or more of work focused on problems of teaching undergraduates.

For a student whose undergraduate preparation does not meet the standards described in (1) and (2c), completion of the first graduate component may require two years of study beyond his bachelor's degree. For example, if his undergraduate preparation in algebra and in analysis is weak, his program for the first graduate component might be as follows:

<u>First Year</u>

Analysis (Mathematics 11)	Analysis (Mathematics 12)
Complex Analysis (Mathematics 13)	Applied Mathematics (Mathematics 10)
Algebra D	Algebra E
Apprenticeship in Teaching	Apprenticeship in Teaching

Second Year

Analysis P	Analysis Q
Topology S	Topology T
Probability (Mathematics 7A) and	Statistics (Mathematics 7B)
OR	
Differential Equations X and	Numerical Analysis Y or Applications Seminar Z
Apprenticeship in Teaching	Apprenticeship in Teaching
	Comprehensive Examination

Most students will have completed some of the undergraduate courses in this program and thus will be able to substitute electives for some of the subjects listed. A student who has a very strong undergraduate major in mathematics will be able to complete the program in one year, for example, by taking the second year of the preceding schedule.

Graduate departments are urged to give careful attention to the proper placement of entering graduate students and to continue to advise them regarding course selections.

COURSE OUTLINES

Analysis

The following section includes suggested outlines for three one-semester graduate courses in analysis:

P. Measure and Integration (two suggested outlines are offered)

Q. Functional Analysis

R. Complex Analysis

Each student should include courses P and Q in his program of study.

P. Measure and Integration

This course provides an introduction to and essential background for Course Q, can be used in Course R, and is naturally useful in more advanced courses in real analysis. We present two outlines, which represent different approaches and a somewhat different selection of material. If presented in the right spirit, a course in Measure and Integration provides insights into the material of lower-division courses that the student will have to teach.

First Outline

1. The limitations of the Riemann integral. Examples of a series that fails to be integrable term-by-term only because its sum is not integrable; of a differentiable function with a nonintegrable derivative. Limitations of integration in general: there is no countably additive, translation-invariant integral for all characteristic functions of sets (the usual construction of a nonmeasurable set will serve).

2. Lebesgue integration on the line. Outer measure; definition of measurable sets by means of outer measure. Measurability of sets of measure 0, of intersections and unions, of Borel sets. Countable additivity. Application: the Steinhaus theorem on the set of distances of a set of positive measure. Measurable functions, Borel measurability, measurability of continuous functions. Egoroff's theorem. Definition of the integral of a bounded measurable function as the common value of $\inf \int \Psi(x)\,dx$ for simple majorants of Ψ of f and $\sup \int \varphi(x)\,dx$ for simple minorants φ. Riemann integrable functions are Lebesgue integrable. Bounded convergence and applications (necessary and sufficient condition for Riemann integrability; $\log 2 = 1 - 1/2 + 1/3 + \ldots$). Integrability of nonnegative functions, Fatou's lemma, monotone convergence, integrability of general functions. A nonnegative function with zero integral is zero almost everywhere.

3. L^p spaces, with emphasis on L^2; motivation from orthogonal series. Schwarz inequality; with little extra effort one gets (via convex functions) the Hölder, Minkowski, and Jensen inequalities. L^∞ as a formal limit of L^p via $(\int f^p)^{1/p} \to \text{ess sup } f$ as $p \to \infty$. Parseval's theorem, Riesz-Fischer theorem. Rademacher functions;

proof that almost all numbers are normal. Convergence of $\Sigma \pm \frac{1}{n}$ and other series with random signs. Proof (by Bernstein polynomials or otherwise) that continuous functions on an interval are uniformly approximable by polynomials; hence, continuous functions are dense in L^p.

4. Differentiation and integration. Proof that an indefinite integral is differentiable almost everywhere and its derivative is the integrand; the Lebesgue set. Equivalence of the properties of absolute continuity and of being an integral.

5. Lebesgue-Stieltjes integral with respect to a function of bounded variation. A rapid survey pointing out what changes have to be made in the previous development. Applications in probability, at least enough to show how to treat discrete and continuous cases simultaneously. Riesz representation for continuous linear functionals on $C[a,b]$.

6. General measure spaces. Definition of the integral and convergence theorems in the general setting; specialization to n-dimensional Euclidean space. Fubini's theorem. Application to convolutions and to such matters as gamma-function integrals and $\int e^{-x^2} dx$. The one-dimensional integral as the integral of the characteristic function of the ordinate set.

7. (If time permits) Complex measures. Decompositions. Radon-Nikodým theorem.

Second Outline

1. Lebesgue integration on the line. F. Riesz's step function approach. Definition of the integral for simple step functions and extension to the class of functions which are limits almost everywhere of monotone sequences of simple step functions. Definition of summable function and fundamental properties of the integral. Extension to complex-valued functions. The basic convergence theorems, including monotone, bounded, and dominated convergence theorems. Fatou's lemma and convergence in measure. Illustrations and

applications: (a) justifications, by bounded convergence, of term-by-term integration of series leading to formulas such as $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots$; (b) use of dominated convergence to perform operations such as $\frac{\partial}{\partial\alpha}\int f(x,\alpha)\,dx = \int \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx$; (c) proof of the analogue of Fatou's theorem for series: $\underline{\lim}_i \Sigma_n a_{in} \geq \Sigma_n \underline{\lim}_i a_{in}$ when $a_{in} \geq 0$. Comparison of the Riemann and Lebesgue integral.

2. Measure and absolute continuity. Measurable functions and measurable sets. Properties of measurable sets. Egoroff's theorem. Cantor's function and the relationship between Lebesgue and Borel sets. Nonmeasurable sets. Proof that the integral of a summable function is a countably additive set function. Almost everywhere differentiability of monotone functions. Review of basic properties of functions of bounded variation. Absolutely continuous functions. Fundamental theorem concerning differentiation of the integral of a summable function. Proof that an absolutely continuous function on an interval is of bounded variation and that its total variation is equal to the L^1-norm of its derivative. Helly's theorem on compactness of families of normalized functions of bounded variation.

3. L^p spaces and orthogonal expansions. Convex functions and the inequalities of Hölder and Minkowski. Proof that the L^p spaces are complete. Theorem: If $\{f_i\}$ is a sequence of measurable functions, if $f_i \to f$ pointwise almost everywhere, and if $\underline{\lim}_i \int |f_i| = \int |f|$, then $\underline{\lim}_i \int |f_i - f| = 0$. Lusin's theorem: f measurable and finite almost everywhere and $\delta > 0$ implies there exists a continuous function φ such that $\varphi = f$ except on a set of measure less than δ. Hence, continuous functions are dense in L^p, $1 \leq p < \infty$. Representation of continuous linear functionals on L^p. Orthonormal systems in $L^p(a,b)$. Bessel's inequality, Parseval's inequality, and the Riesz-Fischer theorem. Proof that the Cesàro means of the Fourier series of a function f in $L^p(0,2\pi)$ converge to f in the L^p-norm $(1 \leq p < \infty)$ and uniformly, provided that f is periodic and continuous. From this latter fact deduce the Weierstrass theorem on polynomial approximation of continuous functions on an interval. The trigonometric functions form a complete

orthonormal system in $L^2(0,2\pi)$.

4. Integration on product spaces. Integration and measure in R^n. Theorems of Fubini and Tonelli. Applications to nonlinear change of variable in multiple integrals.

5. Convolution (optional). If f is summable, then $f(t - x)$ is a measurable function on the plane. By Fubini's theorem, f and g in L^1 implies that $f*g(t) = \int f(t - x)\, g(x)\, dx$, the convolution of f with g, is finite for almost all t, $f*g$ is in L^1, and $\|f*g\| \leq \|f\|\,\|g\|$. In fact, for $p > 1$, $q > 1$, and $0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, we have $L^p * L^q \subset L^r$ and $\|f*g\|_r \leq \|f\|_p\,\|g\|_q$. (Actually, $L^1 * L^p = L^p$ for $1 \leq p < \infty$.) Also, if $1 < p < \infty$ and $p' = p/(p-1)$ and $f \in L^p$, $g \in L^{p'}$, then $f*g$ is bounded. L^1 under convolution is an algebra without unit. Proof that $\widehat{f*g} = \hat{f}\hat{g}$, where $\wedge$ denotes the Fourier transform. Riemann-Lebesgue lemma.

6. General measure theory (optional). Set functions and introduction of abstract measure spaces. Definition of the integral and rapid review of standard theorems. Total variation of measures, regularity properties of Borel measures. Identification of Borel measures on the line with functions of local bounded variation. Absolutely continuous and mutally singular measures and consequences of the Radon-Nikodým theorem. Riesz representation for $C(X)$, X compact.

Q. Functional Analysis

The purpose of this course is to develop some of the basic ideas of functional analysis in a form suitable to applications and to deepen the student's understanding of linear methods in undergraduate mathematics. Whenever possible, topics should be treated and applied in a setting with which the student has some familiarity; main theorems should be supported with concrete and meaningful examples.

1. Metric spaces. Review of topology and metric spaces if necessary. Completion of metric spaces. Method of successive approximations. Proof that a contraction operator on a complete

metric space has a unique fixed point. Application to existence of the solution of a system of linear equations, polynomial equations, initial value problems for ordinary differential equations and integral equations.

2. Normed linear spaces. Examples (not all complete) from sequence spaces, function spaces, and finite-dimensional spaces. Completion of $C[a,b]$ under the L^1-norm. Proof that the unit ball in a normed linear space is compact if and only if the space is finite-dimensional. Equivalence of norms in finite-dimensional spaces.

3. Linear functionals. The dual space of a normed linear space. Computation of the dual for spaces R^n, c_o, ℓ_p $(1 \leq p < \infty)$, and $C[a,b]$. Contrast with algebraic dual. Convex sets and separation of convex sets by linear functionals. Support functionals. Analytic, geometric, and complex forms of the Hahn-Banach theorem. Applications of the Hahn-Banach theorem, such as (a) computation of the distance from a point to a subspace in terms of the linear functionals which vanish on the subspace; (b) the existence of a function in $L^\infty(0,1)$ of minimal L^∞-norm which satisfies the $N + 1$ relations $\int_0^1 t^n f(t)\, dt = a_n$, $n = 0, 1, \ldots, N$; (c) solution of the Hausdorff moment problem for $C[0,1]$; (d) the existence of Green's function for Laplace's equation in the plane for a domain with sufficiently smooth boundary. Principle of uniform boundedness and applications, such as (a) existence of a continuous periodic function on $[-\pi,\pi]$ whose Fourier series fails to converge; (b) the Silverman-Töplitz conditions for a regular matrix summability method; (c) existence of the Riemann-Stieltjes integral $\int_0^1 f\, d\alpha$ for every continuous f implies that α is of bounded variation on $[0,1]$. Weak (not weak*) convergence of sequences in normed linear spaces. Proof that weakly convergent sequences are bounded but not necessarily norm convergent. Characterization of weakly convergent sequences in spaces such as ℓ_p $(1 \leq p < \infty)$ and $C[a,b]$. Elementary introduction to distribution theory.

4. Linear operators. Examples from matrix theory,

differential and integral equations. The closed graph theorem and the interior mapping principle. Notion of an adjoint operator. Inversion of linear operators near the identity. The spectrum and resolvent of an operator.

5. Hilbert spaces. Inner products, orthogonality, orthogonal systems. Fourier expansions, Bessel's inequality, and completeness. Representation of linear functionals. Self-adjoint operators on a real Hilbert space as a generalization of symmetric linear transformations on R^n. Eigenvalues, eigenvectors, invariant subspaces, and projection operators. The spectral theorem for completely continuous self-adjoint operators. (Here the goal is the formula $Ax = \Sigma\, \lambda_k(x,x_k)x_k$, where A is a compact and self-adjoint operator, x is any point, $\lambda_1, \lambda_2, \lambda_3, \ldots$ is the sequence of nonzero eigenvalues, and $x_1, x_2, x_3, \ldots$ is the corresponding set of eigenvectors.) Construction of a one-parameter family of projections E_λ which allows representation of the action of A in terms of a vector-valued Riemann-Stieltjes integral

$$Ax = \int \lambda \; dE_\lambda x.$$

Description (without proof) of the corresponding theorem for the unbounded case. Application of the theory of compact, self-adjoint operators to Sturm-Liouville systems or integral equations with symmetric kernels.

R. Complex Analysis

The amount of material that can be covered in this course depends very much on the amount of knowledge that can be assumed from Mathematics 13. The outline assumes that the student knows this material quite well, but some of the more advanced topics may have to be omitted or treated in less depth. Such topics are enclosed in brackets.

1. Holomorphic functions. (Much of this should be review.) Cauchy's integral theorem in a more general setting than was used in Mathematics 13. According to circumstances, this may be for unions of star-shaped regions, for C^1 Jordan curves, for singular

cells, etc., but not for general rectifiable Jordan curves. Cauchy's integral formula. Taylor and Laurent series. Residue theorem. [Evaluation of some definite integrals which are more sophisticated than those of Mathematics 13.] Classification of isolated singularities, Casorati-Weierstrass theorem. Liouville's theorem. Fundamental Theorem of Algebra. $\int f'/f = 2\pi i\cdot$(number of zeros minus number of poles), with applications to some special cases (number of zeros of a polynomial in a quadrant, for example). Maximum modulus theorem. Schwarz's lemma. Rouché's theorem with some concrete applications (Fundamental Theorem of Algebra again; zeros of $e^z + z$ and other special functions). [Montel's theorem, Phragmèn-Lindelöf theorems.]

2. Harmonic functions. Cauchy-Riemann equations. Mean value property for harmonic functions. Poisson formula and Dirichlet problem for the circle and annulus. Connection with Fourier series and Poisson summability of Fourier series at points of continuity. [Other problems on functions holomorphic in a disk: Abel's theorem, elementary Tauberian theorems.] [Fatou's theorem on radial limits.] [Positive harmonic functions, Herglotz's theorem on the integral representation of holomorphic functions with positive real part in a disk. Harnack's theorem.]

3. Holomorphic functions as mappings. Mapping properties of the elementary functions. Nonconstant holomorphic functions are open. Conformality at points where the derivative is not zero. Only holomorphic functions produce conformal maps. Conformal automorphisms of the disk and the half-plane. Normal families. Proof of the Riemann mapping theorem. [Schwarz-Christoffel formula. Conformal representation of a rectangle on a half-plane. Elliptic functions. Proof of the small Picard theorem.]

4. Analytic continuation. Schwarz reflection principle. Analytic continuation. Permanence of functional equations. Monodromy theorem. [Multivalued functions. Elementary Riemann surfaces.]

5. Zeros of holomorphic functions. Infinite products. Entire functions. Meromorphic functions. The Weierstrass factorization theorem. Mittag-Leffler theorem. Gamma function. [Jensen's formula and Blaschke products.]

[6. Approximation. Runge's theorem and approximation by polynomials. Mergelyan's theorem.]

Topology

The following section includes suggested outlines for a sequence of three one-semester graduate courses in topology:

S. Topology

T. Homology and Multivariable Integration (two outlines are offered)

U. Topology and Geometry of Manifolds

<u>Each student should include courses S and T in his program of study</u>. Students who plan to elect course U must study the first (preferred) outline of T.

S. <u>Topology</u>

We assume that the students have made a brief study of metric spaces, Euclidean spaces, and the notion of continuity of functions in metric spaces. (This material is covered in Sections 4, 5, and 6 of Mathematics 11-12.)

1. <u>Basic topology</u>. Topological spaces, subspace topology, quotient topology. Connectedness and compactness. Product spaces and the Tychonoff theorem. Separation axioms, separation by continuous functions. Local connectedness and local compactness. Metric spaces, completion of metric spaces, uniform continuity. Paracompactness, continuous partitions of unity.

2. <u>Applications to calculus</u>. Use the above results to prove again the basic topological results needed for calculus and the Heine-Borel and Bolzano-Weierstrass theorems.

3. <u>Fundamental group</u>. Homotopies of maps, homotopy equivalence. The fundamental group π_1, functional properties, dependence on base point. Show that $\pi_1(S^1) = Z$.

4. Applications of the fundamental group. Brouwer fixed point theorem for the disk D^2. R^2 is not homeomorphic with R^3. Relevance of the fundamental group to Cauchy's residue theorem. Fundamental Theorem of Algebra.

5. Covering spaces. Covering spaces, homotopy lifting and homotopy covering properties. Regular coverings, existence of coverings, universal covering. Factoring of maps through coverings. Relation with Riemann surfaces.

T. Homology and Multivariable Integration

Preferred Outline

1. Manifolds. Topological manifolds. C^k and C^∞ functions on R^n. Differentiable structure on a topological manifold. Diffeomorphisms. C^∞ partitions of unity for paracompact manifolds.

