Curriculum Burst 40: Two Counterfeit Coins

By James Tanton, PhD, Mathematics, Princeton 1994; MAA Mathematician in Residence

Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins.

A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the remaining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine?

THE QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 10^{th} grade level.

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MATHEMATICAL TOPIC

Combinatorics

COMMON CORE STATE STANDARDS

S-CP.B9: (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 4: **DRAW A PICTURE**

SOURCE

This is question # 21 from the 2011 MAA AMC 10A Competition.

THE PROBLEM-SOLVING PROCESS:

As always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

Probability questions always make me nervous. Phrases like "selected at random without replacement" sound formal and a bit scary.

To get over my nerves let me ask some basic questions about the problem and answer them in my own informal way.

What is the basic set-up of this problem?

We have ten coins, two of which are fake. We make a pile of two and another pile of two and they happen to weigh the same amount.

How is that possible?

Well … It could be that each pile consists of two genuine coins OR if each pile contains one fake and one real coin OR … . Oh. There is no next option: these are the only two ways the two pairs can weigh the same.

What is the question asking?

We are in the situation of the two piles weighing the same. Given that, what are the chances we have the "one-fake/one-real" case happening for each pile?

So what do we need to do?

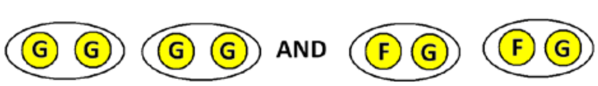
I guess we need to count the total number of ways we could make equal-weighing piles and count the number of those that are of the "one-fake/one-real" type.

I am a very visual person. Let me draw a picture of the coins. (I find it easier to think that way.)

Of course, unlike what I have drawn here, I don't actually see labels on the coins (I can imagine they are on their backs or something). Also, the coins are all jumbled up.

Okay, I have to count all the ways to make two piles of two coins that are of equal weight.

But now I am nervous. If I look at my picture it seems that there is really only one way to make a two "genuine/genuine" pairs and one way to form two "one-real/one-fake" pairs:



Since one case out of two has nothing but genuine coins does this mean that the answer is $1/2$? Umm. That doesn't feel right. Surely with more genuine than fake coins around the chances should sway towards having nothing but genuine coins? (Probability does this sort of thing to me each and every time!)

I've learnt now in probability questions to try to avoid identical looking things. Here we have two "F" coins that look the same and eight "G" coins that are identical. Any of these eight could be used in the pairs shown above, so the counting must indeed be more complicated.

So I am going to ask: Would the problem "care" if the coins happened to have distinguishing labels on them? Say, as:

If I think about it: No it doesn't! We are also making two piles of two. The question has already referred to them as a "first pair" and a "second pair," so those are already distinguished. Okay … everything in this problem – coins, piles - can be considered as labeled. Now let's count.

How can I make two pairs of genuine coins? There are 8 of these coins, two to be identified as "for the first pair", two as "for the second pair" and four as "not used." There are thus $\frac{8!}{2!2!4!} = 420$ all genuine pairs. (See www.gdaymath.com : *Permutations and Combinations* for my non-school approach to counting.)

How can I make two "fake/genuine" pairs?

Of the fake coins, one will be labeled "for the first pair" and one "for the second pair." There are thus $\frac{2!}{1!1!} = 2$ ways to handle the fake coins. Similarly there are $\frac{8!}{1!1!6!} = 56$ ways to handle the genuine coins. This gives a total of $2 \times 56 = 112$ fake/genuine pairs. Thus there are a total of $420 + 112 = 532$ pairs of coins equal in weight, 420 of which are all genuine. The probability we seek is: $\frac{420}{532} = \frac{15}{19}$. That feels better!

Extension: Repeat this question for 5 fake coins and 12 genuine coins, selecting THREE pairs of coins that happen to all have the same weight.

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