Curriculum Burst 10: Maximum Ratio

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Let x and y be two-digit positive integers with mean 60.

What is the maximum value of the ratio $\frac{x}{y}$?

SOURCE: This is question # 7 from the 2011 MAA AMC 12b Competition.

QUICK STATS:

YouTube
Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Algebraic expressions; Arithmetic mean

COMMON CORE STATE STANDARDS

A-SSE.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

A-CED.3: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

A-CED.4: Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure

PROBLEM SOLVING STRATEGIES

ESSAY 1: SUCCESSFUL FLAILING

THE PROBLEM-SOLVING PROCESS:

The key step:

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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Our question has two sentences and my reaction to each is different.

First sentence reaction:

Okay. Not bad. I can process that.

Second sentence reaction:

Whoa! Where's that coming from?!

Okay. Deep breath! I'll start by focusing on the part I am comfortable with.

We have two numbers x and y (actually they are each positive integers, each with two digits) and their mean is 60. So:

$$\frac{x+y}{2}=60.$$

It feels natural to next write:

$$x+y=120.$$

So far so good. But now on to the second sentence.

A good way to deal with something a little scary is to step back from it and ask

Roughly, what are we being asked to do?

The second sentence asks us to …

DO SOMETHING WITH $\frac{x}{y}$.

Okay…. Do what? All we have is $x+y=120$. We can write $x=120-y$ and so:

$$\frac{x}{y}=\frac{120-y}{y}.$$

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REREAD THE QUESTION
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We want the maximum value of $\frac{x}{y}$, that is, of $\frac{120-y}{y}$ (where y is a positive two-digit integer).

We could try substituting in the value $y=10$, and then $y=11$ all the way up to $y=99$ and see which gives the largest output. But that seems like a lot of work.

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AVOID HARD WORK
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The expression $\frac{120-y}{y}$ looks complicated. Is there are way to make it look simpler?

The only algebra I can think to do is to actually divide each of the terms in the numerator by the denominator y:

$$\frac{120-y}{y}=\frac{120}{y}-\frac{y}{y}=\frac{120}{y}-1.$$

We want the maximum value of $\frac{120}{y}-1$.

Woohoo! That will happen when y is as small as it can be! So we are done: Put in $y=10$ to get the largest value $\frac{120}{10}-1=11$.

HANG ON! We have $x+y=120$. So if $y=10$, $x=110$ is not a two-digit number. Sneaky! The smallest y can be is $y=21$ (with $x=99$) and so the largest value of the ratio is actually $\frac{120}{21}-1$. This is, of course(!), the same as $\frac{99}{21}=\frac{33}{7}=4\frac{5}{7}$.

Extension: The geometric mean of x and y is $\sqrt{xy}$, their harmonic mean is $\frac{2}{(1/x)+(1/y)}$, and their quadratic mean is $\sqrt{x^2+y^2}$. (See the Dec2012 Cool Math essay at www.jamestanton.com/?p=1072.)

Suppose x and y are two positive integers, each with at most 4 digits, and with geometric mean 60. What is the maximum possible value of the ratio x/y? (Repeat for harmonic mean 60 and then for quadratic mean 60!)

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.