OF COURSE AND COURSES*

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Mr. President, members of the Assocation, and guests. I am here to retire, but I do not wish to be retiring. Both ends can be accomplished if I address myself to the principal objective of the Mathematical Association of America: that of the steady and imaginative improvement of collegiate education in Mathematics.

Collegiate education in Mathematics must not be construed too narrowly. On the one hand, we cannot ignore high school mathematics; this is the source of our entering students with their manifold difficulties in Mathematics. On the other hand, we cannot neglect graduate instruction in Mathematics; this is the source of our new teachers and the goal of some of our ablest students.

Collegiate education in Mathematics needs the most imaginative and vigorous reform, for it is now beset by numerous troubles and inadequacies.

These are internal troubles. Many of our courses cleave valiantly to a weak and obsolete tradition. Calculus has a perspicuous and beautiful intellectual structure, but its usual presentation is distorted by the unhappy fact that each new "standard" calculus text must copy the weakness of a long line of equally imitative predecessors. Trigonometry is in worse state; the publishers and authors of trigonometry texts conspire to demonstrate in exhaustive detail the combinatorial fact that an infinity of different texts is possible. The demonstration can be given more briefly: just combine all the alternative orderings of the following: define first the trignometric functions of an acute angle, or first those of a general angle; first triangles by logarithms or triangles by natural functions; identities first or equations first, *etc.*, *etc.*, *etc.* Among the standard courses, College Algebra is in perhaps the worst shape. Years ago, when it was first established†, it perhaps made sense, but under the pressure from masses of weaker and weaker entering students, the course has been analytically continued by continued dilutions. Today "College Algebra" stands for a subject which ought to be taught in the high school and which has nothing to do with algebra.

These are only some of the internal troubles of collegiate mathematics. There are also numerous external troubles. On the one hand, the social scientists have discovered that Mathematics can be of use. They properly complain that calculus and linear algebra are currently taught with a view only to the engineering and physical applications. They urge needed reform, and they are under a great temptation to urge too much in the way of special courses addressed primarily to the social scientists. They may fall thereby into the basic error of assuming that one can prepare students to make the necessary new applications

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† The title "College Algebra" is an old one; there are American books under this title by Thomson and Quimby (1880), Bowser (1888) and J. M. Taylor (1889)—and perhaps others earlier.

of novel mathematical ideas by training them primarily in the old applications of older ideas. It cannot be too often reiterated that the aim of collegiate mathematics is the understanding of mathematical ideas *per se*. The applications support the understanding, and not vice versa.

Mathematics, like the rest of the academic community, is also bedevilled by the current fashion of general education. In its basic assumption this fashion was necessary. The emphasis on the great books was a needed corrective to our sorry adherence to text books that were far from great. The emphasis on an education that is broad and common to all students was a necessary answer to the American problem of giving a mass education to masses of students barely removed from illiteracy. However, the general educators cannot avoid one basic observation about the intellectual status of our age: there in fact is no common and accepted conceptual organization of present day knowledge, hence there is no possibility of constructing courses to convey this organization. For this reason and others, the fashion of general education has now been over-extended. As evidence, I cite you the fact that every University president treats general education as the Russians treat science: each president claims that his university discovered it first, or at any rate discovered its only true realization. These enthusiastic presidents miss the point that general education was probably started without benefit of presidents, as at Columbia, which in effect had no president, and as at Chicago, which had a Chancellor and not a president.

In the face of these fashions and troubles, we must keep our minds fixed on the real objectives of collegiate education in Mathematics. We must contrive ever anew to expose our students—be they general students or specialized students—to the beauty and excitement and relevance of mathematical ideas. We must set forth the extraordinary way in which mathematics, springing from the soil of basic human experience with numbers and data and space and motion, builds up a far-flung architectural structure composed of theorems which reveal insights into the reasons behind appearances and of concepts which relate totally disparate concrete ideas.

