Curriculum Burst 54: What's the Domain?

By Dr. James Tanton, MAA Mathematician in Residence

The function f has the property that for each real number x in its domain, $\frac{1}{x}$ is also in its domain and

$$f(x)+f\left(\frac{1}{x}\right)=x.$$

What is the largest set of real numbers that can be in the domain of f ?

QUICK STATS:

Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Functions: Notation, Domain and Range.

COMMON CORE STATE STANDARDS

F-IF-A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then f(x) denotes the output of f corresponding to the input x. The graph of f is the graph of the equation y=f(x).

F-IF-A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 8: SECOND GUESS THE AUTHOR

SOURCE: This is question # 18 from the 2006 MAA AMC 12A Competition.

THE PROBLEM-SOLVING PROCESS:

As always, the first step is …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I feel "thrown" by this question. It looks like a strange, nothing like any other question I have seen before.

What can I possibly do with $f(x)+f\left(\frac{1}{x}\right)=x$? How can I tell what the function is from this?

My first instinct is to skip this question and try a different one!

Deep breath.

The fact that $f(x)+f\left(\frac{1}{x}\right)=x$ is coming out of the blue and looks so strange makes me think that it is NOT coming out of the blue, that it represents something clever the author of this question constructed. Can I use the strangeness of this formula to my advantage?

We need to split "x" into a sum of two related things: $f(x)+f\left(\frac{1}{x}\right)$. Hmm.

Just a random thought: Would $f(x)=\frac{x}{2}$ and $f\left(\frac{1}{x}\right)=\frac{x}{2}$ be meaningful in some way? The second equation has an input of $\frac{1}{x}$, which is hard to think about. But I can be sneaky and think of the input $\frac{1}{1/x}$. We get: $f\left(\frac{1}{1/x}\right)=\frac{1/x}{2}$. That is $f(x)=\frac{1}{2x}$. But $f(x)=\frac{x}{2}$ by the first equation so we get $\frac{x}{2}=\frac{1}{2x}$, giving $x=1$ or $x=-1$. So $f(x)=\frac{x}{2}$ and $f\left(\frac{1}{x}\right)=\frac{x}{2}$ forces $x=\pm 1$.

Well, I don't really know what that previous paragraph means or does for me, but I like what I did in the middle of it – using $1/x$ as an input.

Let's go back to the general situation. We have some function that satisfies the equation:

$$f(x)+f\left(\frac{1}{x}\right)=x$$

for all inputs x in its domain. So let's put in the input $1/x$ as well. Then the equation reads:

$$f(1/x)+f\left(\frac{1}{1/x}\right)=1/x$$

That is:

$$f\left(\frac{1}{x}\right)+f(x)=\frac{1}{x}.$$

Ahh! So for any x in this function's domain, x and $\frac{1}{x}$ both equal $f(x)+f(1/x)$. So we have $x=\frac{1}{x}$ giving $x=\pm 1$. There can be only two possible x-values in the domain of this function. Moreover, our "random thought" in the previous column shows that $f(x)=\frac{x}{2}$, defined for $x=\pm 1$, is an example of a function satisfying the given equation on our domain of $\{1,-1\}$, so everything we've been talking about really can exist.

The answer to this question is:

Largest possible domain $=\{1,-1\}$.

Extension 1: Is $f(1)=\frac{1}{2}$, $f(-1)=-\frac{1}{2}$ the <u>only</u> example of a function defined on $\{1,-1\}$ satisfying $f(x)+f\left(\frac{1}{x}\right)=x$?

Extension 2: Find an example of a function that satisfies $f(xy)=x\cdot f(y)+f(x)\cdot y$ for all positive inputs x and y. (Negative inputs too?)

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.

MAA acknowledges with gratitude the generous contributions of
the following donors to the Curriculum Inspirations Project:

The TBL and Akamai Foundations
for providing continuing support

The Mary P. Dolciani Halloran Foundation for providing seed
funding by supporting the Dolciani Visiting
Mathematician Program during fall 2012

MathWorks for its support at the Winner's Circle Level