2. Functions on manifolds. The ring $\mathfrak{a}(U)$ of C^∞ real-valued functions on an open set U, the ring $\mathfrak{a}(x)$ of germs of C^∞ functions at a point x. Pullbacks of these rings via a C^∞ function. Tangent bundle and cotangent bundle. Bases for tangent and cotangent spaces in a coordinate system. Vector fields, Poisson bracket, flows. Inverse and implicit function theorems. Frobenius' theorem.

3. Applications to differential equations. Relation of vector fields to ordinary differential equations and of Frobenius' theorem to partial differential equations.

4. Differential forms. Differential forms, elementary forms. Exterior multiplication of forms, the differential operator d on forms; $dd = 0$ and d of a product.

5. Applications to classical vector analysis. The algebra of forms on R^3 contains vector algebra and with d contains vector analysis.

6. deRham cohomology. Pullback of forms via a C^∞ map commutes with d. Closed and exact forms, deRham groups as a cohomology theory.

7. Simplicial homology. Simplicial complexes, simplicial

homology. Barycentric subdivision, simplicial approximation theorem. Calculation of π_1 for a simplicial complex. Singular homology and cohomology of a space.

8. <u>Applications of simplicial homology</u>. The Brouwer fixed point theorem for D^n, invariance of domain, and the Jordan curve theorem. (Recall use of the Jordan curve theorem in complex analysis.)

9. <u>Stokes' theorem</u>. Integral of a p-form over a singular p-chain. Proof of Stokes' theorem. This implies that integration induces a bilinear map from singular homology and deRham groups to R. Green's theorem as a special case of Stokes' theorem.

Second Outline

Note: If a student does not plan to take course U, then the following easier version of T may be desirable. This is carried out by working in R^n instead of in general differentiable manifolds, and the result is still a fairly general version of Stokes' theorem.

1. <u>Simplicial homology</u>. Simplicial complexes, barycentric subdivision, simplicial maps, and the simplicial approximation theorem. Simplicial homology theory, functional properties of homology groups. Calculation of homology groups for simple complexes.

2. <u>Differential forms</u>. Differential forms on open sets of R^n. Properties of differential forms, the operator d on forms. Pullback of forms via a C^∞ function. Application to vector algebra and vector calculus. Closed and exact forms, the deRham groups.

3. <u>Singular homology and applications</u>. Singular homology theory. Applications: the Brouwer fixed point theorem, R^n and R^m are homeomorphic if and only if $n = m$, invariance of domain, Jordan curve theorem.

4. <u>Stokes' theorem</u>. Integration of p-forms over differentiable singular p-chains. Proof of Stokes' theorem.

U. Topology and Geometry of Manifolds

1. Chain complexes. Chain and cochain complexes (examples from T), derived groups. Exact sequences, ladders, the 5-lemma. Exact sequences of chain complexes, Bockstein exact sequence. Chain homotopies. Poincaré lemma and cone construction; derived groups of a contractible open set are zero.

2. Riemannian metrics for manifolds. Riemannian metrics for paracompact manifolds. Geodesics: existence and uniqueness. A paracompact C^∞ manifold may be covered with a star-finite covering by geodesically convex sets (so that all sets in the covering and all intersections are contractible).

3. Comparison of homology theories. A lattice L of subsets of X containing $\emptyset$ and X gives a category S with elements of L as objects and inclusions as morphisms. A cohomology theory h on S is a sequence of cofunctors h^q from S to abelian groups, along with natural transformations

$$\delta: h^q(A \cap B) \to h^{q+1}(A \cup B)$$

such that the Mayer-Vietoris sequence is exact. Proof that if L is a star-finite covering of X by open sets, then cohomology theories h, $\hat{h}$ which agree on finite intersections agree on X. Use of these results to deduce deRham's theorem and to prove that simplicial and singular theories agree on a simplicial complex.

4. Global differential geometry. The remainder of the course is devoted to surfaces. Gaussian curvature, spaces of constant curvature. Gauss-Bonnet theorem for surfaces, non-Euclidean geometries.

Algebra

The following section includes suggested outlines for two one-semester graduate courses in algebra:

V. Galois and Field Theory

W. Ring Theory and Multilinear Algebra

These courses are independent of one another and should be offered as electives. Each course outline includes a basic minimal list of topics as well as a list of optional topics from which the instructor is invited to choose.

V. Galois and Field Theory

Note: 1 and 2 are reviews of topics that should have been covered in the previous one-year algebra course D-E outlined on page 453.

1. Review of group theory. The third isomorphism theorem. Definition of simple group and composition series for finite groups. The Jordan-Hölder theorem. Solvable groups. Simplicity of the alternating group for $n > 4$. Elements of theory of p-groups. Theorems: A p-group has nontrivial center; a p-group is solvable. Sylow theory. Sylow theorem on the existence of p-Sylow subgroups. Theorems: Every p-subgroup is contained in a p-Sylow subgroup; all p-Sylow subgroups are conjugate and their number is congruent to 1 modulo p.

2. Review of elementary field theory. Prime fields and characteristic. Extension fields. Algebraic extensions. Structure of $F(a)$, F a field, a an algebraic element of some extension field. Direct proof that if a has degree n, then the set of polynomials of degree $n - 1$ in a is a field; demonstration that $F(a) \cong F[x]/(f(x))$, where f is the minimum polynomial of a. Definition of $(K:F)$, where K is an extension field of F. If $F \subset K \subset L$ and $(L:F)$ is finite, then $(L:F) = (L:K)(K:F)$. Ruler-and-compass constructions. Impossibility of trisecting an angle, duplicating the cube, squaring the circle (assuming π transcendental).

3. Galois theory. The group $G(M/K)$ of K-automorphisms of a field M containing K. Fixed field H' of a subgroup H of $G(M/K)$. Subgroup F' of $G(M/K)$ leaving an intermediate field F fixed. Examples like $Q(\sqrt[3]{2})$ to show that $G(M/K)'$ may be bigger than K. An object is closed if it equals its double prime. If $M \supset F \supset F_1 \supset K$ and $G(M/K) \supset H \supset H_1$, then $[F_1':F'] \leqq (F:F_1)$ and

$(H_1':H') \leqq [H:H_1]$. All finite subgroups of $G(M/K)$ are closed. M/K is galois if $G(M/K)' = K$. Fundamental theorem. Artin's theorem: If M is a field and G is a finite group of automorphisms of M, then M is galois over G'. Extension of isomorphisms theorem. Applications to elementary symmetric functions. Galois subfields and normal subgroups.

4. Construction of galois extension fields. Splitting fields, several characterizations. Uniqueness. Separability. Galois if and only if separable and splitting. Galois closure of intermediate field. Galois group as a group of permutations of the roots. Examples of splitting fields. Explicit calculations of galois groups of equations. Roots of unity. Cyclotomic polynomials. Irreducibility over the rationals. Construction of regular polygons by ruler and compass.

5. Solution of equations by radicals. Definition of radical extension fields. In characteristic 0, if M/K is radical, then $G(M/K)$ is solvable. Tie-up between radical extensions and solving equations by radicals. Insolvability of general equations of degree $\geqq 5$. If f is irreducible over Q, of prime degree p, and has exactly 2 real roots, then its galois group is S_p. Explicit examples. Hilbert's Theorem 90. Form of cyclic extension if ground field contains roots of unity. If $G(M/K)$ is solvable, then M/K is radical.

6. Finite fields. Recall $GF(p)$. A field has p^n elements if and only if it is the splitting field of $x^{p^n} - x$. $M \supset K$, finite fields, implies M is galois and cyclic. Examples from elementary number theory. The normal basis theorem.

Optional Topics

The following topics are listed with no preferential order. They are to be used at the instructor's discretion.

7. Simple extensions and separability. A finite-dimensional extension field is simple if there are only finitely many intermediate fields. M separable and finite-dimensional over K implies M is simple. Purely inseparable extensions and elements. Maximal separable and purely inseparable subfields. Splitting fields are

generated by these. Transitivity of separability.

8. Algebraic closure and infinite galois theory. Definition of algebraically closed field. Existence and uniqueness of algebraic closure (use Zorn's lemma). Point out that one cannot get the Fundamental Theorem of Algebra this way. Infinite algebraic extensions, Krull topology on galois group. Galois group is compact and totally disconnected. Inverse limit of finite groups. Fundamental Theorem of Galois Theory for this case.

9. Transcendental extensions. Algebraically independent subsets of field extensions. Purely transcendental extensions. Transcendental extensions. Transcendence bases treated so that the proof could be used for bases of vector spaces. Usual properties of transcendence bases and transcendence degree. Transcendence degree of composite. Separable generation. MacLane's criterion.

W. Ring Theory and Multilinear Algebra

1. Categories and functors. Introduce the category of sets. Definition of a category. Examples of categories: the category of groups, the category of rings, the category of fields, the category of vector spaces, the category of modules; epimorphisms, monomorphisms, isomorphisms, surjections, injections. Examples to show that an epimorphism is not necessarily surjective and a monomorphism is not necessarily injective. A group as a one-object category whose morphisms are all isomorphisms; similar ways of looking at groupoids and other algebraic systems. Dual of a category, duality, examples. Additive and abelian categories with examples. Functors and natural transformations with many examples, for instance viewing modules as functors. The Yoneda lemma: Nat(Hom(A,-),T) = T(A). Illustrations and examples of universal objects. Definition and elementary properties of adjoint functors. (The language of categories will be useful throughout the course and elementary categorical notions can simplify many proofs.)

2. Introduction to algebraic number theory. Noetherian rings

and their modules. The Hilbert Basis Theorem. Definition of integral elements. Integral closure. Integers in a number field. Examples of quadratic fields. Units. The integers of $Q(i)$, $Q(\omega)$ form a UFD, but the integers in $Q(\sqrt{-5})$, $Q(\sqrt{10})$ do not form a UFD. Fermat's last theorem for $p = 3$ using $Q(\omega)$.

3. Valuation and Dedekind rings. Definition of a discrete valuation ring as a PID with exactly one nonzero prime ideal. Valuation of quotient field associated with discrete valuation ring and converse. Examples of rank one discrete valuations. Various characterizations of discrete valuation rings including: R is a discrete valuation ring if it is a Noetherian domain which is integrally closed and has exactly one nonzero prime ideal. The ring of fractions of a domain with respect to a multiplicative semigroup. R_p for p a prime ideal. A Dedekind ring is a ring R such that R_p is a discrete valuation ring for all prime ideals p of R. Unique factorization of ideals in Dedekind rings. Other characterizations of Dedekind rings. Approximation lemma. If $M \supset K$ are fields with M finite-dimensional and separable over K, and if K is the field of quotients of a Dedekind ring A, then the integral closure of A in M is Dedekind. Integers of a number field are Dedekind.

4. Tensor products. Definition of one-sided module over a ring R. Examples. Free modules. Submodules, quotient modules, exact sequences. Tensor products defined via universal properties. Uniqueness. Existence. Examples, $Z/2Z \otimes Z/3Z = 0$. If R is commutative, tensor product is again an R-module. Tensor product of maps. Behavior of tensor products with regard to exact sequences and direct sums. Examples. Tensor products of free modules and matrix rings. Associativity of tensor product. Tensor product of n modules over a commutative ring, multilinear maps. Tensors. Tensor product of p copies of a free module and q copies of its dual, components in notation of physics. Tensors as defined in physics: R is the ring of C^∞ functions on R^n and M is the R-module of derivations of R. M is free, generated by the usual partials. Transformation of coordinates. Express elements of $M^p \otimes_R M^{*q}$ in terms of two coordinate systems to get usual transformation rules. The tensor algebra and its universal property.

5. Exterior algebra. Multilinear alternating maps. Antisymmetric maps. Definition of the exterior algebra as a homomorphic image of the tensor algebra. Universal property of the exterior algebra. p-vectors. Exterior algebra of a free module over a commutative ring, explicit calculation of a basis and dimension of the module of p-vectors. Prove invariance of vector-space dimension once more. Determinants via exterior algebra. Usual formula for determinant, determinant of transpose = determinant.

6. Structure theory of noncommutative rings. Ring means ring with unit. Simple left module is ring modulo a maximal left ideal. Primitive ideals. Division rings and vector spaces over them. The ring of all linear transformations, both finite- and infinite-dimensional case. Schur's lemma. Density theorem. Wedderburn-Artin theorem. Uniqueness of simple modules. Structure of semisimple Artinian rings. Structure of semisimple modules.

7. Finite group representations. The group algebra. Maschke's theorem: The group algebra of a group of order n over a field of characteristic prime to n is semisimple. Representations and characters. Connection between the decomposition of the group algebra over the complex field and the simple representations. The characters determine the representation.

Optional Topics

The following topics are listed with no preferential order. They are to be used at the teacher's discretion.

8. Radicals of noncommutative rings. Radical = intersection of all primitive ideals = intersection of all left maximal ideals. Equivalent definition of radical. Examples. Behavior of radical under homomorphisms and subring formation. Nakayama's lemma. Artinian rings. Radical is nilpotent in Artinian ring. A ring modulo its radical is a subdirect sum of primitive rings. Connection with semisimple rings.

9. Further group theory. Permutation groups. Linear groups. Structure theory of linear groups. Examples of finite simple groups. Groups defined by generators and relations. Further work on representations of finite groups: one-dimensional representations, the

number of simple characters, orthogonality relations, applications, and examples.

10. Further algebraic number theory. Infinite primes. The product formula. The Dirichlet unit theorem and finiteness of the class number.

Applications

It is essential that the prospective teacher of college mathematics know and appreciate some of the honest applications of the calculus, linear algebra, and probability. Merely as a matter of expediency a teacher of these subjects will have need of convincing examples and illustrations; but, more important, a knowledge of some applications will enable him to know best how to present mathematics and will add an extra dimension to his exposition.

There are many different ways in which the prospective teacher can acquire a background in applications of mathematics. We list here several possibilities which seem highly appropriate; each requires at least one year of course work.

1. Probability and Statistics. Some students will wish to pursue the study of probability and statistics at the advanced undergraduate or graduate level. The year-long course Mathematics 7 [page 79] will serve our purpose well, provided that due emphasis is placed upon applications of these subjects.

2. Differential Equations--Applications. In the pages that follow, three one-semester courses at the advanced undergraduate or beginning graduate level are described:

X. Advanced Ordinary Differential Equations with Applications

Y. Problem-oriented Numerical Analysis

Z. Seminar in Applications

As a source of material in applied mathematics, perhaps no subject is richer than differential equations. Hence, our alternative recommendations for a year's study in applications begin with course X.

A second semester can be chosen from several possibilities. Perhaps the best is the course Mathematics 10 [page 92], using one of the three outlines given in the 1972 CUPM publication

Applied Mathematics in the Undergraduate Curriculum [page 705]. Course Y, if taught in the proper manner, will also be suitable for this purpose. Another alternative for this second semester would be for the mathematics department to offer a seminar (course Z) presenting applications of the calculus, linear algebra, and probability to the physical, biological, and social sciences.

In summary, the suggested requirement for a year of study in applications of mathematics is one of the four sequences: 7A-7B; X-10; X-Y; X-Z. Because of the demand on students' time, we have been compelled to limit this requirement to one year. Nevertheless, we hope that many students will have to opportunity and interest to elect a third semester from among 7A, 7B, 10, X, Y, Z.

X. Advanced Ordinary Differential Equations with Applications

This course is designed to provide background for teaching the topics in differential equations that occur in the lower-division GCMC courses; to give further exposure to applications via one of the most intensively used classical routes; and to provide a first course for students who may be interested in specializing in this area. Because of the nature of the subject, many different good course outlines are possible, but, in any case, emphasis should be put on efficient ways of obtaining from differential equations useful information about their solutions, as distinguished, say, from methods for finding baroque solution formulas of little practical value.

1. Fundamentals. The vector differential equation $\dot{x} = f(t,x)$; prototypes in physics, biology, control theory, etc. Local existence (without uniqueness) by the Cauchy construction, when f is continuous. Prolongation of solutions and finite escape times. Properties of integral funnels (e.g., Kneser's theorem); extreme solutions when $n = 2$. Jacobian matrix of f locally bounded $\Rightarrow$ Lipschitz condition $\Rightarrow$ uniqueness $\Rightarrow$ continuous dependence on initial values and parameters. Effects of stationarity.

2. Numerical integration. Euler, Runge-Kutta, and other methods; elements of error analysis for these methods. Practical machine computation.

3. Linear equations. Discussion of physical and other real-world models leading to linear equations. Linearization. Structure

of the solution set of the vector equation (*) $\dot{x} = A(t)x + b(t)$; variation of parameters formula; the fundamental matrix. Matrix exponentials; thorough treatment of (*), using Jordan canonical form, when A is constant. Applications in engineering system theory. Floquet's theorem.