The program of our sessions today will amply cover the various problems which here arise for the more elementary collegiate courses. I therefore propose to turn my attention to the problems which arise in advanced undergraduate and beginning graduate courses. It is here that the sweeping reforms are necessary; the curriculum in advanced mathematics must be so overhauled that it can set forth the real structure of Mathematics as it is today. This structure can no longer be presented by piecemeal courses, for it is simply no longer true that advanced mathematics can be split neatly into compartments labelled "algebra," "analysis," "geometry," and "applied mathematics."

The fact that these subdivisions have ceased to be relevant may be seen most strikingly by observing how many of the most significant current discoveries in Mathematics refuse to be classified in the old compartments. Basic problems in functions of several complex variables have recently been solved by using the notion of a *faisceau* or "sheaf"—a notion coming straight from algebraic topology. Hilbert's fifth problem has been solved—using techniques derived from topology, from Hilbert-space theory, and from Lie groups. In which compartment does this lie? In algebra, some of the most fruitful advances arise because of the demands for new algebraic tools for use in topology or in algebraic geometry. One is forced to the realization that algebra is not, and indeed never has been, an independent discipline. Modern analysis is replete with distributions, with rings of operators and with representations, so much so that it is difficult to say whether it is analysis, or a wholly new subject.

You may protest that I am talking about research and not about education, and that this should be the business of the Society, not the Association. I answer, first: The character and direction of current research is the best indication of the ideas which we ought to be teaching. I answer, second: One of our main responsibilities is that of training the research mathematicians of the future. American Mathematics has made tremendous strides forward in the last two decades; an essential ingredient in this advance has been the infusion of European mathematical talent. In the decades to come, we must produce a similar infusion on our own and from our own students.

In training research mathematicians the old ideals are not sufficient. When research mathematics was first developed in this country, it was necessary that the emphasis be just on the idea of research: something new, something unpublished, with no especial attention to the significance of the results or their place in Mathematics. Now it is necessary that the research be done with full appreciation of the significance of the various parts of our science, and with full availability of the techniques from the various disciplines which may become necessary even for the apparently most specialized problem.

My topic is "Of Course and Courses;" I mean "of the courses which may be constructed to produce an integrated course in mathematics." As an existence proof for such a course, I offer you the *Elements de Mathematique* by that dipsomaniac, Nicholas Bourbaki. He has achieved a Gallic version of a conceptual integration of mathematics; you may point out that he has paid a high price by way of abstraction and length, but I submit you that the goal is worth the price.

To be more specific, I would like to describe the mathematics curriculum which has been developed by the Department of Mathematics at the University of Chicago. This curriculum is intended to carry serious students of Mathematics from the beginning of the Junior year of college through a Master's degree in Mathematics. The objective is the ambitious one of providing a complete introduction to all those ideas which are basic to Mathematics in the sense that they occur in their several aspects in more than one part of Mathematics.

I must confess at once that our curriculum is still divided for administrative convenience into algebra, geometry, and analysis, but our emphasis is and should be on the use of each of these techniques in the other fields. Some subdivision is necessary because our courses must appear in units of one quarter each, but we view these individual courses—and we hope that the students so view them—as part of a larger pattern.

In algebra we present a sequence of five quarter courses. The first of these is an introduction to abstract algebra by way of number theory, group theory, and vector spaces. That abstract notions require introduction by way of examples is clear; this is done. But abstract notions are not "hard." The young students like to think in these terms. It is a happy omen for the progress of civilization that here (and elsewhere in the curriculum) the beginning student takes more easily to abstractions and generalities than does his professor, who had to understand them the hard way.

The beginning algebra course also offers an ideal place for the introduction of the notion of a mathematical proof. This must be done, even for students who don't aim to be mathematicians. Algebra provides a better locus for this introduction than does the traditional plan of introducing the student to rigor and complex variables at the same time—for this type of introduction led to the assumption that rigor was the same thing as epsilontics. Bringing in rigor with algebra brings it in sooner and in simpler form.

