Curriculum Inspirations

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MAA

Curriculum Burst 11: Logarithms and Exponents

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Let $f(x)=10^{10x}$, $g(x)=\log_{10}\left(\frac{x}{10}\right)$, $h_1(x)=g(f(x))$, and $h_n(x)=h_1(h_{n-1}(x))$ for integers $n\ge 2$.

What is the sum of the digits of $h_{2011}(1)$?

SOURCE: This is question # 17 from the 2011 MAA AMC 10a Competition.
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QUICK STATS:

You Tube
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MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPICS

Exponents and Logarithms; Function notation; Recursive formulas.

COMMON CORE STATE STANDARDS

F-FB.5: Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

F-IF.2: Use function notation, evaluate functions for inputs in their domains and interpret statements that use function notation in terms of a context.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure

PROBLEM SOLVING STRATEGIES

ESSAY 1: **SUCCESSFUL FLAILING: LIST WHAT YOU KNOW**
ESSAY 2: **DO SOMETHING**

THE PROBLEM-SOLVING PROCESS:

As always…

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question looks scary! I see logs and powers and fractions. Just the visual look of the question gives me the heebie-jeebies. Plus I have all the queasy feelings still lingering with me from first learning exponents and logarithms in class. They were far from natural and "obvious" to me.

Okay. Let me settle down and give this query a try.

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Hmm. How do I start?

LIST WHAT YOU KNOW

Well… *logarithm* is another word for *power*: $\log_{10}(x)$ is the power of ten that gives the answer x. If I use logarithms with powers of ten, things collapse and simplify. For example, $\log_{10}(10^N)$ is N - the power of ten that gives the answer 10^N is obviously N !

(See www.jamestanton.com/?p=553 and www.jamestanton.com/?p=556 for some history and mathematics on logarithms.)

We have $f(x)=10^{10x}$ and $g(x)=\log_{10}\left(\frac{x}{10}\right)$ and the question wants us, for starters, to handle $g(f(x))$. Okay,

$$\begin{aligned} g(f(x)) = g(10^{10x}) &= \log_{10}\left(\frac{10^{10x}}{10}\right) \\ &= \log_{10}\left(10^{10x-1}\right) \\ &= 10x-1 \end{aligned}$$

This is $h_1(x)$. We have: $h_1(x)=10x-1$.

[Whoa! The formulas did indeed collapse!]

The next part of the question is scary!

DO SOMETHING

"$h_n(x)=h_1(h_{n-1}(x))$ for integers $n\geq 2$."

I don't know what to make of this. Let's just try playing with some small numbers, small values of n, that is. We've got $h_1(x)=10x-1$. So…

$$\begin{aligned} h_2(x)=h_1(h_1(x)) = h_1(10x-1) &= 10(10x-1)-1 \\ &= 100x-11 \\ h_3(x)=h_1(h_2(x)) = h_1(100x-11) &= 10(100x-11)-1 \\ &= 1000x-111 \\ h_4(x)=h_1(h_3(x)) = h_1(1000x-111) &= 10(1000x-111)-1 \\ &= 10000x-1111 \end{aligned}$$

Okay .. It seems clear what is going on. $h_{2011}(x)=100...00x-11..11$ with 2011 zeros and 2011 ones in the big-digit numbers that appear.

Umm. What does the question want?

The sum of the digits of $h_{2011}(1)$.

Okay … $h_{2011}(1)=100..00-11..11$.

Hmm. What number is this?

Back to smaller examples first:

$10-1=9$

$100-11=89$

$1000-111=889$

$10000-1111=8889$

So $h_{2011}(1)=88...89$ with 2010 eights and one nine. The sum of the digits is $8\times 2010+9=16080+9=16089$.

Extension 1: Find a formula for the sum of the digits of $h_N(1)$. **Extension 2:** Quickly, what is the sum of the digits of $h_{367}(94)$?

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