4. Sturm-Liouville theory. The two-point boundary value problem for second-order self-adjoint equations and how it arises. Existence of eigenvalues. Comparison, oscillation, and completeness theorems. Orthogonal expansions. Green's function. Applications to diffusion and wave equations. Some special functions.

5. Stability. Liapunov, asymptotic, and orbital stability; uniform properties. Basic theorems of Liapunov's direct method. Extensive treatment of the linear case. Applications in control theory.

6. Phase-plane analysis. Geometric treatment of second-order stationary systems. Classification of simple equilibrium points. Closed orbits and Poincaré-Bendixson theory.

Optional Topics

7. Power series solutions. Classification of isolated singularities of linear equations; formal solutions; Frobenius' method. Asymptotic series.

8. Carathéodory theory. (Prerequisite: Lebesgue integration).

Y. Problem-oriented Numerical Analysis

Although the course we have in mind overlaps with standard courses in numerical analysis in some of its material, it differs fundamentally in spirit from such courses. The traditional course in numerical computation is intended to train the student to be able to compute certain specific quantities, such as the approximate value of definite integrals, roots of polynomial and transcendental equations, or solutions of ordinary differential equations, by applying known algorithms to well-formulated specific numerical problems. Courses in contemporary theoretical numerical analysis have tended to emphasize the technical aspects of specialized topics, such as the theory of approximation, spline interpolation, numerical linear algebra, or discrete variable techniques for differential equations; here the stress is on widely applicable computational techniques, their underlying theory, and the errors arising in their application.

A problem-oriented course in numerical analysis starts with real-life problems (from physics, economics, genetics, etc.), develops mathematical models (often in the form of differential or other types of functional equations), analyzes the models, and develops and applies numerical methods to the models in order to get some answers. The student's knowledge of analysis, linear algebra, or differential equations is called upon in the analysis of the model; techniques of numerical analysis are studied and sifted through in the search for applicable methods; specific numerical computations are performed by the student, using a computer; and, finally, the numerical answers are examined in two ways: by means of a theoretical analysis of the errors inherent in the algorithm and in the machine computation, and by a comparison with the original problem to see whether the "answer" (often a table of values of some unknown functions) is a reasonably good approximation to reality.

It seems clear that text materials for Mathematics Y should include books or journal articles on applications (as a source of real problems) and numerical analysis texts (as a source of numerical methods). Sample topics and associated texts are:

a. Problems in the theory of flight. Here one can find mathematical models and their analyses in works such as Theory of Flight Paths by Angelo Miele (Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962). One can apply to the ensuing systems of differential equations techniques found in Discrete Variable Methods in Ordinary Differential Equations by Peter Henrici (New York, John Wiley and Sons, Inc., 1962). One sample problem on these lines can be found in Section 10.9 of Numerical Methods and Fortran Programming by Daniel D. McCracken and William S. Dorn (New York, John Wiley and Sons, Inc., 1964). Although these authors pull a refined model of a simplified flight problem out of a hat--which the instructor in course Y must not do--they examine at length the implications of the properties of the numerical solutions for the behavior of the physical system and use the flexibility of their computer program to vary parameters and do some interesting mathematical experimentation.

b. Control theory. Selected models and analyses from an applied text such as Optimum Systems Control by Andrew P. Sage (Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1968) can lead to problems of numerical solution of partial differential equations, two-point boundary value problems, and problems of numerical linear algebra. There are several suitable sources for numerical methods.

Z. Seminar in Applications

As another approach to applications, we suggest a seminar devoted to applications of the calculus, linear algebra, and probability to the physical, biological, and social sciences. Fortunately, there are now several books which contain a wealth of readily accessible examples. Many of these are referenced in the 1972 CUPM report Applied Mathematics in the Undergraduate Curriculum.

The students would participate in the formulation of scientific problems in mathematical terms and in the interpretation and evaluation of the mathematical analysis of the resulting models. Due emphasis should be given to problems whose analysis rests on the use of the computer. It might be appropriate for the instructor to invite guests who could expose the student to the attitudes of users of mathematics. While such an arrangement would, perhaps, not be a traditional course in applied mathematics, it would allow the students to come into contact with a variety of serious applications of the usual mathematics of the first two undergraduate years. The following illustrate the type of examples we have in mind:

a. The formulation and analysis of a system of differential equations which serves as a model for (i) the interdependence of two species, one of which serves as food for the other, or (ii) a time-optimal navigation problem which requires that a boat be transferred from a given initial position to a given terminal position in minimal time.

b. The formulation and analysis of waiting line and traffic problems involving simple calculus and probability.

c. Elementary matrix analysis associated with chemical mixture problems and mechanical equilibrium problems; matrix eigenvalue problems arising from electrical circuit analysis.

d. The "transportation problem" of making optimal use of a given shipping network to obtain a specified redistribution of commodities. This is, of course, a special case of linear programming.

e. Game theory as applied to games of timing ("duels") in which rewards to competing strategists depend on when certain acts are performed.

APPRENTICESHIP IN TEACHING

Every mathematician is a teacher in the sense that he must explain mathematical ideas to other people--to students, to colleagues, or to the mathematical community at large. For this reason the graduate education of every mathematics student should include a program designed to develop skill in oral and written communication of mathematics. This program should begin as soon as the student enters graduate school and continue at increasing levels of responsibility.

Ultimately, the attitude of the graduate faculty will determine the success of any such program. If effective teaching is regarded as an important and nontrivial function of the department, and if senior mathematicians encourage excellent exposition by precept and personal interest, graduate students and younger faculty will respond accordingly. Every instructor of a graduate class should realize that his course can have a profound effect upon his students in the way it serves to strengthen the attributes of a good teacher.

Because the conditions of undergraduate and graduate instruction vary widely from one university to another, the suggestions given below offer a variety of ways in which the mathematics departments might stimulate more interest in good teaching. Each university is encouraged to create its program individually, seeking to establish an intellectual environment in which teaching and learning flourish together.

Some universities have experimented recently with special programs which bring new teaching assistants to the campus before the start of classes in the fall. Sessions are devoted to a general orientation to graduate and undergraduate study at that university and to the role of the graduate assistant. At least one program runs for the entire summer term and includes an initial involvement with graduate mathematics besides activities in preparation for teaching.

During the first stage of his training, the teaching assistant should be given limited duties, but he should be made to feel that he is a junior colleague in a profession rather than a hired hand in a work crew. At a pace which is adjusted individually to his rate of development, he should progress through a sequence of teaching assignments, acquiring more responsibility and independence as he gains in experience and confidence. He can mark homework papers, conduct office hours for undergraduates, prepare questions for tests, and assist in marking tests.

A prospective teacher can learn much by observing a skillful teacher in an undergraduate class in a subject familiar to the apprentice. This is of particular value in a class of selected students, such as freshman or sophomore honors sections, where the interchange between students and the instructor is lively and challenging.

Regular consultation between an apprentice and his supervisor is essential. Each course supervisor should arrange meetings of all assistants for that course; at these meetings there should be free exchange of ideas concerning problems of instruction, alternate suggestions for presenting specific concepts, proposals for future test questions, and planning the development of the course. In addition to formal consultation, however, supervisors should maintain a running dialogue with apprentices, work with them in the marking of tests, and cooperate in performing with them the day-to-day duties which are an integral part of teaching.

After a graduate student has developed competence in these duties and has acquired a basic feeling for classroom instruction, he should be drawn more actively into teaching by conducting discussion sections, by giving occasional class lectures, or by accepting major responsibility for teaching an appropriate course at an appropriate level. His supervisor should maintain good contact through continued consultation, classroom visitation, and informal discussions. As the assistant matures in his teaching role, direct supervision should be relaxed gradually to encourage him to develop his individual classroom style and techniques; the opportunity for consultation should remain open, but the initiative should pass from the supervisor to the assistant.

Special seminars can also be used to assist students to improve their exposition. Many departments require a proseminar in which graduate students present advanced mathematical topics to fellow students and several members of the faculty. It would be equally appropriate to require each first-year graduate student to present a short series of talks on some phase of undergraduate mathematics which is outside his previous course of study. The objective should be to present the topic at a level suitable for undergraduates, emphasizing clarity in organization and expression rather than making the occasion a mathematical "tour de force."

Another possibility is to assign a few graduate students to experimental projects in undergraduate mathematical instruction instead of assigning them regular classroom duties. For example, they could help to prepare a collection of classroom examples for a calculus course, develop problems to be solved on the computer, or plan and evaluate alternative approaches to specific topics in lower-division undergraduate mathematics.

As indicated in the Program Description, the apprenticeship in teaching should constitute approximately one fourth of the total work load of a student during his first graduate component. It is conceivable that some of these activities, such as seminars, can qualify for academic credit. But whether or not academic credit is granted for this phase of graduate work, the student's performance as a teacher should be evaluated, and an informal departmental record should be kept in sufficient detail to show the work done and the level of competence attained.

Finally, any program of increased attention to the teaching role of prospective mathematicians has budgetary implications which cannot be ignored. One additional cost is for increased faculty time devoted to supervising teaching assistants. Another is for stipends for graduate students if the number of apprentice teachers is expanded. But if the quality of mathematics instruction improves in the future as a result of such efforts, the money will have been well spent. Fortunately, there is reason to believe that imaginative proposals to improve the quality of teaching by graduate students can attract the additional financial support needed to make them effective.

Although many graduate students welcome an opportunity to teach and thereby to become self-supporting, the stipend itself is not an adequate incentive for good teaching. This incentive can best be provided by the persistent concern of established mathematicians that teaching be excellent throughout the department.

QUALIFICATIONS FOR TEACHING UNIVERSITY-PARALLEL MATHEMATICS COURSES IN TWO-YEAR COLLEGES

A report of

The *ad hoc* Committee on Qualifications for a Two-Year College Faculty in Mathematics

August 1969

INTRODUCTION

CUPM has published reports on the qualifications needed by teachers of the GCMC curriculum and on the adaptation of that curriculum to the circumstances of university-parallel programs in two-year colleges (Qualifications for a College Faculty in Mathematics (1967) and A Transfer Curriculum in Mathematics for Two-Year Colleges (1969)). The present report is an effort to describe the qualifications desirable for faculty members teaching courses in the university-parallel or transfer programs in two-year colleges.

Our comments and recommendations are addressed to administrators of two-year colleges, to university mathematics departments, to mathematics teachers in two-year colleges, and to those contemplating careers as mathematics teachers in two-year colleges. The concluding section of our report offers specific advice to each of these four groups.

We discuss the qualifications of teachers of the following set of courses, whose subject matter can be thought of as a working definition of university-parallel mathematics.

Mathematics 0. Elementary Functions and Coordinate Geometry. A one-semester course in coordinate geometry and the properties of the elementary functions.

Mathematics A. Elementary Functions and Coordinate Geometry (with Algebra and Trigonometry). A more slowly paced version of Mathematics 0 in which are embedded some topics from high school algebra and trigonometry. This course is to be thought of as extending over more than one semester.

Mathematics B. Introductory Calculus. An intuitive one-semester course covering the basic concepts of single-variable calculus.

Mathematics C. Mathematical Analysis. A two-semester course completing the study of elementary calculus.

Mathematics L. Linear Algebra. A sophomore-level one-semester introduction.

Mathematics PS. Probability and Statistics. An elementary one-semester course (not having calculus as a prerequisite) suitable for students in business and social sciences.

Mathematics NS. The Structure of the Number System. A two-semester course recommended by the CUPM Panel on Teacher Training for beginning the preparation of elementary school teachers. [Since this report was written, the recommendations on teacher training have been revised. See the 1971 publication Recommendations on Course Content for the Training of Teachers of Mathematics, page 158.]

The detailed discussion of these courses will be found in the CUPM report A Transfer Curriculum in Mathematics for Two-Year Colleges, mentioned above. In addition, this report suggests that, under certain circumstances, it may be advisable for a two-year college to offer additional courses and suggests a selection from among the following: further courses for elementary school teachers; finite mathematics; a calculus-based course in probability; numerical analysis and intermediate differential equations (or differential equations with topics from advanced calculus). For suggested work in computing, see the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics, page 563.

Our recommendations are intended to apply to all instructors who teach any such university-parallel courses. We are aware of the great importance in two-year colleges of courses in mathematics for students in occupational and technical curricula and of courses designed for students lacking even basic mathematical skills. We are also aware of the existence of difficult and challenging pedagogical and curricular questions related to such courses. We have chosen to wait until there is a better resolution of these questions before seeking to formulate recommendations about the proper qualifications for teaching courses that are not parallel to those commonly offered by four-year colleges and universities. [Some of these questions are discussed in A Course in Basic Mathematics for Colleges.]

The university-parallel role of the two-year college is of increasing importance in the educational system. For example, in California 86% of all freshmen in publicly supported institutions in 1966-67 were in two-year colleges. The percentage of college students enrolled in two-year colleges has been increasing rapidly nationwide. There are already a number of universities in which the junior class is larger than the freshman class. Moreover, a majority of students entering two-year colleges intend to continue their education at least to the bachelor's degree. Thus, university mathematics departments must recognize that the university-parallel courses taught in two-year colleges are becoming an integral part of the university program in mathematics.

Conversely, recent trends in four-year institutions are placing new demands on teaching of mathematics in two-year colleges. As students come to colleges with better preparation in mathematics, many courses are moving downward toward the freshman year. It should be recognized that those now being trained or hired as teachers in two-year colleges must be prepared to deal at some time in the near future with subjects that are now thought of as belonging to the junior or senior years.

The degree most commonly held by mathematics teachers in two-year colleges is the master's degree, but this degree is of such varying quality that it is scarcely useful as a measure of qualification for appointment, promotion, or tenure. We feel it necessary to make recommendations which are independent of degrees held or of total credit hours earned in mathematics courses but which deal, rather, with the substance of the mathematical training of prospective faculty members.

It should be understood that no academic program or degree in itself qualifies an individual to teach effectively at any level unless this preparation is accompanied by a genuine interest in teaching and by professional activities reflecting continuing mathematical growth. These activities may assume many forms:

a. taking additional course work

b. reading and studying to keep aware of new developments and to explore new fields

c. engaging in research for new mathematical results (even if unpublished)

d. developing new courses, new ways of teaching, and new classroom material

e. publishing expository or research articles

f. participating in the activities of professional mathematical organizations

This list reflects our conviction that an effective teacher must maintain an active interest in the communication of ideas and have a dedication to studying, learning, and understanding mathematics at levels significantly beyond those at which he is teaching.

A two-year college mathematics department, whose staff members are engaged in activities such as those described above and have the academic qualifications to be described below, should have confidence in its ability to provide the quality of teaching required of it.

THE FORMAL EDUCATION OF MATHEMATICS TEACHERS IN TWO-YEAR COLLEGES

The university-parallel courses in mathematics that a teacher in a two-year college should be able to teach effectively have been described in the previous section. We shall now set forth our recommendations for the mathematical qualifications for the teachers of these courses.

This mathematical background falls into two distinct components: a basic component which consists of a strong mathematics major program and a graduate component which embodies the requirement that a teacher at a two-year college must have a knowledge of mathematics well beyond that which he will be asked to teach.

Basic Component

The basic component of mathematics courses for the two-year college teacher is most succinctly described as a solid grounding in analysis and algebra, with additional courses in geometry, computer science, and probability providing greater breadth of knowledge.

We assume that the prospective teacher has mastered the following lower-division undergraduate material, as described in the CUPM publication Commentary on A General Curriculum in Mathematics for Colleges.

a. Calculus courses in one and several variables, including an introduction to differential equations (Mathematics 1, 2, 4)

b. The fundamentals of computer science, including experience in programming as well as the use of a computer [See, for instance, the course C1, page 563]

c. A semester course in linear algebra employing both matrices and a basis-free, linear transformation approach (Mathematics 3)

In addition, the prospective teacher should attempt to obtain as many of the following upper-division courses as he can at the undergraduate level.

a. A semester course in advanced multivariable calculus, covering differential and integral vector calculus, including the theorems of Green and Stokes, and an introduction to Fourier series and boundary value problems (Mathematics 5, first version)

b. A year's work in abstract algebra, treating the important algebraic systems (groups, rings, modules, vector spaces, and fields) and thoroughly developing the basic concepts of homomorphism, kernel, and quotient construction, with applications and consequences of these ideas. (This course is described in the CUPM report Preparation for Graduate Study in Mathematics, page 453. See also the courses Mathematics 6M and 6L in Commentary on A General Curriculum in Mathematics for Colleges, page 65 .)

c. A thorough year's course dealing with the important theorems in real analysis, with emphasis on rigor and detailed proofs. The treatment should use metric space notions and should lead to a detailed examination of the Riemann-Stieltjes integral. (Mathematics 11-12)

d. A semester course in complex analysis, covering Cauchy's theorem, Taylor and Laurent expansions, the calculus of residues, and analytic continuation, with application of these ideas to transforms and boundary value problems (Mathematics 13)

e. A semester course in applied mathematics. The student should be introduced to applications of mathematics in order that his teaching might better reflect the relevance of mathematical ideas. [The courses described in the 1972 publication <u>Applied Mathematics in the Undergraduate Curriculum</u> are suitable.]

f. A semester course in which the student studies some geometric subject such as topology, convexity, affine and projective geometries, differential geometry, or a comparative investigation of Euclidean and non-Euclidean geometries (Mathematics 9)

g. A year's course in probability and statistics that reflects the growing importance of this subject to engineering and the biological, social, and management sciences [Mathematics 7 or the course outlined in <u>Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists</u>, page 642]

If the student has not completed all of the upper-division courses of this strong mathematics major as an undergraduate, then he should cover material comparable to that of the omitted courses during his graduate training.

