Curriculum Burst 29: Three-Digit Reversal

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The hundreds digit of a three-digit number is 2 more than the units digit.
The digits of the three-digit number are reversed, and the result is subtracted from the original three-digit number.
What is the units digit of the result?

SOURCE: This is question # 22 from the 2010 MAA AMC 8 Competition.

QUICK STATS:

YouTube
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MAA AMC GRADE LEVEL

This question is appropriate for the 8th grade level.

MATHEMATICAL TOPICS

Additive inverses.

COMMON CORE STATE STANDARDS

7.NS.1c: Understand subtraction of rational numbers as adding the additive inverse, $p-q=p+(-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 3: ENGAGE IN WISHFUL THINKING

THE PROBLEM-SOLVING PROCESS:

As always …

> **STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question takes a little time to process! We have a three digit number "abc" (like 154 or 753, for example), with hundreds digit two larger than the units digit. (Oops! Like 351 or 694, instead.) We then reverse this three-digit number to get "cba" and subtract:

$$\begin{array}{rccc} & a & b & c \\ - & c & b & a \\ \hline = & ? & ? & ? \end{array}$$

We want the units digit of the result.

Okay … so the digit a is two more than the units digit c. Let's write $a = c + 2$. Our calculation now looks like:

$$\begin{array}{rccc} & c+2 & b & c \\ - & c & b & c+2 \\ \hline = & ? & ? & ? \end{array}$$

ENGAGE IN WISHFUL THINKING

Okay. Look at the sum.

We have $c+2$ hundreds from which we subtract c hundreds. This leaves 2 hundreds.

We have b tens from which we subtract b tens. That leaves zero tens.

We have c ones from which we subtract $c+2$ ones. That leaves -2 ones(!).

So this gives the following answer… ???

$$\begin{array}{rccc} & c+2 & b & c \\ - & c & b & c+2 \\ \hline = & 2 & 0 & -2 \end{array}$$

The final digit is -2? Hmm.

Well, let's try some examples:

$$\begin{array}{r} 351 \\ -\ 153 \\ \hline =198 \end{array} \qquad \begin{array}{r} 694 \\ -\ 496 \\ \hline =198 \end{array} \qquad \begin{array}{r} 705 \\ -\ 507 \\ \hline =198 \end{array}$$

Is the answer always 198? Is that the same number as $2\,|\,0\,|\,-2$?

When we write a three-digit number "abc" we really mean a hundreds, b tens, and c ones. So the actual number is $100a + 10b + c$.

So the computation:

$$\begin{array}{rccc} & c+2 & b & c \\ - & c & b & c+2 \\ \hline = & ? & ? & ? \end{array}$$

is really:

$$100(c+2) + 10b + c - 100c - 10b - (c+2)$$

This equals:

$$\begin{aligned} &100c + 200 + c - 100c - c - 2 = \\ &= 200 - 2 \\ &= 198 \end{aligned}$$

So yes! It is two hundreds and negative two ones, and that equals 198, for sure, every time! The final digit is sure to be 8.

Question: Is it meaningful to consider a final digit of -2 the same as a final digit of 8? This question seems to suggest so! To what does a final digit of -6 correspond?

> **Extension:** Lulu was asked to compute $6872839 + 1439456$ and 5629×3. She wrote the following:
>
> $$\begin{array}{rccccccc} & 6 & 8 & 7 & 2 & 8 & 3 & 9 \\ + & 1 & 4 & 3 & 9 & 4 & 5 & 6 \\ \hline = & 7 | & 12 | & 10 | & 11 | & 12 | & 8 | & 15 \end{array}$$
>
> $5629 \times 3 = 15|18|6|27$
>
> Explain why she is absolutely correct mathematically. Show how to convert her answers to numbers the rest of the world would understand.

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