Curriculum Burst 27: Hats and Gloves

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> In a room, $\frac{2}{5}$ of all the people are wearing gloves, and $\frac{3}{4}$ of the people are wearing hats. What is the minimal number of people in the room wearing both hats and gloves?

SOURCE: This is question # 20 from the 2010 MAA AMC 8 Competition.

QUICK STATS:

YouTube

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MAA AMC GRADE LEVEL

This question is appropriate for the 8^{th} grade level.

MATHEMATICAL TOPICS

Probability and Statistics: Counting sizes of sample spaces.

COMMON CORE STATE STANDARDS

7.SP.8b: Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 10: GO TO EXTREMES

THE PROBLEM-SOLVING PROCESS:

As always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

The first thing I notice about this question is that I am not told how many people are in the room! This seems strange. How can I possibly answer a question about an actual count of people?

Actually, I am being asked about the minimal possible number of people in the room each wearing both a hat and gloves. Is that better?

Might we assume then we have the smallest total number of people in the room too? Is that helpful?
(Making this assumption doesn't feel like it helps.)

Hmm. I don't know. But it does seem like we are following the strategy:

GO TO EXREMES!

Okay then, what is the smallest possible number of people in the room? Whatever the number is, we know that $\frac{2}{5}$ of them are wearing gloves (so the count of people better be a multiple of five) and $\frac{3}{4}$ of them are wearing hats (so the count of people in the room must also be a multiple of four). The smallest number that is both a multiple of four and of five is 20. So let's go with this minimal count of just 20 people in the room. (And if we run into a pickle later on, we can adjust this number to 40, or to 60, and so on.)

Now what? Two-fifths of 20 is eight. We have 8 people wearing gloves. And $\frac{3}{4}\times 20$ gives 15 people wearing hats. (Is it cold in this room?) So that's a total of $15+8=23$ people out of 20 wearing either hat or gloves?

No. There must be overlap, just as the question asks for.

I guess we've just argued with another extreme:

> Suppose there are 0 people wearing both items. This means we have 8 glove wearers and 15 hat wearers, which is impossible as there are only 20 people in the room.

What's the next extreme case for this line of reasoning?

Suppose there is exactly 1 person wearing both. That leaves 7 people wearing gloves alone and 14 people wearing hat alone. This is impossible as $1+7+14$ is larger than 20.

Plodding on …

Suppose there are exactly 2 people wearing both. This leaves 6 people wearing gloves alone and 13 people wearing hat alone. This is again impossible as $2+6+13>20$.

Suppose there are exactly 3 people wearing both. This leaves 5 people wearing gloves alone and 12 people wearing just a hat. And bingo! $3+5+12=20$

That's it! A bigger number of people in the room would only make this count larger, and we proved – by actually checking - that there can't be just 0, 1 or 2 people wearing both items. 3 is, for sure, the absolute minimal number.

Extension: a) In the original problem, what is the largest possible fraction of people wearing neither hat nor gloves?

b) In a very cold room $\frac{2}{5}$ of the people are wearing gloves, $\frac{3}{4}$ of the people hats and $\frac{5}{7}$ of the people scarves. What is the minimum number of people wearing all three items? What is the largest possible fraction of people not wearing any of these items?

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