Graduate Component

Graduate (one-semester) courses which are especially appropriate for the graduate component are:

P Measure and Integration

Q Functional Analysis

R Complex Analysis

S General Topology

T Homology and Multivariable Integration

U Topology and Geometry of Manifolds

V Galois and Field Theory

W Ring Theory and Multilinear Algebra

X Advanced Ordinary Differential Equations with Applications

Y Problem-oriented Numerical Analysis

Z Seminar in Applications

Of these, P, S, and X should be in the program of every prospective two-year college teacher.

Detailed descriptions of these courses are given in the CUPM report A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates, page 113. The program presented there is designed to prepare teachers to function in the first two years of a four-year college with occasional teaching of upper-division courses. It does not differ greatly from our program: In the four-year college report, instead of P, S, and X, the courses P, Q, S, and T are regarded as essential, and the material on probability and statistics, applied mathematics, and differential equations serves as a pool of courses on the applications of mathematics from which a year sequence is to be elected by the student. These differences are due to the fact that a two-year college teacher has less access to the services of experts in specific fields and, consequently, needs a somewhat broader training.

It should be emphasized that course X in differential equations is not a second undergraduate course in the subject, but is to be a genuine graduate course at least on the same level as the course in measure and integration. The graduate course in applied mathematics which the Committee most strongly favors is one (not yet commonly offered) which stresses the formulation and analysis of mathematical models in diverse fields, using the calculus, probability, and linear algebra of the first two undergraduate years. Course Z is of this type.

Students who plan to continue into advanced graduate work and to specialize in some area of pure mathematics are advised to take as many as possible of the other courses in the list. Other students may substitute electives to obtain a deeper knowledge of some other area of mathematics or computer science.

Undergraduate mathematics, especially in the lower division, is heavily slanted toward real analysis; courses in general topology and measure theory provide essential background for teaching courses in calculus and probability. If further work in analysis is elected, we recommend a study of functional analysis (course Q) in preference to complex analysis (course R) as a sequel to measure theory, as the former will further develop the ideas of linear algebra and the concept of uniform convergence. In spite of its importance for more advanced work in pure mathematics, a second year of abstract algebra to follow the strong undergraduate algebra course described in the basic component is not recommended as essential for teachers of mathematics in a two-year college.

The graduate component of courses should be augmented by two particular features to prepare the prospective teacher for the two-year college mathematics faculty.

First, a year's work focused on the problems of lower-division undergraduate teaching, such as an apprenticeship in teaching as described below, preferably carried out at a nearby two-year college.

Second, a comprehensive examination designed specifically to test the breadth and depth of a candidate's understanding of mathematics relevant to the undergraduate curriculum.

A student who has a strong undergraduate major in mathematics will be able to complete the program in one year, even if he has not completed all of the courses listed in the basic component. For a student whose prior training is not substantially that of the basic component, the completion of the graduate component may require two years of study beyond his bachelor's degree. For example, if his undergraduate major does not include strong preparation in algebra and analysis, his program might be as follows:

First Year (both semesters)
Abstract Algebra
Real Analysis (Mathematics 11-12)
Probability and Statistics
Apprenticeship in Teaching

Second Year
<u>First Semester</u>
Measure and Integration (P)
Complex Analysis (Mathematics 13)
Topology (S)
Apprenticeship in Teaching

<u>Second Semester</u>
Applied Mathematics (Z or one of the courses outlined in <u>Applied Mathematics in the Undergraduate Curriculum</u>)
Advanced Differential Equations (X)
Apprenticeship in Teaching
Comprehensive Examination

The mathematical background in the graduate component, if satisfactorily completed, will permit the student to teach with confidence the university-parallel courses of the two-year college. Moreover, new courses, as they arise, should be well within his competence to prepare.

<u>Apprenticeship in Teaching</u>

An important component in the training of teachers of mathematics for two-year colleges is an understanding of the teaching and learning processes as they apply to these institutions. One of the best ways for the potential instructor to gain this kind of knowledge and experience is through a supervised teaching activity. This activity preferably should take place in a two-year college, but it can, if necessary, be carried out in a four-year institution in appropriate courses.

Most value will be obtained if apprentice teachers receive experience in a variety of courses involving a heterogeneous group of students with differing career aspirations, comparable to the situation that they will encounter in most two-year colleges.

The success of an apprenticeship program will depend significantly upon the attitude of the graduate faculty. If effective teaching is regarded as an important function of the department, and if senior mathematicians encourage excellence in teaching by precept and by example, the apprentice teachers will respond accordingly.

The work assignment of the apprentice should be carefully graduated and should always involve close contact with and supervision by a senior colleague. The apprentice should have frequent opportunities to go over purposes, methods, and content with his supervisor. Arrangements should be made for frequent post-teaching conferences in which the teaching and learning problems encountered are reviewed and solutions suggested. This can be done individually or in a group for all apprentices in the program. Valuable contributions can be made to such seminar sessions by mathematics instructors from two-year colleges and by experts in curricular construction and evaluation.

In total, the apprenticeship in teaching should constitute approximately one quarter of the work load of the student during his graduate experience.

Adequate budgetary provisions should be made for the extra burden of the apprenticeship program on the senior mathematicians, as well as financial support for the apprentices.

An apprenticeship system has a great potential for preparing two-year college mathematics teachers having a real attachment to the discipline and an understanding of the values and the rewards of the teaching profession. Done poorly, it will discourage candidates from the field. Done well, it will attract and retain competent and interested persons.

COMPOSITION OF A TWO-YEAR COLLEGE MATHEMATICS FACULTY

Although mathematics teachers at two-year colleges are called upon to teach specialized courses for a variety of students (remedial, general education, technical-occupational), our attention in the present report continues to be focused upon qualifications of persons who teach courses in the university-parallel curriculum.

It is our recommendation that all teachers of university-parallel courses at a two-year college have the mathematical preparation equivalent to our graduate component. Although the university-parallel courses that a two-year college teacher may be called upon to offer today are principally like those described in the introduction under 0, A, B, and C, it seems reasonable to expect that courses in finite mathematics, linear algebra, probability and statistics, and mathematics for prospective elementary school teachers will be standard offerings in two-year colleges in the near future. Our recommendation reflects a belief that the teacher of university-parallel courses should have mathematical training well beyond the course he is teaching. Moreover, the mathematical background we recommend will permit all faculty members to participate in knowledgeable discussions of curricular changes, both internally and with faculty members of four-year colleges and universities. The mathematical preparation we recommend will permit a faculty member to prepare new courses with confidence. Moreover, it will provide the individual faculty member with a basis for effective participation in mathematical organizations, which in turn will help him to maintain the intellectual curiosity and interest in mathematics that is essential to a successful mathematics teacher.

The Committee believes that a universally well-qualified faculty for the university-parallel courses is most important, with each member able to make a contribution in all of the ways already indicated. We do not, however, envision that all two-year college staff members will have exactly the same mathematical background. The choice available for individual preferences in the graduate component allows a staff which includes people with varying interests, and hence people especially well prepared to teach linear algebra or probability and statistics or computer science.

A two-year college may not be able at this time to recruit all such staff members from candidates with preparation equivalent to our recommended graduate component. In this case, they might seek on a temporary basis new candidates who have the mathematical preparation equivalent to our basic component; these candidates could be assigned to teach courses 0, A, and B. New faculty members whose qualifications are not equivalent to our graduate component should be required to augment their mathematical background so that in time they will be better prepared to have responsibility for any of the university-parallel courses.

RECOMMENDATIONS TO FOUR GROUPS

a. To Administrators of Two-Year Colleges

These recommendations are addressed to those who appoint and promote two-year college faculty members and to those who, through accreditation and certification, influence the setting of qualifications for such teachers.

Although it has been traditional for college policies on the appointment, promotion, and tenure of faculty to include certain degree requirements, it is a fact that the course requirements for a particular degree in mathematics vary considerably from one institution to another; even minimum standards are not well defined. This is especially true of the master's degree. The Committee strongly encourages those concerned to note that this report recommends a set of courses which prospective members of a mathematics faculty should have taken. Successful completion of these courses should insure that the faculty member is adequately prepared, in terms of subject matter, to teach university-parallel courses.

The Committee urges all administrators to recognize proficiency in the content of the courses recommended in this report rather than academic degrees as a basis for faculty appointments and advancement. For example, graduate mathematics training of secondary school teachers, customarily and properly, differs from the training we have described. The Committee suggests that faculty members in mathematics be relied upon to determine the degree of proficiency possessed by those under consideration. Furthermore, it is recommended that orientation programs be developed for new faculty members who have had no previous experience teaching in two-year colleges.

b. To University Mathematics Departments

University mathematics departments should realize from the preceding sections that the major role in the training of mathematics instructors for two-year colleges is theirs. They must accept responsibility for establishing formal programs for the training of new instructors for two-year colleges and for retraining instructors who are now teaching in these institutions.

We believe that this can be done within existing frameworks of mathematics departments whose course offerings approximate in depth the detailed outlines to which we have referred. For such departments, this will not require extensive changes in curricula, except possibly for the introduction of a program for apprenticeship in teaching. However, it is necessary that the mathematics faculty be fully aware of the particular complexion, problems, and status of two-year colleges throughout the country. The mobility of instructors suggests the need for a national point of view. Moreover, in order to fulfill their responsibility, the university faculty must recognize and respect the basic role of two-year colleges and be mindful of the problems that will be faced by mathematics instructors in two-year colleges.

c. To Those Currently Teaching Mathematics in Two-Year Colleges

All college teachers of mathematics, at one time or another, find it necessary to supplement their own mathematical training. Rapid changes are taking place in college mathematics. Hence, increasing numbers of college teachers are continuing their mathematical development by individual study and additional formal course work in mathematics.

Teachers of mathematics in two-year colleges should find that the course outlines referred to in this report provide useful guidelines for individual study, faculty seminars, and additional course work.

To serve the mathematical needs of two-year college students, a faculty member must maintain an awareness of contemporary curricula in both secondary schools and four-year colleges. He will find that the recommended courses provide a basis for effective communication with staff members of mathematics departments of four-year colleges. Personal knowledge of mathematics courses at four-year colleges is needed in order to be aware of the demands that will be made upon students after they transfer. This knowledge and the recommended strong preparation in mathematics make possible the necessary continuous evaluation and development of mathematics courses in two-year colleges.

d. To Prospective Teachers of Mathematics in Two-Year Colleges

The two-year college teacher of university-parallel mathematics courses has the responsibility for training students with a wide variety of goals. Some could be mathematicians, some scientists or engineers; there are others who will use mathematics in economics, psychology, or other social sciences. One who intends to teach mathematics in a two-year college could well use our description of a program of mathematics courses and apprenticeship in teaching as a guide in planning his own graduate work. He should also be aware that the program we have outlined is substantially different both in nature and extent from what we would regard as an optimal graduate program for teachers in secondary schools.

RECOMMENDATIONS ON COURSE CONTENT
FOR THE TRAINING OF TEACHERS OF MATHEMATICS

A Report of

The Panel on Teacher Training

August 1971

TABLE OF CONTENTS

THE PURPOSE OF THIS REPORT

This report presents an outline of the Panel's recommendations for the minimal college preparation for teachers of school mathematics based upon its assessment of those significant changes that have taken place or can be expected to take place in school curricula during the 1970's.

The nature of school mathematics is, of course, far from static, and the forces for change are many. The past 25 years have produced a phonomenal increase in the quantity of known mathematics, as well as in the variety and depth of its applications. This growth has been reflected in our total culture, which has become increasingly mathematical, a trend which is certain to continue. It is inevitable and proper that these changes will be reflected in the content of school mathematics, as well as in the way it is taught.

Thus, in the past ten years we have seen a flurry of activity directed toward improving the mathematics curricula in our schools. The pace of change alone demands that those engaged in such activity periodically review their efforts. A decade seems to be an appropriate period for such a review.

It is reasonable to ask what specific changes in mathematics and mathematics education during the past decade impel us to modify our previous recommendations.

The dependence of western civilization on technology has long been evident, and it has been recognized that mathematics supports the physical and engineering sciences upon which technology thrives. More recently new applications of mathematics to the biological, environmental, and social sciences have developed. Statistics and probability have emerged as important tools in these applications. Indeed, as ordinary citizens we frequently encounter surveys and predictions that make use of probability and statistics, so that an intelligent existence demands our understanding of statistical methods. Consequently, many secondary schools have begun to teach probability and statistics and, as we shall show, there are compelling reasons to teach these subjects in the elementary grades.

But aside from the specific mathematics, e.g., statistics, which is brought to bear upon applications, the applications are interesting in themselves, so that it is pedagogically sound to incorporate them in the mathematics program. Thus, our new recommendations emphasize the applications of mathematics.

Many of these new applications have been aided, perhaps even made possible, by modern computers, and the teaching of computer science and computational mathematics are becoming commonplace in our colleges. Computer programming is now a part of many junior and senior high school mathematics programs, and it has been discovered that the notion of a flowchart for describing an algorithmic process is a useful pedagogical device in the teaching of elementary mathematics, as well as an efficient device for prescribing a computer program. Thus, computers are influencing mathematics education at all levels, and we have attempted in preparing this report to assess this influence and to recommend measures for increasing it.

It is a fact that change induces change. For instance, an important aspect of curricular change over the past decade has been emphasis on the understanding of mathematical concepts. As a result, we have learned that abstract concepts can be assimilated at a much earlier age than was previously thought possible. Thus, we are less reluctant today to suggest that elementary notions of probability may be useful in explaining ideas about sets and rational numbers than we were a decade ago to suggest that elementary ideas about sets might be useful in helping children to understand the process of counting. Furthermore, the curricular revisions of the past decade have led to improved training programs which have produced elementary school teachers who are more confident about presenting mathematical topics. It is our purpose in this report to take advantage of today's teachers' new attitudes and skills in order to meet new challenges.

Finally, our recommendations are intended to reflect improved preparation over the past decade of entering college freshmean.

THE OBJECTIVES OF TEACHER TRAINING

The Panel believes that the following objectives of mathematical training are important:

1. Understanding of the concepts, structure, and style of mathematics

2. Facility with its applications

3. Ability to solve mathematical problems

4. Development of computational skills

These statements deserve amplification.

It is our belief that the disciplined, rational man has the best chance of becoming independent, mature, and creative, and that the development of these qualities is a lifelong process. The intellectual discipline of mathematics contributes in a unique way to this development. We identify two reasons why this is so. First, mathematical concepts are necessarily rooted in man's awareness of the physical world. Understanding mathematics allows him to relate more efficiently to his environment. Second, a person's understanding of a concept depends upon its meaningful relation to and firm grounding in his personal experience, as well as upon his awareness of its role in a system of interrelated ideas. As he learns to relate concepts to one another in an orderly fashion he becomes better organized and he improves his ability to abstract and to generalize, that is, to recognize a concept in a variety of specific examples and to apply this concept in differing contexts. We believe that the study of mathematics can directly benefit this process of personal organization. We therefore regard it as essential that mathematics be taught at all levels in such a way as to emphasize its concepts, structure, and style.

It is possible, of course, to study, to appreciate, and even to practice mathematics by and for itself, but people who can and wish to do this are rare. For most of us an important value of mathematics is its applicability to other scientific disciplines. The recent fruitfulness of mathematics in this regard has already been mentioned, but this is really in the tradition of mathematics, which has repeatedly responded to other disciplines that seek to apply its theories and techniques. It should also be recognized that the sciences in their turn have stimulated the development of new fields of mathematics. Thus, we believe that students of mathematics should acquire an understanding of its wide applicability in various fields, and for this reason, applications should be emphasized in every course.

When we speak of facility with applications we mean the ability to recognize and delineate a mathematical model of a physical, social, biological, or environmental problem. Being able to solve the related mathematical problem is a skill which we also regard as important. Effective mathematics instruction must include development of the ability to attack problems by identifying their mathematical setting and then bringing appropriate mathematical knowledge to bear upon their solution.

