Curriculum Burst 19: Eliminating Roots

By Dr. James Tanton, MAA Mathematician in Residence

Which of the following is equal to $\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}$?

(A) $3\sqrt{2}$ (B) $2\sqrt{6}$ (C) $\frac{7\sqrt{2}}{2}$ (D) $3\sqrt{3}$ (E) 6

SOURCE: This is question # 16 from the 2010 MAA AMC 10a Competition.

QUICK STATS:

YouTube

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MAA AMC GRADE LEVEL

This question is appropriate for the 10^{th} grade level.

MATHEMATICAL TOPICS

Algebra

COMMON CORE STATE STANDARDS

A-SSE.2: Use the structure of an expression to identify ways to rewrite it.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGIES

ESSAY 3: **ENGAGE IN WISHFUL THINKING**

THE PROBLEM-SOLVING PROCESS:

As usual …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question gives me the heebie-jeebies. It looks very scary. How am I meant to evaluate something as complicated as that?

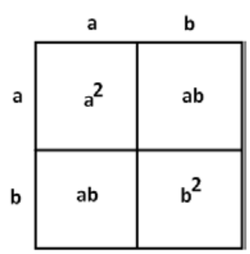
The question would be so much easier if the square roots weren't there.

ENGAGE IN WISHFUL THINKING

Okay then. I wish the square roots weren't there! How do I eliminate square roots? Answer: By squaring. Let's do it! Let's work out

$$\left(\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}\right)^2.$$

Comment: I am not going to fall into the common trap of thinking $(a+b)^2$ is a^2+b^2. I remember the picture below that makes it clear that $(a+b)^2=a^2+b^2+2ab$.



Here goes. $\left(\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}\right)^2$ equals:

$$9-6\sqrt{2}+9+6\sqrt{2}+2\sqrt{9-6\sqrt{2}}\cdot\sqrt{9+6\sqrt{2}}$$

Ohh. Is that better?

It simplifies a little to:

$$18+2\sqrt{9-6\sqrt{2}}\cdot\sqrt{9+6\sqrt{2}}$$

Should I square again and try to eliminate the next set if square roots? (I have a feeling square roots will stick around if I do. Hmm.)

What can I do with what I have so far?
We can reduce the product of roots to a single root using $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$. This gives:

$$18+2\sqrt{\left(9-6\sqrt{2}\right)\left(9+6\sqrt{2}\right)}$$

Oh. That is too beautiful! Look under the square root sign and see the a product of the form $(x-y)(x+y)$. This is just x^2-y^2, the difference of two squares formula (backwards, I suppose). Our expression is:

$$18+2\sqrt{9^2-\left(6\sqrt{2}\right)^2}$$

which is

$$18+2\sqrt{81-36\cdot 2}$$
$$=18+2\sqrt{81-72}$$
$$=18+2\sqrt{9}$$
$$=24$$

Cool! But that is not in the list of answers.

Actually, that's not the answer because we started by squaring the quantity. The answer squared is 24, so this means $\sqrt{9-6\sqrt{2}}+\sqrt{9+6\sqrt{2}}=\sqrt{24}$, which is option (B).

Extension: We decided not to square $18+2\sqrt{9-6\sqrt{2}}\cdot\sqrt{9+6\sqrt{2}}$ and try to eliminate the roots again. But if we give the quantity a name, say, $18+2\sqrt{9-6\sqrt{2}}\cdot\sqrt{9+6\sqrt{2}}=k$, write $\sqrt{9-6\sqrt{2}}\cdot\sqrt{9+6\sqrt{2}}=\frac{1}{2}(k-18)$ and then square, we might be in good stead to continue with the problem after all. Are we?

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