Curriculum Burst 35: Filling a Grid

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A checkerboard of 13 rows and 17 columns has a number written in each square, beginning in the upper left corner, so that the first row is numbered $1,2,\ldots,17$, the second row $18,19,\ldots,34$ and so on down the board. If the board is renumbered so that the left column, top to bottom, is $1,2,\ldots,13$, the second column $14,15,\ldots,26$, and so on across the board, some squares have the same numbers in both numbering systems. Find the sum of the numbers in these squares (in either system).

SOURCE: This is question # 16 from the 2000 MAA AMC 12 Competition.

QUICK STATS:

YouTube

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MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Arithmetic Sequences

COMMON CORE STATE STANDARDS

F-BF.2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

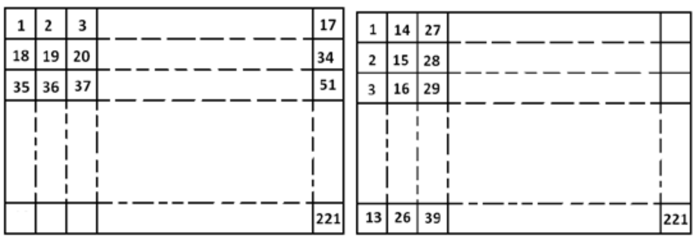
ESSAY 1: **SUCCESSFUL FLAILING**

THE PROBLEM-SOLVING PROCESS:

As always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I think I understand the question. We have two numbering schemes for a 13×17 grid: One filling in the numbers $1,2,3,\dots$ across the rows and one across the columns. (The final number is $13\times17=221$ in each case.)
Here are sketches of the two schemes.

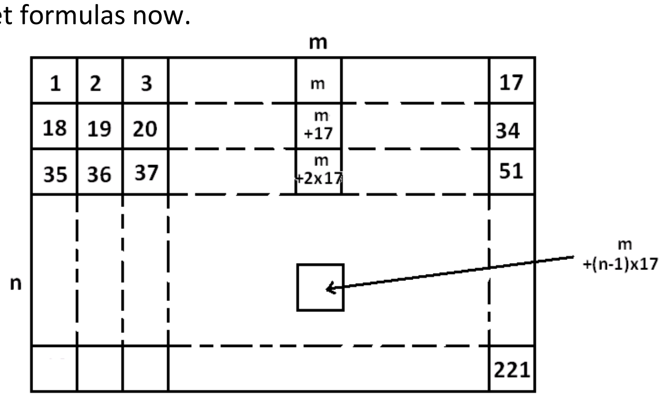


I certainly see the two numbers 1 and 221 in the same position for each scheme. There are probably more. The question wants me to find them all and sum them. Hmm.

I could fill in each table completely and just look for matches. That would be one approach that would work – but it doesn't feel fun.

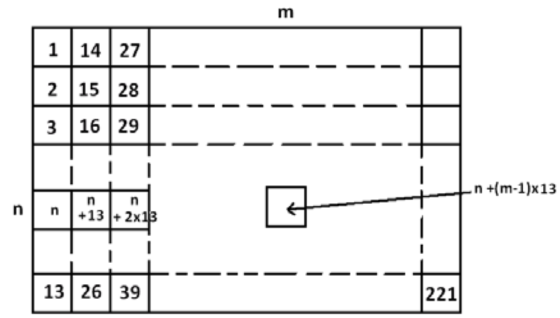
Can I come up with formulas for the numbers in each numbering scheme?

Look at the numbers in the first grid. The numbers in the first row are just $1,2,\dots,13$. The numbers in the second row are just these numbers plus 13. The numbers in the third row are increased by an additional 13. I think I can get formulas now.



The entry in the nth row and mth column for the first numbering scheme is $m+17(n-1)$.

Similar thinking shows that the number in this position is $n+13(m-1)$ for the second scheme.



We want to see when these entries are equal in value.

$$m+17(n-1)=n+13(m-1)$$

$$16n=12m+4$$

$$4n=3m+1$$

This equation wants a multiple of four to be one more than a multiple of three. I don't know what to do but just plug in values for n and see if there is a value of m that works.

$n=1$ gives $4=3m+1$ and $m=1$ works.
$n=2$ doesn't work.
$n=3$ doesn't work.
$n=4$ gives $m=5$.
$n=5$ and 6 don't work.
$n=7$ gives $m=9$
$n=8$ and 9 don't work.
$n=10$ gives $m=13$
$n=11$ and 12 don't work.
$n=13$ gives $m=17$, which are both as large can be.
We have them all!

But these are only the m,n values of entries. We need the actual values of the entries themselves.

$n=1,m=1$ is the entry: $m+17(n-1)=1$
$n=4,m=5$ is the entry: $m+17(n-1)=56$
$n=7,m=9$ is the entry: $m+17(n-1)=111$
$n=10,m=13$ is the entry: $m+17(n-1)=166$
$n=13,m=17$ is the entry: $m+17(n-1)=221$

The sum of these entries is
$1+56+111+166+221=555$. Phew!

Extension: Dare I ask about numbering schemes in $11\times13\times17$ rectangular prisms?

Curriculum Inspirations is brought to you by the Mathematical Association of America and MAA American Mathematics Competitions.