The sum of the base-10 logarithms of the divisors of $10^n$ is 792. What is $n$?

**QUICK STATS:**

**MAA AMC GRADE LEVEL**
This question is appropriate for the 12th grade level.

**MATHEMATICAL TOPICS**
Logarithms, Summation

**COMMON CORE STATE STANDARDS**

**F-BF.5** Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

**MATHEMATICAL PRACTICE STANDARDS**

**MP1** Make sense of problems and persevere in solving them.
**MP2** Reason abstractly and quantitatively.
**MP3** Construct viable arguments and critique the reasoning of others.
**MP7** Look for and make use of structure.

**PROBLEM SOLVING STRATEGY**

**ESSAY 1:** Engage in successful flailing

**SOURCE:** This is question # 23 from the 2008 MAA AMC 12B Competition.
THE PROBLEM-SOLVING PROCESS:

As always, the best first step is ...

**STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question feels scary! I know the words in the question – logarithm, divisor (that is, factor) – and I know what I need to do – list all the factors of \(10^n\), take their logarithms, add, and set this sum equal to \(792\). But that task seems overwhelming and potentially very ugly. Hmm.

Let’s engage in the art of “successful flailing.” Let’s follow the start we have and just see how it goes. If it is too awkward it might nonetheless shed light on an alternative route to follow.

Okay. We need to find the divisors of \(10^n\) and compute their logarithms. What are the factors of \(10^n\)?

Well \(10^n = 2^n \cdot 5^n\) and any factor must be a product of a power of 2 and a power of 5. The factors are thus the numbers of the form \(2^a \cdot 5^b\) with \(a\) and \(b\) each integers between 0 and \(n\). The list of factors is:

\[1, 2, 5, 2^2 \cdot 5, 2 \cdot 5^2, 2^3 \cdot 5, 2^2 \cdot 5^2, 2 \cdot 5^3, \ldots, 2^n \cdot 5^n.\]

Alright, so let’s now take their logarithms. That’s not too bad: \(\log (2^n \cdot 5^n) = a \log 2 + b \log 5\).

Next is to add all these logarithms – and that’s the part that feels hard. Let’s just start writing it out and see if any structure to the sum becomes clear.

Following the list of factors above we get the sum:

\[
\log (1) + \log (2) + \log (5) + 2 \log (2) + \log (5) + 2 \log (5) + 3 \log (2) + 2 \log (5) + \log (5) + \cdots
\]

This looks like a mess! We need to count how many \(\log (2)\) s are in this sum and how many \(\log (5)\) s. Hmm. That seems hard.

Maybe the problem is how I chose to list my factors. Let’s rewrite the list so that is more focused on the 2s.

We have the factors: \(1, 2, 2^2, 2^3, \ldots, 2^n,\)

and the factors: \(5, 2 \cdot 5, 2^2 \cdot 5, 2^3 \cdot 5, \ldots, 2^n \cdot 5,\)

and the factors: \(5^2, 2 \cdot 5^2, 2^2 \cdot 5^2, 2^3 \cdot 5^2, \ldots, 2^n \cdot 5^2,\)

all the way up to: \(5^n, 2 \cdot 5^n, 2^2 \cdot 5^n, 2^3 \cdot 5^n, \ldots, 2^n \cdot 5^n.\)

Now the sum of logarithms in the first list is:

\[
\log (1) + \log (2) + 2 \log (2) + \cdots + n \log (2) = 0 + \log (2) + \cdots + n \log (2) = \frac{1}{2} n(n+1) \log (2)
\]

(See [http://www.jamestanton.com/?p=1006](http://www.jamestanton.com/?p=1006) to explain the summation used here.) The sum of logarithms for the second list is:

\[
\log (5) + \log (5) + \log (2) + \log (5) + 2 \log (2) + \cdots + \log (5) + n \log (2) = (n+1) \log (5) + \frac{1}{2} n(n+1) \log (2)
\]

And for the third list it is:

\[
(n+1) \log (5) + \frac{1}{2} n(n+1) \log (2).
\]

And so on up to the final list which has logarithm sum:

\[
(n+1) \log (5) + \frac{1}{2} n(n+1) \log (2).
\]

I am not sure where this is going (it still looks messy), but the total sum of logarithms is the sum of all these answers and this is:

\[
(n+1) \log (5) + \frac{1}{2} n(n+1) \log (2)
\]

We can rewrite this as:

\[
(n+1) \log (5) + \frac{1}{2} n(n+1) \log (2)
\]

\[
= \frac{1}{2} n(n+1)^2 \left( \frac{\log (5) + \log (2)}{2} \right)
\]

\[
= \frac{1}{2} n(n+1)^2 \log (10) = \frac{1}{2} n(n+1)^2.
\]

Whoa! All we need do now is set this equal to \(792\) and see what \(n\) is!

\[
n(n+1)^2 = 2 \cdot 792.
\]

Oh heavens! We need to solve a cubic equation? Perhaps there is something special about \(792\)? Since \(7 + 2 = 9\) I see that \(792 = 11 \times 72\), so

\[
2 \cdot 792 = 2 \times 11 \times 72 = 11 \times 144 = 11 \times 12^2.
\]

That’s it. We have \(n = 11\). Crazy!

**Extension:** Let \(d(N)\) be the count of factors of the number \(N\). Prove that the product of the divisors of \(N\) equals \(N^{d(N)/2}\).

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