Finally, computational skill is essential. Without it the student cannot learn to solve mathematical problems, to apply mathematics, or to appreciate even its simplest concepts and structures. Although we normally think of this skill as including speed and accuracy in applying the common algorithms of arithmetic and algebra, we should keep in mind the fact that it also includes the ability to estimate quickly an approximate result of a computation. Each course should not only provide systematic practice in computation but should also inculcate in the student the skill and habit of estimating.

This report recommends courses that we believe prospective teachers should study in order to help them achieve these objectives, both for themselves and for their future students. First we explain briefly why we believe that teachers need much more mathematical education than most of them are now getting, and why their training needs to be of a special kind in certain cases.

There have been recent improvements in the certification requirements for elementary school teachers, but in our opinion they continue to be inadequate or inappropriate in many cases. Some states require a semester or a year of "college mathematics" without indicating what sort of mathematics this should be. These practices appear to be based on the assumption that little or no special training in mathematics is needed to teach in an elementary school. This assumption has always been unrealistic, and in the present context of rapidly changing and expanding curricula it is wholly untenable.

In some elementary schools the rudiments of algebra, informal geometry, probability, and statistics are already being taught in addition to arithmetic. But even if only arithmetic is taught, the teacher needs sound mathematical training because his understanding affects his views and attitudes; and in the classroom, the views and attitudes of the teacher are crucial. An elementary school teacher needs to have a grasp of mathematics that goes well beyond the content and depth of elementary school curricula.

Similarly, a Level II or III teacher's understanding of mathematics must exceed, both in content and depth, the level at which he teaches. Within the next decade it is to be expected that secondary school teachers will be asked to teach material which many of our present teachers have never studied.

We therefore recommend courses for all teachers which will not only insure that they thoroughly understand the content of the courses they must teach, but will also prepare them to discuss related topics with able and enthusiastic students. These college courses must also prepare teachers to make intelligent judgments about changes in content, pace, and sequence of mathematics programs for their schools, and to have the flexibility of outlook necessary to adjust to the curriculum changes which will surely take place in the course of their professional careers.

THE RECOMMENDATIONS

These recommendations concern only the preparation of teachers of elementary and secondary school mathematics. Whereas in 1961 these teachers were classified into three groups, we find it convenient to use four classifications:

LEVEL I. Teachers of elementary school mathematics (grades K through 6)

LEVEL II-E. Specialist teachers of elementary school mathematics, coordinators of elementary school mathematics, and teachers of middle school or junior high school mathematics (roughly grades 5 through 8)

LEVEL II-J. Teachers of junior high school mathematics (grades 7 through 9)

LEVEL III. Teachers of high school mathematics (grades 7 through 12)

These classifications are to be taken rather loosely, their interpretation depending upon local conditions of school and curricular organization. It will be noted that the various classifications overlap. This is a deliberate attempt to allow for local variations.

The reader should note that the training for Level I teaching is a separate program, while, except for their Level I content, the curricula for the further levels form a cumulative sequence.

The recommendations of this report are not motivated by a desire to meet the demands of any special program of mathematics education or the goals of any particular planning organization. We consider our recommendations to be appropriate for any teachers of school mathematics, including teachers of low achievers.

Level I Recommendations

The applications of mathematics, the influence of computers, and the changes wrought in the 1960's in the teaching of mathematics prompt us to revise our 1961 Recommendations for the Training of Teachers of Mathematics at all four levels. At Level I in particular we are aware that new teaching strategies designed to facilitate and enrich learning are being adopted or are the subject of experimentation. Such strategies impose on elementary school teachers the necessity of a deeper understanding of the school mathematics curriculum than is required by conventional teaching methods and increase the teacher's need for knowledge of mathematics well beyond the level at which topics are treated in the elementary classroom.

We believe that this deeper understanding can be better achieved if mathematics is taught, and understood, from the earliest stages as a unified subject. The function concept, for instance, should serve as a unifying thread in elementary school mathematics, and elementary intuitive geometry should be taught for its connections with arithmetic as well as for its own sake. The applications of mathematics reflect its unity and offer an opportunity to illustrate its power. For instance, the notion of a finite sample space in probability can be used at a very elementary level to illustrate the idea of a set (of outcomes), and the probability of an event can motivate the need for rational numbers. Simple statistical problems yield practice in computing with both integers and rational numbers as well as in applying probability theory to practical situations. Finally, the use of flowcharts helps to explain the elementary algorithms of arithmetic as well as to prepare the student for later study of computer programming.

Thus, while the development of the number system should remain the core of the elementary school curriculum and of the content of Level I courses, there are other crucial topics which ought to be contained in the Level I sequence. We stress this point by listing these topics in the following recommendations.

Recommendations for Prospective Level I Teachers:

> We propose that the traditional subdivision of courses for prospective elementary school teachers into arithmetic, algebra, and geometry be replaced by an integrated sequence of courses in which the essential interrelations of mathematics, as well as its interactions with other fields, are emphasized. We recommend for all such students a 12-semester-hour sequence that includes development of the following: number systems, algebra, geometry, probability, statistics, functions, mathematical systems, and the role of deductive and inductive reasoning. The recommended sequence is based on at least two years of high school mathematics that includes elementary algebra and geometry.
>
> We further recommend that some teachers in each elementary school have Level II-E preparation. Such teachers will add needed strength to the elementary school's program.

Our suggestion of an integrated course sequence represents a very important change which these recommendations are intended to bring about, but there are certain to be questions on how this can be accomplished. We attempt in the sequel to provide our answer to such questions.

There are many ways in which to organize the appropriate material into integrated course sequences, and we encourage experimentation and diversity. Two possible sequences of four 3-semester-hour courses are described in detail in the course guides [page 175]. For reference we list their titles here:

Sequence 1	Sequence 2
1. Number and Geometry with Applications I	1. Number Systems and Their Origins
2. Number and Geometry with Applications II	2. Geometry, Measurement, and Probability
3. Mathematical Systems with Applications I	3. Mathematical Systems
4. Mathematical Systems with Applications II	4. Functions

These sequences differ in the ordering of topics and in the degree of integration, yet both conform to our idea of an integrated course sequence. It may be helpful to discuss briefly, without any attempt at being comprehensive or any desire to be prescriptive, the rationale which led us to these integrated sequences.

The dual role of numbers in counting and measuring is systematically exploited in each of the four-course sequences. In one direction we are led to arithmetic, in the other to geometry. The extension of the number system to include negative numbers may then be explained by reference to both the counting and measuring models. The study of rational numbers may likewise be motivated through measurement and counting, and the arithmetic of the rationals finds justification and natural applications in elementary probability theory. Geometrical considerations lead to vector addition on the line and in the plane--and then in space--and the Pythagorean Theorem leads naturally to irrational numbers.

The arithmetic of decimals can be presented as the mathematization of approximation. Also, the algorithms of elementary arithmetic lead naturally to flowcharts and to a study of the role of computers.

Length, area, and volume have computational, foundational, and group-theoretical aspects. Extensions of ideas from two to three dimensions constitute valuable experience in themselves, are useful in developing spatial intuition, and also help in understanding the nature of generalization in mathematics.

Graphs of functions of various kinds, theoretical and empirical, may be studied, incorporating intuitive notions of connectedness and smoothness.

The function concept plays an important role in an integrated curriculum. Counting, operations on numbers, measurement, geometric transformations, linear equations, and probability provide many examples of functions, and common characteristics of these examples should be noted. Ideas such as composition of functions and inverse functions may then be introduced and illustrated by algebraic and geometric examples. Of course, detailed formal discussion of the function concept should come only after examples of functions have been mentioned in various contexts in which it is useful to do so.

Similarly, the elementary notions of logic such as logical connectives, negation, and the quantifiers should be treated explicitly only after attention has been called informally to their uses in other mathematical contexts. Indirect proofs and the use of counterexamples arise naturally and may be stressed when the structure of the number systems is examined. However, in the final stages of a prospective elementary teacher's training it is useful to return to logic in a more explicit way for the purpose of summarizing the roles of inductive and deductive reasoning in mathematics and providing examples of deductive systems in geometry, algebra, number theory, or vector spaces.

Finally, the references to algorithms in the foregoing paragraphs emphasize the pervasive role of computing and algorithmic techniques in mathematics and its applications. The use of flowcharts for describing algorithms is becoming commonplace in the elementary school. Moreover, flowcharts are proving to be an important educational tool in teaching elementary and secondary school students to organize their work in problem solving. These ideas should therefore be encountered by a prospective teacher in his mathematics training. We recommend that computing facilities be made available, so that he will also have an opportunity to implement some algorithms and flowcharts on a high-speed computer using some standard computing language.

The course sequence should include many references to applications outside mathematics. This is self-evident for probability theory, but it is important to stress this over the whole spectrum of topics studied. In particular, the function concept itself provides many opportunities to underline the significance of mathematical formulations and methods in our study of the world around us.

At the conclusion of the course sequence the prospective teacher should understand the rational number system and the necessity, if not the method, of enlarging it to the real number system. He should be familiar with elementary linear geometry in two and three dimensions. In his study of the integers and the rational numbers, he should understand the essential role played by the properties of the addition and multiplication operations and the order relations in justifying and explaining the usual computational algorithms, the factorization theory of whole numbers, and the methods of solution of equations. He should thereby, and through experience with algebraic structures encountered in geometry, acquire an appreciation of the importance of abstraction and generalization in mathematics.

He should know something of the basic concepts and the algebra of probability theory, and he should be able to apply them to simple problems. He should grasp the idea of an algorithmic process and understand a bit about computers and how one programs them. We expect him to appreciate something of the role of mathematics in human thought, in science, and in society. We hope finally that he can learn all this in such a way that he will enjoy mathematics and the teaching of it, and that he will desire to continue to study mathematics.

Level II Recommendations

In the years since the first set of recommendations was made, dramatic changes have taken place in the mathematics of junior high school. These changes are in depth, as witnessed by greater emphasis on logic and mathematical exposition, and in breadth, as witnessed by the increased amount of geometry and probability. They make it necessary to re-examine the background needed by a teacher at this level. Moreover, the intermediate position of the junior high school requires of teachers at this level an appreciation of the mathematics of the elementary school as well as knowledge of the mathematics of the high school.

Finally, it seems desirable that there be two kinds of teachers in the middle school or the junior high school: those who concentrate on the transition from the elementary school and those who concentrate on the transition to the high school. For this reason we give two sets of recommendations for this level.

Level II-E Recommendations

These recommendations are for students who begin with Level I preparation and pursue further training to qualify them to be either specialist teachers of elementary school mathematics, or coordinators of elementary school mathematics, or teachers of middle school and junior high school mathematics. There should be some teachers with Level II-E preparation in each elementary school. The recommended program is:

A. <u>The Level I program</u>. (A student who is already prepared for calculus may omit the course on functions of the second sequence of courses listed on page 165 .)

B. <u>An elementary calculus course</u> (e.g., Mathematics 1 [page 44]). At this level all teachers need an introduction to analysis and an appreciation of the power that calculus provides.

C. <u>Two courses in algebra</u>. The courses in linear and modern algebra are identical to those described under C of the Level III recommendations, page 171 .

D. <u>A course in probability and statistics</u>. This course is identical to the first course under D of the Level III recommendations.

E. <u>Experience with applications of computing</u>. This recommendation is identical to that under F of the Level III recommendations.

F. <u>One additional elective course</u>. For example, a further course in calculus, geometry, or computing.

Level II-J Recommendations

These recommendations provide a special curriculum for the training of junior high school teachers which is slightly less extensive than that for Level III. The recommended program is:

A. Two courses in elementary calculus (e.g., Mathematics 1 and 2, page 44). Greater emphasis on calculus is desirable for this level because teachers at the upper level of the junior high school must see where their courses lead.

B. Two courses in algebra. The courses in linear and modern algebra are identical to those described under C of the Level III recommendations, page 171.

C. One course in geometry. Either of the two courses described under E of the Level III recommendations will suffice.

D. A course in probability and statistics. This course is identical to the first course described under D of the Level III recommendations.

E. Experience with applications of computing. This recommendation is identical to that under F of the Level III recommendations.

F. Review of the content of courses 1 and 2 of Level I (either sequence) through study or audit. There is a problem with the interface between the elementary school and the junior high school. In part this is caused by the fact that, traditionally, junior high school teachers are prepared for secondary school teaching and hence are little aware, at first, of their students' capabilities and preparation. We therefore believe that some sort of orientation to the mathematical content and spirit of the elementary school mathematics program is necessary to equip the Level II-J teacher properly. Two means have been considered to meet this need. One method would be to give an additional course in college to the prospective Level II-J teacher, a course which would be a streamlined version of courses 1 and 2 of the Level I program. On the whole, we prefer this solution although it makes the plan of study rather long. The second method would be to encourage schools to supply the new Level II-J teacher with elementary school texts to read, and to require him to visit classes and to talk with elementary school teachers, especially those in grades 4 to 6. A combination of both of these methods might prove most effective.

G. Two elective courses.

Items A through E supply the bare essentials. Greater breadth and greater depth are both to be desired. In order to give the teacher freedom to pursue his interests, electives are suggested, with further courses in computing, analysis, algebra, and geometry having high priority. Teachers with Level III preparation can meet the requirements listed above by fulfilling the intent of F.

Level III Recommendations

Although the mathematics of the senior high school has not changed as dramatically in the past ten years as has that of the elementary school, yet there are significant directions of change which make new recommendations desirable. These are: (1) a gradual increase in the volume and depth of mathematics taught at the secondary level which brings with it an increased occurrence of calculus (with the Advanced Placement program), (2) an increasing use of computers in mathematics courses and as an adjunct in other courses, and (3) an increasing realization that applications should play a more significant role.

Our recommendations, while designed primarily to specify minimum requirements for prospective high school teachers, have also been constructed with a view to maintaining, as far as possible, comparability of standards between prospective teachers and prospective entrants to a graduate school with a major in a mathematical science. We want to maintain a freedom of choice for the student to go in either direction. While the program we recommend for prospective teachers will leave the student with a deficiency in analysis and in algebra in order to meet the CUPM recommendations for entry to graduate school, the prospective graduate student in mathematics would normally need courses in geometry and in probability and statistics to meet our recommendations for teachers. We regard it as a matter of great importance that a program for teachers should be identical to the one offered to other mathematics majors, except for a few courses peculiarly appropriate to prospective high school teachers.

Before detailing the recommendations, some remarks on the role of applications, the computer, and on the problem of teaching geometry are in order.

Every experienced teacher knows that mathematics must begin at the concrete level before it can proceed to a more theoretical or abstract formulation. It is assumed that topics in the courses under discussion will contain a judicious mixture of motivation, theory, and application. A purely abstract course for teachers would be madness, but a course in calculation with no theory would not be mathematics. In addition to including applications where possible in mathematics courses, there is a need for introducing some specific study of the lore of mathematical model building, in order to provide the framework of ideas within which specific applications can be placed in their proper perspective. The idea of a mathematical model of a "real" situation and the associated techniques and rationale of the model building process have developed as a sort of folk knowledge among mathematicians and users of mathematics, and now an effort is being directed toward making these ideas more explicit and including them in the curriculum. The course Mathematics 10 described in A General Curriculum in Mathematics for Colleges (1965) [see page 92] was such an effort, but only now are detailed descriptions of such a course appearing (see Applied Mathematics in the Undergraduate

Curriculum, page 705). As these efforts begin to affect the high school curriculum, where much of the material belongs, it becomes more urgent that the future high school teacher receive appropriate preparation. Conversely, the preparation of teachers to communicate these ideas will accelerate the improved treatment of applications in high schools.

Computers have already had a phenomenal impact on the high school mathematics curriculum in supplementing, and in part replacing, traditional formal methods by algorithmic methods. As access to digital computers becomes more common, one can expect both the flavor and content of high school mathematics courses to change dramatically. In schools where such facilities are already available, it has become clear that opportunities for experimentation and creative outlet, using the computer as a laboratory device, are within the reach of many students whose mathematical ability, motivation, or background would preclude any comparable experience in a formal mathematical setting. Moreover, it has been found that certain abstract mathematical ideas are understood and appreciated more completely when experience is first obtained through the use of a computer. Algorithmic and numerical techniques should therefore be given strong consideration in all courses in which they are appropriate; and, wherever computing facilities are available, use of the computer should be a routine part of these courses.

The nature of high school geometry continues to change. Changes over the past decade have mainly been toward remedying the principal defects in Euclid's Elements that are related to the order, separation, and completeness properties of the line, but more recently there has developed an entirely new approach to geometry that links it strongly to algebra. This approach is now finding its way into the high school geometry course. A teacher should be prepared to teach geometry either in the modern Euclidean spirit or from the new algebraic point of view. Thus we are recommending that he take two geometry courses at the college level.

