Curriculum Burst 59: A Complex Minimum

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A function f is defined by $f(z)=(4+i)z^2+\alpha z+\gamma$ for all complex numbers z, where α and γ are complex numbers and $i^2=-1$. Suppose that $f(1)$ and $f(i)$ are both real. What is the smallest possible value of $|\alpha|+|\gamma|$?

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QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 12^{th} grade level.

MATHEMATICAL TOPICS

Function Notation and Functions, Complex Numbers

COMMON CORE STATE STANDARDS

N-CN.1 Know there is a complex number i such that i2=−1, and every complex number has the form a+bi with a and b real.

N-CN.2 Use the relation *i*^2=−1 and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

N-CN.3 Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

N-CN.6 Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 2: **DO SOMETHING**

SOURCE: This is question # 19 from the 2008 MAA AMC 12B Competition.

THE PROBLEM-SOLVING PROCESS:

As always, start with …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

Whoa! That's my emotional reaction. This question looks very scary and strange and my first impulse is to skip it and to try another question.

Deep breath!

Let's read the question again slowly.

We have a "complex function" f with $f(z)=(4+i)z^2+\alpha z+\gamma$ for each complex number z. Here α and γ are themselves complex numbers.

What does that mean?

Well, α and γ complex numbers means $\alpha=a+ib$ and $\gamma=c+id$ for some real numbers a,b,c,d.

That f is a "complex function" means that f takes a complex number z as an input and gives a complex number as an output. (In fact, the output associated to z is $(4+i)z^2+\alpha z+\gamma$. This looks scary.)

We are told that $f(1)$ and $f(i)$ are both real. Well, I can at least work those out and do something!

$$\begin{aligned} f(1) &= (4+i)\cdot 1^2+\alpha\cdot 1+\gamma \\ &= 4+i+a+ib+c+id \\ &= 4+a+c+i(1+b+d) \end{aligned}$$

$$\begin{aligned} f(i) &= (4+i)\cdot i^2+\alpha\cdot i+\gamma \\ &= -4-i+ia-b+c+id \\ &= -4-b+c+i(-1+a+d) \end{aligned}$$

Hmm. These are both real, which mean their imaginary parts are zero.

$$1+b+d=0$$
$$-1+a+d=0$$

Now I've lost track of the question! What was the question?

What is the smallest possible value of $|\alpha|+|\gamma|$?

I have to remember now that the modulus of a complex number is just its distance from the origin:

$$|\alpha|=|a+ib|=\sqrt{a^2+b^2}$$
$$|\gamma|=|c+id|=\sqrt{c^2+d^2}.$$

So we want the smallest possible value of their sum:

$$\sqrt{a^2+b^2}+\sqrt{c^2+d^2}.$$

Heavens!

Well, I don't know what to do other than to work with the two equations we have.

From:

$$a=1-d \text{ and } b=-1-d$$

we get

$$\begin{aligned} &\sqrt{a^2+b^2}+\sqrt{c^2+d^2} \\ &\quad=\sqrt{(1-d)^2+(-1-d)^2}+\sqrt{c^2+d^2} \\ &\quad=\sqrt{2+2d^2}+\sqrt{c^2+d^2}. \end{aligned}$$

Well, this will be as small as it can be if $c=0$ and $d=0$ (so that $\alpha=a+ib=(1-0)+i(-1-0)=1-i$ and $\gamma=c+id=0$). Then we get the value $\sqrt{2+0}+\sqrt{0+0}=\sqrt{2}$. This must be the answer.

For some reason this doesn't feel satisfactory. Do we get to chose the values of a,b,c and d? That is, do we get to chose the values of α and γ? I suppose we do, but I am not clear if that is clear in the question. Hmmm.

Extension: Does the quadratic formula still work for complex quadratic equations?

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