The second and third algebra courses deal completely with vector spaces and linear transformations with especial attention to such topics as the invariant description of linear transformations (elementary divisors and the Jordan and rational canonical forms), the properties of quadratic forms (in particular, the principal axis theorem and its geometrical meaning), and finally the various relevant general notions, such as invariants, equivalence, and dual vector spaces. These topics belong early in the training of a mathematician. The proper treatment of calculus for functions of several variables requires vector ideas; the budding statistician and the coming physicist need them; modern analysis is unthinkable without the notion of linear dependence and all that flows from it. Throughout these courses the infusion of a geometrical point of view is of paramount importance. A vector is geometrical; it is an element of a vector space, defined by suitable axioms—whether the scalars be real numbers or elements of a general field. A vector is not an n-tuple of numbers until a coordinate system has been chosen. Any teacher and any text book which starts with the idea that vectors are n-tuples is committing a crime for which the proper punishment is ridicule. The n-tuple idea is not "easier," it is harder; it is not clearer, it is more misleading. By the same token, linear transformations are basic and matrices are their representations.

The fourth algebra course reverts to abstract algebra proper, with a treatment of rings, homomorphisms, ideals, groups, normal subgroups, the Jordan-Hölder theorem, and the Sylow theorems. The important notion of tensor product of groups and of spaces is often included and probably should always be there, for it is needed to understand modern geometry and modern algebraic topology. It is proved that every finitely generated abelian group is a direct sum of cyclic groups. Indeed, this theorem is an almost ideal example of an algebraic "structure theorem." I wish that I could report that this theorem is always established as a special case of the corresponding theorem about modules over a principal ideal ring, but I must confess that the latter notion goes down hard—even though it is necessary as a connection between the groups treated in this course and the canonical forms for linear transformations from the previous course. In any event, abstract algebra is done with emphasis on the basic idea of homomorphism—the object of algebra is not just the study of a mathematical system *per se*, but of mappings of one such system into another. This idea is not restricted to algebra, and thus underlines again the conceptual unity of Mathematics.

The fifth course of the algebra sequence deals with the Galois theory—one of the most beautiful examples of a self-contained mathematical discipline, and one of the most convincing demonstrations of the power of the notions "homomorphism" and "automorphism." The course terminates with the basic structure theorems for linear algebra, which are the models for current developments in the structure of rings and in the study of rings of operators on a Hilbert space.

The geometry sequence consists of three courses. The first of these takes up analytic geometry, already treated to some extent in the calculus course, and carries it further. At the same time, analytic projective geometry appears, not in the flowery decadence which this subject reached in its American heyday, but as a necessary introduction of geometrical ideas of duality and of locus, and as a first demonstration that geometry starts with the space of ordinary experience but has the fertility to conceive new spaces representing and extending that experience.

The second geometry course deals with the foundations of geometry. The axiomatics of projective geometry, with the introduction of coordinates on the basis of these axioms, is one of the most beautiful instances of the power of axiomatic method, and at the same time emphasizes how geometry leads to algebra and how abstract notions like those of endomorphism have concrete meanings. The course continues with n-dimensional projective geometry—collineations, correlations, and the classification of hyperplanes. Finally, it turns to non-euclidean and inversive geometry, where these geometries are given in terms of subgroups of the projective group, thus illustrating again the relevance of group theory.

The third geometry course treats differential geometry. Here again we see the inadequacies of the "standard" course in this subject in comparison with the actual state of Mathematics. It is no longer sufficient to consider curves, surfaces, curvature, torsion, and first and second fundamental forms, all as an elegant application of the calculus. One must pay attention to differential geometry as it now is: with this in view, the course omits some of the more extended and uninteresting parts of the classical doctrine and instead provides an introduction to ideas of differential geometry in the large (the four vertex theorem, the theorem on turning tangents, *etc.*) and to the modern ideas of differentiable manifolds. The geometry on a surface is just not adequate if it is done only "locally"—the consideration of changes of coordinate systems must build up to the notion of a differentiable manifold, and the study of the first and second differential form must lead to the notion of exterior differential forms on a manifold. The business of the young mathematicians is with ideas, and these are the ones he must meet in this field, the sooner, the better.