The minimum preparation of high school teachers of mathematics should include:

A. Three courses in calculus. Mathematics 1, 2, and 4 [page 44] are suitable. This recommendation assumes that the student has the necessary prerequisites. It is also desirable to take advantage of the growing role of computers in introducing mathematical concepts.

B. One course in real analysis. Mathematics 11 [page 93] would be satisfactory provided that the instructor is aware that his students' primary interest is teaching.

C. Two courses in algebra. One of these should treat those topics in linear algebra that are essential for the understanding of geometry and that have become crucial in applications, especially to the social sciences. Mathematics 3 [page 55], with careful attention to examples, would suffice. The second algebra course should be an abstract algebra course approximating Mathematics 6M [page 68]. Again, opportunities should be found to incorporate geometrical ideas that motivate and illustrate various algebraic structures (e.g., groups of symmetries, groups of transformations, rings of functions).

D. Two courses in probability and statistics. The first of these should begin with intuitive notions of probability and statistics derived from the real world. Mathematical model building and the relationship of mathematics to the real world should be considered. Calculus may be required in the latter part of this first course. The second course will treat those additional and more advanced topics normally included in a statistics sequence. In Preparation for Graduate Work in Statistics [page 459] the Statistics Panel of CUPM has described two courses that are close to what we have in mind. Throughout the two courses, care should be taken to include an analysis of some statistical studies which have appeared on the public scene and should make explicit some of the misinterpretations that are possible. Applications (in particular, applications to decision theory) should be drawn from such fields as medicine, education, business, and politics. The range and realism of problems can be enhanced if students are able to use computers.

E. Two courses in geometry. One course emphasizes a traditional approach by concentrating on synthetic methods and a careful study of the foundations of Euclidean geometry with a brief treatment of non-Euclidean geometry. The other course is strongly linked to linear algebra, includes an investigation of the groups of transformations associated with geometry, and is explicitly related to other parts of mathematics. Examples of such courses are given on page 86.

F. Experience with applications of computing. This should involve learning the use of at least one higher level programming language such as BASIC or FORTRAN. For this purpose we recommend a formal course such as C1 [page 563], but the experience may also be obtained independently or in other courses that make use of computers.

G. One course in applications. This should place heavy emphasis on mathematical models in the physical or social sciences. Examples of appropriate outlines can be found in Applied Mathematics in the Undergraduate Curriculum, page 705.

Nine of these 12 items should be included in the undergraduate program of every prospective high school teacher, namely A, B, C, F, and one course each from D and E. The remaining courses may be deferred to his post-baccalaureate study, although consideration should be given to including them among the electives in his undergraduate program. A list of possible elective courses is included at the end of the following paragraph. Were it not for our view that an undergraduate program should permit maximum flexibility in choosing a career and as much latitude as possible for every student to express his own interests in acquiring the proper breadth in his area of concentration, we would have specified that all of the 12 items be included in the undergraduate program of a prospective teacher. Fortunately, it is now becoming commonplace for Level III teachers to continue their mathematical education at the graduate level. Indeed, this is mandatory in many instances through permanent certification requirements or through the salary schedules and policies of individual school systems.

In structuring his undergraduate mathematics program, a student will naturally choose his electives after reflection upon his career goals. If, for example, a prospective high school teacher wishes to pursue graduate study in mathematics, he will necessarily choose additional courses in algebra and analysis beyond those which we have mentioned in our recommendations for Level III teachers. We include below a partial list of electives which would suitably extend our recommendations for the training of high school mathematics teachers.

Real Variables (Mathematics 11-12, page 93)

Complex Variables (Mathematics 13, page 97)

Numerical Analysis (Mathematics 8, page 83)

Abstract Algebra (Mathematics 6M, 6L, page 65)

Geometry and Topology

Number Theory

Foundations of Mathematics

Logic and Linguistics

OTHER ASPECTS OF TEACHER TRAINING

In accordance with our charge, we have made recommendations only on the content of the teacher training curriculum. We are, however, well aware that there are other crucial aspects of a teacher's overall preparation. In discussing these matters we hasten to observe that, just as we do not believe in any sharp distinction between teacher-trainees and other students in respect to the content of their mathematics courses, so we insist that these other aspects are relevant to all mathematics instruction. We believe only that they deserve more emphasis for teacher-trainees than for other mathematics majors.

Communication is of the essence in mathematics, and prospective teachers must pay special attention to all of the ways in which mathematics is most effectively communicated. They should be led to regard mathematics as a creative activity--something which one does rather than merely something which one learns. The active participation of the student in the process of discovering and communicating mathematical ideas is crucial for his real understanding. Courses should be taught in ways that foster active student involvement in the development and presentation of mathematical ideas.

Development of skills in writing and reading and speaking and listening should be an explicit part of teacher training at every stage, and not only in mathematics courses. These, like any other skills, can be developed only through constant and active involvement of the student in practices which exercise these skills. Thus, his regular courses, reading courses, clubs, or seminars should stress opportunities for two-way communication of mathematical ideas.

It is also important for teachers to continue to study and to do mathematics throughout their professional lives. This is closely related to, but goes beyond, the processes of communication mentioned above, for a willingness to grow reflects an enthusiasm that often transcends other skills in communicating mathematics.

Other aspects of communication which are not dealt with in this report are those relating to behavioral objectives and to special teaching methods and aids. While the Panel agrees that these are very important matters, it feels that they demand a much more complex effort and a totally different expertise, and might properly be the subject for another study. An excellent volume, which explores "the educational and psychological problems in the selection, organization and presentation of mathematics materials at all levels from the kindergarten through the high school," is the Sixty-ninth Yearbook of the National Society for the Study of Education, entitled "Mathematics Education."

The relationship of mathematics to other studies is another important matter not touched upon in this report except insofar as we have recommended the study of applications and mathematics. Indeed, we believe that every mathematics teacher should develop skill in other subjects which make use of mathematics.

Finally, we share a widespread concern for the special education of the culturally disadvantaged child and of the child whose achievements, for whatever reason, are below accepted standards. Such children require specially trained teachers. We do not know what form this training should take, but we feel that this is a proper concern of CUPM for the future.

Introduction

Two sequences, each consisting of four courses, are outlined here in detail. One reason for presenting two different sequences is to illustrate our earlier claim that there are various ways of organizing the material. We have no desire to be prescriptive or definitive with respect either to course content or to the ordering of material. The outlines are to be construed as models only of the content and depth of coverage that we believe will be possible in the best circumstances. We do believe that the material presented approximates that which a really first-rate teacher of elementary school mathematics should know.

We regard either sequence as a way of achieving an integrated curriculum. In the first sequence the courses do not emphasize any particular single area of the traditional curriculum. Thus, arithmetic and geometry are both developed throughout the entire sequence. Each figures prominently in all four courses. In the second sequence, on the other hand, the emphasis is on number systems in the first course, geometry in the second, mathematical systems and induction in the third, and functions in the fourth. In both sequences each course contains topics from most of the areas identified as essential in the recommendation on page . By the end of the second course in either sequence the student will have met most of the topics that we consider essential for the elementary teacher, although not at the depth or in the detail preferred. Indeed, throughout both sequences the reader must be careful to interpret the statements of topics to be covered as referring to a treatment appropriate to the level of the student, and not a definitive treatment such as would be accorded to such a topic if encountered at a higher level. Typical places where there is danger of misinterpretation are Section 5 of Course 2 in the first sequence (Operational Systems and Algebraic Structures) and Section 1 of Course 2 in the second sequence (Intuitive Non-metric Geometry).

The unification of the four courses of each sequence requires the use of a common language for the expression of mathematical ideas. For instance, the concepts of set, function, and operation are introduced early and used throughout. Logical terms are introduced and used where appropriate.

Each section of a course guide has a suggested time allocation stated as a percentage of the course. These time allocations indicate first the balance of the sections within the course, second the depth and detail of treatment of the topics listed under each section heading. Thus they should enable the reader to judge the level of treatment and avoid the danger, already referred to, of giving a more comprehensive (or, perhaps, more superficial) treatment than intended.

It must be kept in mind that a prospective teacher is profoundly influenced by what he observes and experiences as a student.

Later his own methods and philosophy of teaching will reflect that experience. Hence, it is of paramount importance that these courses be conducted in a manner which encourages active participation in mathematical discovery. Frequent and substantial assignments which expose and drive home the attendant manipulative and computational skills should also be the rule.

SEQUENCE 1

Sequence 1 consists of four courses:

1. Number and Geometry with Applications I

2. Number and Geometry with Applications II

3. Mathematical Systems with Applications I

4. Mathematical Systems with Applications II

Course 1 begins with some intuitive geometry so that the concept of a number line is available immediately. Then arithmetic and geometry are developed throughout the entire sequence; the interaction of these two areas of mathematics enriches both subjects. Nevertheless, there are occasions when each area is developed within its own context. In particular, the algebra of the rational numbers is applied extensively to the theory of probability and statistics without reference to the geometric aspects of the number line.

A feature of this sequence is the adoption, to a limited extent, of the spiral approach. Thus, certain notions, such as extensions of the number system and the group of rigid motions, reappear several times, each time at a higher level of sophistication and with enhanced mathematical knowledge at the disposal of the student. An important practical advantage of this approach is that the student who takes only two or three courses of the sequence will have met most of the important mathematical concepts. In such cases, however, the depth of understanding is less than desired.

Some simple logic is introduced where appropriate to enhance the student's understanding. At the end of the sequence the student should understand the nature of mathematical reasoning and proof.

Again, we stress the importance of interpreting the Course Guides in the spirit of the comments made in the Introduction.

Course 1. Number and Geometry with Applications I

1. Elementary Ideas of Space, Measurement, and the Number Line (20%)

2. The Rational Number System and Subsystems (60%)

3. Probability, Statistics, and Other Applications (20%)

Course 1 is concerned with the study of the rational number system. It commences with the most intuitive geometrical notions, from which attention is focused on the number line and its role as a representation of the set of whole numbers. This approach enables the arithmetical and geometrical aspects of elementary mathematics to be developed from an integrated standpoint emphasizing their complementarity. It is a particular feature of the number line that negative integers are thereby immediately suggested and readily studied; this insures that the development of the number system follows a path which is mathematically natural. The twin approach also enriches the scope for interpretation of the operations of arithmetic. At this stage these operations and their properties are motivated through physical models.

Applications of the arithmetic of the nonnegative rationals to the most intuitive ideas of probability and statistics are given. Further applications of the arithmetic should be a feature of the course.

The language should be informal, but it should be such that a transition to precise mathematical language can be naturally effected. In particular, the student should be prepared for the function concept through the use of appropriate language. Those students who have not already met the notions of sets and functions may require more explicit introductions to these concepts.

1. Elementary Ideas of Space, Measurement, and the Number Line (20%)

Intuitive development of geometric figures in the plane and space, first as idealizations of familiar objects and then as sets of points; an intuitive development of incidence relations and some simple consequences; congruence developed by use of slides and flips of models of figures leading to turns as another means of preserving congruence and with attention also to parallelism, perpendicularity, and symmetry; consideration of measurement of segments with various units and the beginning notion of approximation; informal introduction of the number line.

2. <u>The Rational Number System and Subsystems</u> (60%)

Introduction of the set $W = \{0, 1, 2, 3, ...\}$* of whole numbers, from sets of objects; addition in W from disjoint union, multiplication in W from cartesian products (treated informally); counting as the link between sets and numbers; place value systems and decimal numeration of whole numbers (reinforced by examples with nondecimal bases); properties of operations in W from observed properties of operations on sets, including order properties; algorithms for computation in W (include use of flowcharts); simple closed and open mathematical sentences, including inequalities.

Coordinatization of the half-line with W; addition in W as a vector sum, using slides of the number line; subtraction in W as a slide to the left to introduce the set Z of integers; properties of addition and order properties in Z; mathematical sentences in Z. (Multiplication by negatives is delayed until the set of rationals is developed.)

Factorization in W, prime factorization, unique prime factorization, some simple divisibility criteria, the Euclidean algorithm; multiplication and division of integers by whole numbers introduced through experiences with the number line; greatest common divisor and least common multiple.

Introduction of the set Q of rationals through division by n, $n \in W$ and $n \neq 0$, as shown on the number line; change in scale of the number line and its use in measurement; equivalence classes of symbols for rationals; coordinatization of the line with Q; introduction to the question of completeness.

Addition in Q suggested by slides of the number line; multiplication of a rational by a whole number suggested by slides of the number line; multiplication by a positive rational suggested by stretching and shrinking; multiplication by a negative integer suggested by multiplication by a whole number followed by a flip; multiplication in Q; algorithms for computation in Q^+, properties of addition and multiplication in Q, and order properties in Q.

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* A glossary of symbols is included on page 202.

Decimal numeration of Q; percentages; integer exponents; scientific notation; orders of magnitude; algorithms and flowcharts for computation in Q; mathematical sentences.

3. Probability, Statistics, and Other Applications (20%)

Examples of statistical experiments in finite event spaces and their outcome sets, leading to counting procedures for determining the number of outcomes of various kinds of compound events (use tree diagrams); sampling problems with and without replacements, leading to combinatorial devices for counting samples; relative frequencies; assignment of probabilities to singleton events and to disjoint unions and intersections of events through addition and multiplication in Q^+.

Other applications, e.g., measurement, constant rate, profit and loss, expectation and risk, percentages, estimation, significant figures, and approximation.

Course 2. Number and Geometry with Applications II

1. Functions (5%)
2. The Rational Number System and Subsystems (20%)
3. Geometry (35%)
4. Real Numbers and Geometry (10%)
5. Operational Systems and Algebraic Structures (10%)
6. Probability, Statistics, and Other Applications (20%)

Course 2 is designed to give a genuine mathematical treatment of ideas introduced and studied at a more intuitive level in Course 1. The language of functions is established; this enables the solution of linear equations over Q and Z to be investigated systematically. Then the interrelationship between geometry and algebra again becomes evident in the study of symmetries, rigid motions, and sets with operations. At the same time, questions connected with measurement are studied, thus insuring that the material can be usefully applied; explicit reference to problems of approximate calculation involving large amounts of data can lead to consideration of computer programs.

The Pythagorean relation prepares the way for the introduction of irrational numbers and a preliminary discussion of real numbers.

The ideas here are difficult, and no attempt should be made to give complete proofs; nevertheless, the topic should be explored extensively.

Algebraic structures are defined, but studied only in familiar examples (including modular arithmetic). Further study of probability and statistics is included, beginning with a study of permutations and combinations which employs the function concept and presents systematic counting procedures.

1. Functions (5%)

The function concept (motivated by examples from Course 1), one-one and onto properties of functions; relations (motivated by examples from Course 1) with emphasis on equivalence and order relations. Binary operations as functions on cartesian products of form $X \times X$. Power sets; unions, intersections and complements as operations; the functions $2^X \to 2^Y$ and $2^Y \to 2^X$ induced by a function $X \to Y$.

2. The Rational Number System and Subsystems (20%)

Review of properties of Q with some arithmetical proofs; properties of Z; Z as an ordered integral domain; realization that Z is not closed under division, with the closure of Z leading to Q.

Coordinatization of the line with Z and then with Q; coordinatization of the plane with Z^2 and then Q^2; and coordinatization of space with Z^3 and Q^3.

3. Geometry (35%)

Review of geometric figures as idealizations of familiar objects and as sets of points in space; review of rigid motions and symmetry; review of congruence, parallelism, and perpendicularity. Groups of symmetries of an equilateral triangle and a square.

Rigid motions as functions that preserve lengths of segments; classification of rigid motions; composition and inverses; intuitive understanding of group properties of the group of rigid motions.

Review of measurement of segments with various units; principles of measurement; principles of measurement applied to length, area, angle measurement, volume; approximation; formulas for measurement related to rectangles, triangles, right prisms, pyramids.

Approximate calculation. Pythagorean relation through the formula for the area of a rectangle.

4. Real Numbers and Geometry (10%)

Adequacy of Q for physical measurement; inadequacy of Q for representing lengths of segments demonstrated by construction of segments with irrational measures; distance between points in Q^2 (use of Pythagorean relation). Introduction of real numbers in terms of nested intervals and nonterminating decimals; location of irrational points on the number line; infinite decimals regarded as sequences of approximating rationals; informal definition of addition and multiplication in R by means of approximating terminating decimals; distance in R^2; use of unique factorization to prove the irrationality of $\sqrt{2}$, $\sqrt{3}$, etc.

Coordinatization of the line, plane, and space with the real numbers; distance in R^2. Solution of linear equations and inequalities and graphs of solution sets in R^2; solution of linear equations in R^3 as intersections of planes.

