

Curriculum Inspirations

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MAA American Mathematics Competitions



Curriculum Burst 13: Painted Cube

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Each face of a cube is painted either red or blue, each with probability $1/2$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

SOURCE: This is question # 23 from the 2004 MAA AMC 10b Competition.

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPICS

Probability; Counting

COMMON CORE STATE STANDARDS

- S-CP.2:** Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.
- S-CP.5:** Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.
- S-CP.9 (+):** Use permutations and combinations to compute probabilities of compound events and solve problems.

MATHEMATICAL PRACTICE STANDARDS

- MP1** Make sense of problems and persevere in solving them.
- MP2** Reason abstractly and quantitatively.
- MP3** Construct viable arguments and critique the reasoning of others.

PROBLEM SOLVING STRATEGIES

ESSAY 1: [SUCCESSFUL FLAILING](#)



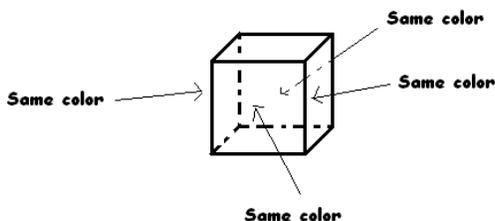
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THE PROBLEM-SOLVING PROCESS:

The key step is always ...

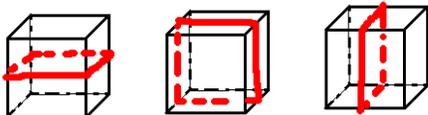
STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I don't feel too overwhelmed by this question. Its first part does seem a bit fussy with the word "independently," but I can read past it and see that the question is simply about painting the faces of a cube. We want four faces in a "ring" around the cube all the same color.



The top and bottom faces can be any color we want. So basically all we have to do is figure out how many different colorings there are for the top face, the "ring" and the bottom face, and compare that to the total number of colorings possible. Is that right?

Hang on! We get to move the cube and then place it on the surface. So actually there are three types of "rings" to consider?



Or maybe this doesn't matter? Now I am confused!

Alright. Let's state two obvious cases that work for the question and I can count.

The cube is all red: 1 way
The cube is all blue: 1 way

The next easiest case would be:

One face blue, five red.

Since I can move the cube, does this mean there is only 1 way for me to paint the cube in this pattern? Hmm. But

there are six faces, so maybe this can be done 6 different ways? Okay. I am very confused!

I've always found probability questions hard. I never seem to know when things are to be considered as the same or as different!

Okay...deep breath. My trouble in this question is whether or not to consider the faces of the cube as equivalent or distinct. So let me ask this ... Does the question care if the faces are different, say they were numbered one through six like a die? Would painting the faces of a die red and blue, philosophically, be exactly the same question? Well... actually yes. The mathematics doesn't "care" if the cube happens to be a die. Let's assume it is one!

One face blue, five red: 6 ways
One face red, five blue: 6 ways

Two opposite faces blue, four red: 3 ways.
Two opposite faces red, four blue: 3 ways.

So there are a total of twenty colorings that possess a "ring" of the same color.

And as there are 2 ways to color face one, 2 ways to color face two, and so on, all way to 2 ways to color face six, there are $2^6 = 64$ possible colorings in total. The probability we seek is $\frac{20}{64} = \frac{5}{16}$.

Oooh! The details we skimmed over. Did we attend to them: each face is painted "with probability $1/2$ " and color of each face is determined "independently"? This is just saying each face-color is equally likely to be chosen and there is no bias among the possible painting schemes for the faces. We assumed this in our counting. Phew! We were meant to. We're good!

Extension: How many essentially different two-color face-painting patterns are there for a cube with identical blank faces - if the cube is held fixed in space? If the cube can be moved and tossed about? If the cube can be moved and tossed about and viewed in a mirror?