Finally I turn to analysis. The ideas of calculus are presented to freshmen and sophomores, as usual. We attempt to treat calculus with proper attention to rigor and more rapidly than is the custom, this by trimming some of the barnacles which have accumulated through the years. One just doesn't need an infinity of different applications of the definite integral! The calculus for several variables is a hard nut; for example, the proper treatment of Stokes' and Green's theorem really requires the notion of exterior forms; I must confess that we have not yet found a good way to introduce these ideas where they belong in calculus.

The final course in our calculus sequence covers various topics in advanced calculus. It starts with the idea of uniform convergence for series; this is then applied to establish the standard results for Fourier series. The idea of successive approximation is then introduced and used to provide existence and continuity proofs for differential and integral equations. The course terminates with a survey of the methods of complex variable theory up through contour integration, leaving the more sophisticated treatment of these topics for a later course on the subject.

The further reaches of analysis can no longer be treated in isolation from other topics in Mathematics; properly construed, they rest essentially upon algebra and topology and in turn fructify these subjects. Hence the student next takes a sequence of two courses in topology. The first of these, on point sets and metric spaces, starts with the basic algebra of sets, including cartesian products, and develops the cardinal and ordinal numbers and the technique of using Zorn's lemma in its various forms. Then comes a study of metric spaces, including completeness and compactness; the power of the notion of a metric is illustrated by showing how uniform convergence can be realized as convergence in the metric of suitable function space. This treatment of metric spaces presents the ϵ-δ technique of analysis in its proper setting and motivates the more general notions of topology.

The latter ideas are covered in the second course, starting with the definition of a topological space and continuing through the study of the separation axioms, connectivity, and compact spaces. The connection with metric spaces is re-established by means of metrization theorems. The possibility of various topologies on function spaces is used to illustrate the breadth and power of the general notion of a topological space, while the compactification theorems illuminate the processes of constructing new spaces from old ones. The course ends with the Weierstrass-Stone approximation theorem and, when possible, with a brief treatment of fundamental groups and of covering spaces. The last topic in particular has great merit, with application on the one hand to topological groups, and on the other to algebraic topology, where it serves as an introduction to the more general notions of fibre bundles.

On this basic array of topological instruments the student is then well prepared to take up the further essential topics in analysis; a systematic study of complex variable theory (one quarter) and an examination of measure theory and Lebesgue and Stieltjes integrals (one quarter). The traditional material of a course in real variables is thus subdivided and dispersed—as it should be. It appears in part in the more general notions of metric spaces, and in part in more advanced specialized courses.

This then is the curriculum which my colleagues at the University of Chicago have laid out to cover mathematical ideas on a broad front. For the student who aims to teach, it provides a sound knowledge of what mathematics is about. For the student going further in Mathematics, it provides a broad base and technical equipment for the attack on further ideas—functional analysis, topological groups, algebraic topology, differentiable manifolds, algebraic geometry, the structure of rings, Lie groups, and the rest. The basic requirement on a sound curriculum is precisely that it give the necessary background for these and other studies, and thereby exhibit the unity of Mathematics.

In so outlining the curriculum which has been set up at Chicago, I do not wish to claim that it is perfect or unique. It has some gaps (for example, an introduction to partial differential equations). Other quite different organizations of material could be made; indeed some of these organizations could better stress the relation between various ideas. I claim then no perfection, I submit rather that this is but one first approximation to the pressing need for basic mathematical training. One must design modern and coherent curricula, cleared of traditional impedimenta and providing rapid access to new and general ideas, so that they can then be applied in special domains. I submit that this objective is a vital one and urge that you go and do better.

MOTIVES AND TRENDS IN MATHEMATICS*

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1. The main idea of this paper. The purpose of this paper is to set forth the thesis that in basic discoveries in science generally, and in mathematics particularly, the ordinary motives of men do not seem to operate. For example, if we list as ordinary motives (1) the desire for personal prestige, (2) the wish to create something useful—that is, the utility motive—and (3) (strongest of all)

* Condensed from a paper presented at the annual meeting of the Southeastern Section of the Mathematical Association of America at Alabama Polytechnic Institute, March 13–14, 1953.