Curriculum Burst 33: A Trigonometric Length

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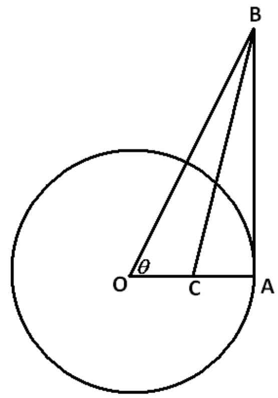
A circle centered at O has radius 1 and contains the point A. Segment $\overline{AB}$ is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on $\overline{OA}$ and $\overline{BC}$ bisects $\angle ABO$, then $OC =$



(A) $\sec^2\theta - \tan\theta$ (B) $\frac{1}{2}$ (C) $\frac{\cos^2\theta}{1+\sin\theta}$

(D) $\frac{1}{1+\sin\theta}$ (E) $\frac{\sin\theta}{\cos^2\theta}$

SOURCE: This is question # 17 from the 2000 MAA AMC 12 Competition.

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

YouTube Click here for video

MATHEMATICAL TOPICS

Geometry and Trigonometry

COMMON CORE STATE STANDARDS

G-SRT.8: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-SRT.11: Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.

MP3 Construct viable arguments and critique the reasoning of others.

MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 1: SUCCESSFUL FLAILING

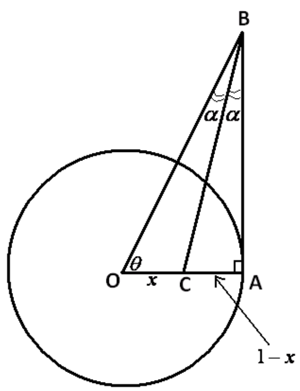
THE PROBLEM-SOLVING PROCESS:

The key first step …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question looks scary. Actually, it is more the list of answers that look scary – the question and the diagram look like a fairly standard textbook type question.

Let me start by marking on the diagram some basic information.



A radius and a tangent always meet at 90°, I've marked that. I've marked and named congruent angles (from bisecting $\angle ABO$), and I called the length we seek x (and consequently $CA = 1 - x$). That feels good. Now what?

I do a faint memory of an "angle bisector theorem" from geometry, but I can't recall what it is right now.

We certainly have two right triangles in this picture: triangles OAB and CAB, which is handy for trigonometry! I am not sure what to do with them though.

My worry is that the length we seek, x, is part of the NON-right triangle ACB. Can we use trigonometry on non-right triangles? Since all the answers presented to use involve $\sin\theta$, $\cos\theta$ and $\tan\theta$, I guess the answer is YES! Hmm.

Alright, what trigonometry applies to non-right triangles? All I can think of is the Law of Sines and the Law of Cosines. Do I want to play with either of those?

Since we have two named angles in triangle OCB, maybe the version of the Law of Sines to use is:

$$\frac{x}{\sin\alpha} = \frac{BC}{\sin\theta}.$$

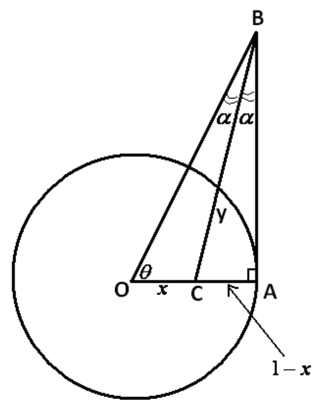
If we use with the Law of Cosines, then we might go with

$$x^2 = OB^2 + BC^2 - 2 \cdot OB \cdot BC \cdot \cos\alpha$$

or with

$$BC^2 = OB^2 + x^2 - 2x \cdot OB \cdot \cos\theta.$$

These seem too complicated! The Law of Sines makes use of only one unknown length. Let's label BC as y and try playing with the Law of Sines.



We have $\frac{x}{\sin\alpha} = \frac{y}{\sin\theta}$ giving $x = \frac{\sin\alpha}{\sin\theta} y$. Maybe this is still horrid: we need to compute y and $\sin\alpha$ … somehow!

Oh .. hang on! There are two α s in the picture. If we look at the right triangle CAB we see $\sin\alpha = \frac{1-x}{y}$. This gives $y\sin\alpha = 1 - x$ which is exactly what we need!

$$x = \frac{y\sin\alpha}{\sin\theta} = \frac{1-x}{\sin\theta}.$$

Consequently $x\sin\theta = 1 - x$ giving $x = \frac{1}{1+\sin\theta}$, option (D). All fell into place!

Extension:
a) Can this problem be solved using the Law of Cosines? (Do you see that $OB = \sec\theta$?)

b) Look up the angle bisector theorem in geometry and use it to solve the problem!

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