Intuitive treatment of the perimeter and area of a circle; π as an irrational number.

5. Operational Systems and Algebraic Structures (10%)

Review of properties of the operations of addition and multiplication and the order relation in the number systems W, Z, and Q; defintions of group, ring, integral domain, and field, with examples drawn from subsets of Q, groups of rigid motions, groups of symmetries of a figure, power sets.

Arithmetic mod m as an operational system; contrasted with W, Z, and Q (closed under additive inverses, closed under multiplicative inverses when m is prime, divisors of zero when m is not prime, absence of compatible order relation); applications to the arithmetic of W (e.g., Fermat's theorem, Wilson's theorem, casting out 9's).

6. Probability, Statistics, and Other Applications (20%)

Permutations and combinations; randomness of a sample and tests of randomness; examples of applications of random sampling (using random number tables) to estimation of populations, quality control, etc.

Measure of central tendency in lists of data and possible measures of spread of data.

Random walks and their applications; assigning probabilities to compound events with applications; conditional probability.

Other applications, e.g., area, volume, weight, density; constructing an angle whose measure is p/q times the measure of a given angle.

Course 3. Mathematical Systems with Applications I

1. The Rational Number System (15%)
2. The Real Number System (5%)
3. Geometry (35%)
4. Functions (15%)
5. Mathematical Language and Strategy (15%)
6. Probability, Statistics, and Other Applications (15%)

In Course 3 the process of extending the number system from the whole numbers to the rationals, which has been explained and motivated in previous courses from both geometrical and arithmetical considerations, is carried out as a piece of formal algebra. Polynomials are also studied. Then algebra and geometry each provide examples of small deductive systems to illustrate the nature and power of the axiomatic method. The Pythagorean relation is available in this course, so that Euclidean geometry may be carried out in the coordinate plane. Vector notation and methods are studied, and there is a discussion of the generalization of coordinate geometry to three dimensions.

Algebraic concepts are also exemplified by the study of the group of Euclidean motions and certain subgroups, thereby enriching the notion of subgroup with concrete examples. The trigonometric, exponential, and logarithmic functions are defined and studied in their own right and in view of their applications. There are some explicit discussions in this course on mathematical methods, covering such topics as proof, conjecture, counterexample, and algorithms, together with a review of appropriate logical language. Probability theory is itself developed from an axiomatic standpoint, experience of its practical nature having been gained in previous courses.

1. The Rational Number System (15%)

Formal construction of Z from W and of Q from Z.

Polynomials as functions and forms; Q[x] and Z[x] as rings; the degree of a polynomial; substitution; quadratic equations.

Description of proof by induction in W with examples; application to number-theoretic properties of Z, e.g., divisibility properties, Fundamental Theorem of Arithmetic; Euclidean algorithm; similar applications to Q[x]; remainder theorem.

2. The Real Number System (5%)

Review of the coordinatization of the line with R; approximation of reals by rationals; addition and multiplication in R through rational approximation; the field of real numbers.

3. Geometry (35%)

Review of coordinate geometry of the real plane; distance; graphs of linear equations in two variables.

Plane vectors from translations; vector addition as the composition of translations; multiplication by scalars; vector equations of a line (with resulting parametric and general form of the equations); conditions for parallelism and perpendicularity in coordinate representation; appropriate generalization of these ideas to three dimensions.

Examples of groups and subgroups drawn from geometry, e.g., the group of similarities with the subgroup of rigid motions, the group of rigid motions with the subgroup of rotations about a point, the group of rotations with the subgroup of cyclic permutations of a regular polygon.

Similarities and the representation of similarities as composites of magnifications and rigid motions; similar figures. Constructing the points which separate a segment into n congruent parts.

Equations of circles and the beginning notions of trigonometric functions.

Small deductive system in plane geometry, e.g., incidence properties from postulates of incidence or some constructions from postulates of congruent triangles, or some angle-measure properties from postulates of incidence, parallelism, and segment and angle measures.

4. Functions (15%)

Review of real-valued functions and their graphs; inverses of invertible functions; graph of an invertible function and its inverse.

Beginning notions of exponential functions and some of their properties; applications of these ideas, e.g., growth and decay; logarithmic functions; application of logarithmic functions to approximate calculation and construction of a slide rule.

Definitions and graphs of the trigonometric functions with emphasis on periodicity.

5. Mathematical Language and Strategy (15%)

Review of the language of connectives, the common tautologies, and the relation of some of these notions to set union, intersection, complementation, and inclusion.

Universal and existential quantifiers; denial of a mathematical statement, counterexamples.

Examples from previous sections of direct and indirect proof and of proof by induction; selected new topics to illustrate proof by induction (e.g., binomial theorem for positive integer exponents; number of zeros of a polynomial; sums of finite series); explicit contrast with inductive inference, role of hypothesis, conjecture.

Examples of algorithms and flowcharts.

6. Probability, Statistics, and Other Applications (15%)

Postulates for a discrete probability function and some consequences proved for probabilities of compound events; random walks and their applications.

Computation of measures of central tendency and variance; simple intuitive notions of statistical inference, tests of significance, frequency distributions, passage from discrete to continuous variables, normal distribution; application of statistical inference to real-life situations, e.g., opinion polls, actuarial tables, health hazards.

Course 4. Mathematical Systems with Applications II

1. Geometry (30%)

2. The Real and Complex Number Systems (10%)

3. Operational Systems and Algebraic Structures (40%)

4. Probability, Statistics, and Other Applications (20%)

Course 4 consists of a systematic study of precalculus mathematics. Linear algebra in R^2 and R^3, as vector spaces and as inner product spaces, leads on the one hand to matrix algebra and on the other to the standard trigonometric identities. The algebraic method in geometry is contrasted with the synthetic method. The real numbers R are presented as the completion of the rationals, and the extension of R to the field C of complex numbers is motivated and described.

Abstract algebra occurs in the course--a beginning study is made of abstract group theory--but emphasis is on familiar examples of the various algebraic systems, for example, the integral domain of polynomials over Z, Q, Z_p (p prime). The key notion of homomorphism of algebraic structures is introduced; among the examples treated are the logarithmic and exponential functions which are seen to be mutually inverse isomorphisms.

Frequency distributions form the main topic of the probability and statistics component; although the course remains essentially concerned with discrete probability spaces, the normal distribution is mentioned here. Applications of the preceding theory are made to problems of approximation and error.

1. Geometry (30%)

Coordinatization of space with R^3, distance in space, first-degree linear equations in three variables; vectors in space, vector addition, scalar multiples of vectors in R^2 and R^3, description of the vector spaces R^2 and R^3; norms of vectors, inner product, definition of the inner-product (or Euclidean) space R^3, relation of cosine to the inner product; definition of the vector product in R^3, triple scalar product and volume; the idea of closeness in R^3, with some of the simpler topological properties of this metric space.

Definitions of linear transformations of R^2 and R^3; orthogonal transformations; matrix representation of linear transformations of R^2 and conditions for orthogonality; matrix multiplication suggested by composition of linear transformations; representations of rotations in the plane by orthogonal matrices, leading to the standard trigonometric identities.

Invertible linear transformations of the plane with coordinate representations; rigid motions, magnifications, and other subgroups of the group of invertible linear transformations; representation of similarities in R^2 by matrices.

Analysis of the roles of synthetic and analytic methods in geometry, e.g., properties of circles, coincidence properties of triangles.

2. <u>The Real and Complex Number Systems</u> (10%)

Algebraic extensions of Q; algebraic and order properties of R; discussion of the completeness of R.

Extension of the real number system to the field C of complex numbers; failure of order relations in C; graphical representation of C in R^2.

3. <u>Operational Systems and Algebraic Structures</u> (40%)

System of polynomial forms over Q, its integral domain properties, factorization; Euclidean algorithm and appropriate flow diagram; factor theorem; elementary theory of polynomial equations; comparison with theory for polynomials over Z_p (p prime), Z.

Exponents, extension of exponential functions over Q to functions over R, with graphs; rational functions over Q, over Z_p (p prime); Newton's method of approximating zeros of polynomials (no differential calculus) and appropriate flow diagram.

Subgroups; Lagrange's theorem; applications to elementary number theory (Fermat's theorem and Euler's theorem); commutative groups, quotient groups of commutative groups; application to Z_n.

Homomorphisms of algebraic structures with many examples; identification of those which are one-one, onto; defintion of isomorphism as invertible homomorphism; one-one and onto homomorphisms are isomorphisms; isomorphic systems, e.g., Z_4 and rotational symmetries of a square, the positive reals under multiplication and the real numbers under addition.

4. Probability, Statistics, and Other Applications (20%)

Review of sample spaces, probability functions, random walks; discrete binomial distributions; statistical inference and tests of significance; other frequency distributions with applications, e.g., rectangular, Poisson; normal distribution (treated descriptively).

Application of the number system Q to problems in scaling, ratio, proportion, variation; approximation, errors in approximation, errors in sums, errors in products; Bayesian inference.

SEQUENCE 2

Sequence 2 consists of four courses:

1. Number Systems and Their Origins

2. Geometry, Measurement, and Probability

3. Mathematical Systems

4. Functions

Each course contains topics from most of the areas identified in the general description of the Level I Recommendations (algebra, the function concept, geometry, mathematical systems, number systems, probability, deductive and inductive reasoning). While interrelationships among these topics are explored in the spirit of an integrated curriculum, each of the four courses, nevertheless, has a special emphasis or focus. The emphasis in the first course is on number systems, the second on geometry, the third on mathematical systems, and the fourth on functions.

Although the special character of each course can be suggested in a few words, it is a mistake to assume that any course is narrowly defined by its title. In fact, by the end of the second course the student will have met the full breadth of topics considered essential for the elementary teacher. He will not, however, have reached the depth of understanding desired.

Again, we stress the importance of interpreting the Course Guides in the spirit of the comments made in the Introduction.

Course 1. Number Systems and Their Origins

1. Sets and Functions (15%)

2. Whole Numbers (45%)

3. First Look at Positive Rational Numbers (10%)

4. First Look at Integers (5%)

5. The Systems of Integers and Rationals (25%)

Course 1 features integration of arithmetic and algebra with supplementary assistance from geometry. The number line is thought of as a convenient device for representing numbers, order, and operations. Rational numbers are introduced in the context of comparing discrete rather than continuous sets (though brief reference is also made to the rational line). Algebraic similarities and differences between the number systems are emphasized. In particular, the systems of integers and rationals are studied in parallel. Algorithms, flowcharts, and manipulative rules for the various number systems are not only justified by referring to physical or schematic models but also are seen as consequences of the algebraic structural properties of the number systems. The whole number system receives much attention, as its algebraic properties (and its algorithms) recur in only slightly altered form in the systems of integers, rationals, and reals.

1. Sets and Functions (15%)

Review, at an intuitive level, of the basic concepts associated with sets and functions in order to establish the language and notation that will be used throughout the course. (For most students this will be a review of things they have seen repeatedly since junior high school.) Set concepts covered are: membership, inclusion, and equality for sets; various ways of describing sets (rosters, set-builder notation, Venn diagrams); special subsets that often lead to misunderstanding (empty set, singletons, the universal set); common operations on sets (intersection, union, complementation, cartesian product); illustration of the above concepts in various ways from real objects and from geometry.

The connection between set operations and logical connectives; for example, "or," "and," and "not" are related to union and intersection and complement while the inclusion relation "$A \subset B$" is related to the implication "$x \in A \Rightarrow x \in B$." (In this first introduction of logic, the treatment should be very brief and informal, but the language is necessary for subsequent use.)

The major function concepts to be covered include an intuitive rule-of-assignment definition; various ways of specifying this rule (arrow diagram, table, graph, set of ordered pairs, formula); notions of domain and range; input-machine-output analogy; one-one and onto properties; one-one correspondences between finite sets and between infinite sets; brief look at composition and inverses with a view toward later ties to rational arithmetic. (Real and geometric examples should be used).

2. Whole Numbers (45%)

Whole numbers are motivated by a desire to specify the "size" of finite sets, numerals and numeration systems by the inadequacy of verbal "symbols" for numbers; the Hindu-Arabic numeration system contrasted with historical and modern artificial numeration systems in order to emphasize the roles of base and place value; order among whole numbers related to the process of counting and to the existence of one-one or onto functions between finite sets; order in W represented schematically by the position of points on a whole number line. In the construction of this "line" the concept of congruent point pairs arises naturally.

The operations of addition and multiplication related, with counting as the link, to the set operations of union and cartesian product (e.g., the use of multiplication in determining the area of a rectangle); multiplication also related to repeated addition and to determining the number of outcomes in a multi-stage experiment (the product formulas for C^n_r and P^n_r might be illustrated); addition and multiplication represented schematically in the usual vector fashion (slides and stretches) on the number line.

The algorithms of whole-number arithmetic justified initially (as in the elementary classroom) by reference to manipulating and grouping finite sets. (A flow diagram for division via repeated subtraction can be given.) The whole numbers with their operations and order now viewed as a mathematical system, the algebraic properties of which are motivated by reference to finite sets and set operations; the algorithms of whole number arithmetic re-examined from an internal point of view and justified on the basis of notational conventions and fundamental algebraic structural principles. The importance of estimating products and quotients should be emphasized as the algorithms are studied.

3. First Look at Positive Rational Numbers (10%)

Fractions motivated by a desire to compare two finite sets. If a probabilistic flavor is desired, compare a set of favorable outcomes with a set of possible outcomes; if a more conventional approach is desired, "ratio" situations can be used. With fractions viewed as operators, addition continues to correspond, in a sense, to disjoint union, while multiplication corresponds to composition; fractions represented schematically as points or vectors on a number line, and the operations viewed vectorially; rules for manipulating fractions motivated initially from physical or schematic representations. Rational numbers appear as abstractions of equivalence classes of fractions, and some algebraic properties of the system of rational numbers can be motivated by physical examples; the algebraic structure of Q^+ is not explored in detail. (The embedding of W in Q^+ considered briefly with a light touch.) A careful structural investigation deferred until the full system Q appears.

4. First Look at Integers (5%)

Integers suggested by some real situation, e.g., profit-loss, up-down, etc.; addition corresponds to an operation in the given situation; multiplication by a positive integer considered as repeated addition. Again the number line is used as a schematic representation; the various uses of the symbol " - " clarified; some algebraic properties identified and motivated; the embedding of the whole numbers in Z introduced, but treated only lightly.

5. The Systems of Integers and Rationals (20%)

Following a brief review of the concepts of open sentence, variable, replacement set, truth set, equation, and solution set (or perhaps only the last two), a systematic, parallel exposition of the algebraic structures of Q and Z in terms of solutions to equations is possible ($-a$ represents the unique solution to $a + x = 0$,

$1/a$ represents the unique solution to $ax = 1$ $(a \neq 0)$, $b - a = b + (-a)$, $b/a = b \times 1/a$); all the familiar rules for manipulating minus signs and fractions follow. The two important unique representations of rationals--as fractions in lowest terms and as (all but one type of) repeating decimals--can be illustrated and computational rules for (finite) decimals justified; several non-repeating infinite decimals described. The work on fractions in lowest terms will involve a certain amount of number theory, which should be done on an ad hoc basis. Review of the concepts of divisibility and prime; the Fundamental Theorem of Arithmetic illustrated and then assumed. (In Course 3 this principle may be proved.)

Course 2. Geometry, Measurement, and Probability

1. Intuitive Nonmetric Geometry (25%)

2. Intuitive Metric Geometry (25%)

3. Probability (20%)

4. Further Geometry (20%)

5. The Real Number System (10%)

In Course 2 the system of positive rational numbers reappears in two new contexts: in the context of geometric measurement, where continuous sets are being compared, and in the context of probability, where discrete sets are being "measured." Thus the system of positive rational numbers and the concept of measurement act as unifying threads. But the major blocks of new content covered are in geometry and probability.

Many opportunities for tying together these areas present themselves. For example, in the initial work in geometry which inevitably is concerned with establishing terminology and notation, combinatorial problems can be inserted to make the content more interesting. Later a probabilistic technique for approximating π could be given. Also, while studying probability, geometric representations can be given for many situations. For example, random processes are simulated by spinners, and experiments involving repeated trials are represented by random walks.

The geometry and the probability in this course are presented in a rather intuitive, nondeductive fashion. The main purpose here is to present the elementary facts in these areas, not to investigate their logical structure. Course 3 re-examines both areas from a more rigorous, deductive point of view.

The field of real numbers is also discussed to some extent in this second course.

