Curriculum Burst 37: Counting Evens

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> How many even integers are there between 200 and 700 whose digits are all different and come from the set $\{1,2,5,7,8,9\}$?

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 10th grade level.

MATHEMATICAL TOPICS

Combinatorics

COMMON CORE STATE STANDARDS

S-CP.B9: (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 7: **PERSEVERANCE IS KEY**

SOURCE: This is question #13 from the 2011 MAA AMC 10A Competition.

THE PROBLEM-SOLVING PROCESS:

Of course, as usual, we start with …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question doesn't feel too scary. After all, the concepts in it are familiar:

> An even integer must end with 0, 2, 4, 6 or 8.

In this question, of these, only 2 and 8 are available to us.

> For an integer to lie between 200 and 700 it must be a three-digit number whose first digit is 2, 3, 4, 5, 6 or 7.

(Does the word "between" include the ends: 200 and 700? If so, 7 could be a first digit, I suppose.)

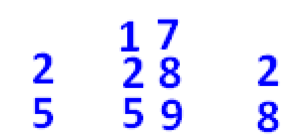
Of these only 2, 5 and 7 are available.

So our job is to count the number of three-digit numbers with last slot 2 or 8, and first slot 2, 5 or maybe 7. There are no restrictions on the value of the middle slot. Here's a picture of the three slots.

So what about a first digit of 7?

Actually, as I think about this, I see that this ambiguity is moot: Since 0 is omitted as a digit for use, 200 and 700 are not in our considerations. There cannot be a number with first digit 7 no matter how we interpret the word "between." (The author of the question was either clever, or just lucky that this ambiguity turns out to be a non-issue.)

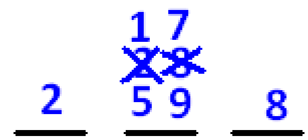
Okay … we have to create a three-digit number with last slot 2 or 8, and first slot 2 or 5, and middle slot 1, 2, 5, 7, 8 or 9.



It is tempting to say that since we have 2 options, 6 options and 2 options, there are $2\times6\times2=24$ such numbers. But this is not right because the question says we cannot repeat digits (I just reread the question!). Integers such as 228 and 858 are not permitted.

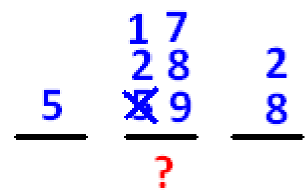
Alright … I guess we have to split our counting task into cases:

<u>The first digit is 2</u>: This means the last slot must be 8 and the middle digit 1, 5, 7 or 9.



There are **four** integers of this type.

<u>The first digit is 5</u>: The last digit is 2 or 8, two options.



Whichever digit is chosen for the last position, we see that there will be only four options remaining for the choice of middle digit. Thus the total count of integers of this type is $1\times4\times2$, that is, there are **eight** permissible integers beginning with a five.

That's it. But chugging along with care we've got the problem licked! There are **TWELVE** even integers between 200 and 700 of the type required by the question.

Extension:

a) How many of the twelve integers are multiples of three? (And how many are multiples of six?)
b) How many of the twelve integers are multiples of four?
c) How many are simultaneously multiples of three and of four?

Further Extension: Of <u>all</u> the three-digit numbers possessing no repeat digits, how many are multiples of three? How many are multiples of four? How many are multiples of twelve?

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