Let \( f(x) = ax^2 + bx + c \), where \( a, b, \) and \( c \) are integers. Suppose that \( f(1) = 0, 50 < f(7) < 60, 70 < f(8) < 80, \) and \( 5000k < f(100) < 5000(k + 1) \) for some integer \( k \). What is \( k \)?

**SOURCE:** This is question # 20 from the 2011 MAA AMC 12a Competition.
THE PROBLEM-SOLVING PROCESS:

The right place to begin...

**STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I feel like I can “see through” this question. It is about a quadratic \( ax^2 + bx + c \) (and I have studied quadratics in great depth in algebra II) with a whole bunch of complicated details that, in the end, seem only to be about plugging in numbers. That feels do-able. So I am just going to cross my fingers and follow my nose on this one and just start with the strategy...

**DO SOMETHING**

Okay, reading through the question now with care, I see we have a quadratic:

\[
f(x) = ax^2 + bx + c.
\]

And we are first told: \( f(1) = 0 \). No problem, this means:

\[
a + b + c = 0.
\]

Next we are told some complicated things about \( f(7) \) and \( f(8) \). Well...

\[
f(7) = 49a + 7b + c
gen \quad f(8) = 64a + 8b + c
\]

I am not sure what’s next. What specifically are we being told about \( f(7) \) and \( f(8) \)?

Now \( 50 < f(7) < 60 \) is telling me that \( f(7) \) is a number in the 50s. (Is it obvious that \( f(7) \) is an integer?) And \( 70 < f(8) < 80 \) says \( f(8) \) is an integer in the 70s.

Let me write:

\[
49a + 7b + c = \text{fifty something}
gen \quad 64a + 8b + c = \text{seventy something}
\]

We still have:

\[
a + b + c = 0
\]

I am not sure where this is taking me. But it does look like a system of three equations in three unknowns (with extra “unknownishness” of where exactly I am in the fifties and the seventies!)

Shall we just try some standard algebra: subtract one equation from another to eliminate a variable? We should make use of the equation with the zero in it.

Subtracting this third equation from the first gives:

\[
45a + 7b + c = \text{fifty something}
\]

\[
a + b + c = 0
\]

\[
48a + 6b = \text{fifty something}
\]

Helpful? Hmm. Subtracting the third equation from the second gives:

\[
63a + 7b = \text{seventy something}
\]

I am still not sure if this is at all helpful.

**ENGAGE IN WISHFUL THINKING**

We have:

\[
48a + 6b = \text{fifty something}
\]

\[
63a + 7b = \text{seventy something}
\]

If we knew what the actual numbers are on the right, we could then solve for \( a \) and \( b \) and use \( c = -a - b \) to find \( c \). Then we would know \( f(x) \) completely and we could just compute \( f(100) \) to solve the problem! Is there any way to know those numbers?

Oh heavens! \( 63a + 7b \) is a multiple of seven, and it must be a multiple of seven in the seventies (and not be 70 itself). It can only be 77!

What about \( 48a + 6b \)? It is a multiple of six in the fifties. It can only be 54! (The author of this question was very clever!)

Solving gives \( a = 2 \), \( b = -7 \) and \( c = 5 \). So

\[
f(100) = 2(100)^2 - 7(100) + 5 = 19,305\]

which is between the third and fourth multiples of 5000. So \( k = 3 \). Wow!

**Extension:** Design an equally clever problem like this, but for a cubic!