1. Intuitive Nonmetric Geometry (25%)

Geometry viewed as the study of subsets of an abstract set, called space, whose elements are called points; the subsets are called geometric figures; drawing conventional pictures of points, lines, and planes suggests incidence relations. Some of the logical connections between various incidence properties explored. (The 4-point geometry might be introduced here, but axiomatics should not receive much emphasis in this course.) The standard terminology associated with incidence (collinear, coplanar, concurrent, parallel, skew, ...) reviewed in a combinatorial context (e.g., into how many pieces is the plane partitioned by n lines no three of which are concurrent and no two of which are parallel?). Further geometric figures (half lines; rays; open, closed, and half-open segments; half planes, half spaces, plane and dihedral angles and their interiors and exteriors) defined in terms of the basic figures--points, lines, planes; the set operations; the intuitively presented notions of (arcwise) connectivity and betweenness. The standard notation for these figures reviewed, using combinatorial problems for motivation (e.g., how many angles are "determined" by n points no three of which are collinear?).

The ideas of polygonal and nonpolygonal curves, simple curves, simple closed curves, and the interior of a simple closed curve presented intuitively along with a few other topological and geometrical concepts such as dimension of a figure, boundary of a figure, convexity; polygons, polyhedra, and Euler's formula.

Congruence introduced intuitively in this nonmetric setting as meaning same size and shape. At this first contact, the notions of measurement and distance avoided. The natural development of ideas seems to be: congruence which leads to a process of measuring which in turn suggests the existence of a distance function. Perhaps here, but probably more appropriately in Course 3, a definition of congruence in terms of distance can be given. Congruence of segments, angles, and other plane and spatial figures, with perpendicularity introduced in terms of congruence of adjacent angles.

Congruence in the plane viewed in terms of intuitive notions of rigid motions of the plane (slides, turns, flips, and their compositions). Symmetries of figures in terms of invariant point sets and rigid motions. The composition of rigid motions is a rigid motion and the inverse of a rigid motion is a rigid motion. The group concept introduced to tie together algebra and geometry. A fuller treatment of transformation geometry is suggested in Course 4.

2. Intuitive Metric Geometry (25%)

The process of measuring described in terms of filling up the set to be measured with congruent copies of a unit and counting the number of units used. Illustrated for segments, angles, and certain plane and spatial figures. Integrally nonmeasurable figures (with respect to a given unit) introduced and the positive rationals used as operators endowed with stretching-shrinking or replicating-partitioning powers. The rational number line reinterpreted in terms of segments and lengths, briefly showing the existence of rationally nonmeasurable segments and the real number line; a non-repeating infinite decimal exhibited and the theorem on decimal representation of irrationals recalled. The assignments of numbers to figures viewed as functions; observation that such measure functions are additive and invariant under congruence. The domain of segment measure functions extended to the domain of polygonal curves by additivity; perimeters computed. The additivity property applied to partitioning techniques for finding area; some familiar area and volume formulas derived (triangles, parallelograms, prisms, pyramids). The formula $A = \ell \times w$ for rectangles with irrational dimensions illustrated by drawing inscribed and circumscribed rectangles with rational dimensions. Plausible limiting arguments presented for circles and spheres; irrationality of π. The angle-sum theorem for triangles verified experimentally and then extended to convex n-gons by triangulation; the subsequent results about the various angle measures in regular polygons applied to making ruler-protractor drawings. Use of these measuring instruments suggests investigation of practical versus ideal measurement. Various units of length, area, angle, volume measurements and conversion factors relating them; the inevitability of approximation in practical measurement and the usage of such terms as "greatest possible error," "precision," "accuracy," "relative error." The various notational conventions in use for reporting how good an approximation is: significant digits, ± notation, interval of measure, scientific notation.

3. Probability (20%)

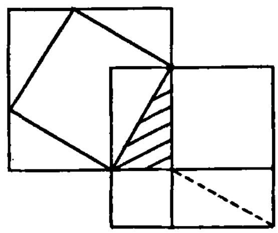
Various single and multi-stage experiments with discrete sample spaces considered and represented geometrically (trees, walks, spinners); large sample spaces and events "counted" using permutation and combination techniques. The terminology--sample space, outcome, event--compared with the terminology of geometry--space, point, figure; the assignment of probabilities to events compared with the assignment of lengths, areas, etc., to geometric figures. Both involve a comparison of two sets; the rational numbers are the indicated algebraic system. A priori assignment of probabilities (from shape of die, partitioning of spinner, constituency of urn, ...) compared with a posteriori assignment (long-range stability of relative frequency of events). The assignment of probabilities to events in terms of the point probabilities of their constituent outcomes, in the finite case, leading to additivity of probability measures; Venn diagrams used to illustrate the connection between set operations and logical connectives and to suggest the useful formulas

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{and} \quad P(A) = 1 - P(A').$$

Conditional probability and independent events; problems involving multiplication along the branches of a tree. The connection between C_r^n and the number of paths of a certain kind in the plane; the formula $C_r^n p^r (1 - p)^{n-r}$ for r successes in n repeated trials deduced and used; Pascal's Triangle (if not described earlier). The work on probability should include many exercises, as this may be the first contact with the subject for many prospective elementary school teachers.

4. Further Geometry (20%)

The simplest straightedge-compass constructions reviewed and related to the parallel postulate and congruence conditions for triangles (some proving of triangles congruent appropriate here). Each straightedge-compass construction technique compared with a ruler-protractor drawing technique for the same figure. Some plausibility argument for the Pythagorean Theorem, perhaps the one suggested by this sketch.



The simplified congruence condition for right triangles stated. Some work on square roots is appropriate here: a proof, based on the Fundamental Theorem of Arithmetic, that $\sqrt{n}$ is irrational or a whole number; an algorithm or two for computing rational approximations to $\sqrt{n}$; a remark that the existence of square roots in R but not in Q is a tipoff that R must have some extra fundamental property that Q does not have; geometric construction of a segment with irrational length, with respect to a given unit. Projection techniques for drawing similar triangles reviewed. Applications to scale drawing and to straightedge-compass construction of the rational number line. Similarity conditions for triangles analogous to the congruence conditions for triangles; for the case of right triangles the "AAA" similarity condition reduces to just "A." This suggests the calculation of trigonometric ratios and their use in indirect measurement. The Law of Cosines as a generalization of the Pythagorean Theorem.

5. <u>The Real Number System</u> (10%)

A brief review of the properties of the reals in some nonformal way--e.g., R includes Q; enjoys the same basic properties of $+$, $\times$, $<$ that Q does but has one more property, namely, the least upper bound property. Use of this property to suggest but not prove the existence of $\sqrt[n]{a}$ for all $a \in R^+$ and all $n \in Z^+$. Some work with rational exponents.

Course 3. Mathematical Systems

1. The System of Whole Numbers (25%)
2. Fields (25%)
3. Geometry (25%)
4. Probability-Statistics (25%)

In this course certain portions of algebra, geometry, and probability are studied more deeply in a systematic, deductive way. Proof receives more emphasis than in Courses 1 and 2. Some of the concepts of logic itself receive explicit treatment, along with new results in algebra, geometry, and probability.

1. The System of Whole Numbers (25%)

The algebraic and order properties of W reviewed; the well-ordering principle introduced. Symbol $a|b$ defined; proofs of some simple divisibility theorems such as $a|b \Rightarrow a|bc$, $a|b$ and $a|c \Rightarrow a|(b + c)$. Various simple divisibility criteria of the base ten numeration system derived. Primes and prime factorization; the sieve of Eratosthenes; checking for prime divisors of n only up to $\sqrt{n}$; Euclid's theorem and Wilson's theorem. Unsolved problems such as Goldbach's conjecture and the twin primes problem. Other interesting odds and ends, e.g., figurate, deficient, abundant, and perfect numbers.

The important concepts of common and greatest common divisor (GCD) introduced and the existence, uniqueness, and linear combination expressibility theorems for GCD derived. Techniques for finding the GCD (by listing all divisors, by the Euclidean Algorithm, from known prime factorizations); the least common multiple (LCM), its existence and uniqueness proved, its relation to the GCD, and several techniques for finding it. (A flow diagram for the Euclidean

Algorithm is appropriate here.) The concept of relative primeness and the lemma stating that $p|ab \Rightarrow p|a$ or $p|b$ (p prime), leading to a proof of the Fundamental Theorem of Arithmetic. Whether a rigorous proof of this theorem should be given using the well-ordering principle or mathematical induction, is debatable. The use of the Fundamental Theorem in reducing fractions and in demonstrating the existence of irrationals. Euler's φ-function might be defined and the theorems of Euler and Fermat illustrated.

2. Fields (25%)

Field defined using Q and R as prototypes. Subtraction and division in a field defined and their usual properties derived; Z_p (p prime) defined and shown to be a field; various properties of subtraction and division illustrated again in this context. Possibly enough group theory interposed (Lagrange's theorem, order of element theorem) to prove the theorems of Euler and Fermat. In the context of solving a linear equation over a field, several concepts of logic can be studied; statement, equality, variable, open sentence, reference set, truth set; the unsolvability of $x^2 = 2$ over Q contrasts with its solvability over R; the completeness property of R recalled. The statement of this property depends on the concept of order; the definition of an ordered field abstracted from familiar properties of Q and R. Simple order properties deduced; Z_p shown to be unorderable (in any decent sense). The logical connectives and their relation to the set operations, within the context of solving inequalities over (say) R. Equivalence of open sentences; equivalence transformations. The Archimedean and density properties for R derived; a short excursion into limits (optional). The intermediate value theorem cited, behavior of polynomials for large $|x|$ illustrated; existence and uniqueness of positive n^{th} roots deduced.

3. Geometry (25%)

Incidence axioms suggested by the Euclidean plane and space stated as abstract axioms and exhibited in a finite model; simple incidence theorems proved and interpreted in both models. Illustrate in the context of segments how the natural genesis of concepts: congruence (superposition) → process of measurement → distance function, can be reversed in a formalization of geometry: postulated distance function → congruence defined in terms of it. The intuitive notion of betweenness used to define rays and segments; betweenness also defined in terms of distance; the ruler postulate; proofs of a few elementary betweenness properties. (The depth to which this Birkhoff-SMSG approach is carried is a matter of taste. For the future elementary teacher it might be more appropriate to do most of the deductive work in the spirit of Euclid, pointing out from time to time an implicit betweenness or existence assumption.) Possible subjects for short deductive chains include: triangle congruence and straightedge-compass constructions; parallels, transversals, and angle sums; area postulates and an area proof of the Pythagorean Theorem.

Some flavor of other modern approaches to geometry, with attention restricted to the plane: coordinatization of the plane, vector addition and scalar multiplication of points, lines as subspaces and their cosets, vector and standard equations for lines; Pythagorean Theorem and its converse, Law of Cosines, perpendicularity, dot product, norm, distance; isometry, orthogonal transformation, matrix representation of linear transformations, classification of orthogonal transformations, decomposition of an isometry into a translation and an orthogonal transformation. Alternatively, a coordinate-free study of transformation groups.

4. Probability-Statistics (25%)

Review of the "natural" development of the terminology and basic concepts of outcome, sample space, event, point probability function, and probability measure; axioms for a probability measure. The possible backward rigorization in probability of the intuitive notion of "equally likely outcomes" as "outcomes having the same probability" compared with the backward rigorization in geometry of "congruent segments" as "segments having the same length." It might be worthwhile to digress in more generality on equivalence relations, partitions, functions, and preimages. After a few deductions from the axioms $[P(\emptyset) = 0, \quad P(A') = 1 - P(A), \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)]$, less emphasis is placed on deduction and further probabilistic concepts and techniques are presented. The idea of <u>simulation</u> by urn models, balls in cells, spinners, and the use of random number tables illustrated through a wide variety of problems. Permutations and combinations reviewed. More formal attention to combinatorial identities such as

$$2^n = (1 + 1)^n = \sum_{i=0}^{n} C_i^n, \qquad C_{k-1}^{n-1} + C_k^{n-1} = C_k^n,$$

$$\sum_{i=j}^{n-1} C_j^i = C_{j+1}^n.$$

More work on conditional probability, repeated trials, and random walks. Simple problems in hypothesis testing (e.g.: Ten tosses of a coin result in eight heads. With what confidence can you reject the hypothesis that the coin is honest?). Elementary expectation problems where expectation is thought of simply as a weighted average (e.g.: How much should one expect to win on a roll of a die if the payoff when n turns up is n^2 dollars?). The same problem posed using a nonsymmetric spinner instead of a die. Common distribution functions; measures of spread.

Course 4. Functions

1. Real Functions (10%)
2. Algebraic Functions (20%)
3. Exponential and Logarithmic Functions (15%)
4. Transformations and Matrices (20%)
5. Trigonometric Functions (15%)
6. R^2 and the Dot Product (20%)

The purpose of this course is two-fold: to present a satisfying culmination to the four-course sequence and to prepare the student to continue in mathematics with a calculus course such as Mathematics 1 [page 44]. Both of these goals are met in the context of a course centered around the function concept. Preparation for calculus is accomplished by studying special real functions, namely, the elementary functions; study of special functions of the plane (affine transformations) provides an appropriate dénouement of the four-course sequence by bring together arithmetic, algebra, and geometry in a transformation approach to plane geometry.

1. Real Functions (10%)

Brief review of the general concept of function from both the rule-of-assignment and ordered-pair points of view; specialization to the case where the domain and range are real numbers. Simple examples of functions--some artificial, some from science or business. Graphing and reading graphs. Graphs of real functions make it easier to think of them as mathematical objects in their own right, subject to operations as are other mathematical objects. Addition, subtraction, multiplication, division, and composition of functions viewed graphically as well as algebraically. Various additive and multiplicative groups of functions. One could look for rings, vector spaces, or even algebras of functions if that much abstract algebra is available.

2. Algebraic Functions (20%)

Real functions specialized to polynomial functions with emphasis on linear and quadratic functions. Slope and equations of straight lines, zeros of polynomials, and the factor theorem. Graphical interpretation of linear functions in the plane; geometric interpretation of linear functions on the number line in terms of stretches, shrinks, slides, and flips. (This suggests considering analogous functions of the plane and presents a natural opening for the discussion of general mappings of the plane and then the special types of mappings which are important for geometry, namely, translations, rotations, magnifications, and reflections. A discussion of transformation geometry may be included at this point.)

Increasing and decreasing real functions; there is no analogous concept for plane functions since the plane is not ordered. The completeness of the real number system and the Intermediate Value Theorem done at an intuitive level; invertible functions, both in the context of functions of the plane and in the context of real functions.

For real functions this leads to work with roots and rational exponents. Quadratic equations and various explicit algebraic functions.

3. Exponential and Logarithmic Functions (15%)

It is not reasonable to give a rigorous development of exponential functions. After adequate study of a^x for x rational and after some geometric motivation, the existence of a^x for x real should be assumed. The "laws of exponents" need emphasis. Other isomorphisms and homomorphisms recalled. Graphs of exponential functions and combinations thereof; logarithm functions defined as inverses of exponential functions, their properties derived from the properties of exponential functions; a minimal amount of computational work with common logarithms.

4. Transformations and Matrices (20%)

Having just completed computational work with some real functions, one can naturally ask whether various functions of the plane can also be given a concrete numerical representation. This leads to linear algebra and the study (in R^2) of vectors, dependence, independence, basis, linear transformation, matrix representation of linear transformations, matrix multiplication, and transformation composition.

5. Trigonometric Functions (15%)

The sine and cosine functions introduced by recalling the trigonometric ratios (Course 2); their definitions in terms of the winding function. The other trigonometric functions defined, graphs drawn, and questions of periodicity and invertibility entertained. Rotation matrix derivation of the addition formulas for sines and cosines. Proof of other trigonometric identities. The Pythagorean Theorem and its converse recalled and the Law of Cosines proved as a generalization.

6. R^2 and the Dot Product (20%)

The dot product motivated by the Law of Cosines as a measure of perpendicularity. The chain of ideas from dot product through length to metric traced. The central role played by this metric in currently popular axiomatic developments of geometry. The geometric and algebraic significance of the determinant function for 2×2 matrices.

Glossary of Symbols

SYMBOL	MEANING
W	$\{0, 1, 2, \ldots\}$, the set of whole numbers
Z	The set of integers
Q	The set of rational numbers
R	The set of real numbers
C	The set of complex numbers
Z^+, Q^+, R^+	The set of positive elements of Z, Q, R, respectively
Z_n	The set of integers modulo n
$x \in A$	The element x belongs to the set A
$A \subset B$	The set A is a subset of set B
$A \times B$	$\{(x,y)\colon x \in A \text{ and } y \in B\}$
A^2	$A \times A$
A^3	$A \times A \times A$
$\Rightarrow$	implies
$a \mid b$	a divides b
GCD	Greatest common divisor
LCM	Least common multiple
C_r^n	$\dfrac{n!}{r!(n-r)!}$
P_r^n	$\dfrac{n!}{(n-r)!}$
2^A	The set of all subsets of A
$A[x]$	The set of all polynomials in one indeterminate with coefficients in A