Curriculum Burst 60: A Lopsided Pyramid

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> A pyramid has a square base $ABCD$ and vertex E. The area of square $ABCD$ is 196, and the areas of ΔABE and ΔCDE are 105 and 91, respectively. What is the volume of the pyramid?

QUICK STATS:

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MAA AMC GRADE LEVEL

This question is appropriate for the 12^{th} grade level.

MATHEMATICAL TOPICS

Geometry of Solids

COMMON CORE STATE STANDARDS

G-GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.

PROBLEM SOLVING STRATEGY

ESSAY 1: ENGAGE IN SUCCESSFUL FLAILING

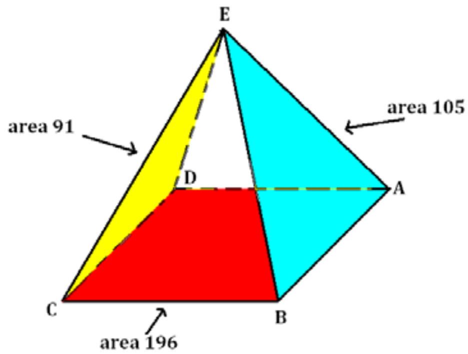
SOURCE: This is question # 18 from the 2008 MAA AMC 12B Competition.

THE PROBLEM-SOLVING PROCESS:

As always, start with …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

I feel I understand the question. We have a square pyramid with two opposite faces whose areas we know. (We also know the area of the square base.)



Our job is to find the volume of the pyramid, which is given by $V = (1/3) \times 196 \times H$, where H is the height of the pyramid. The challenge is to find H.

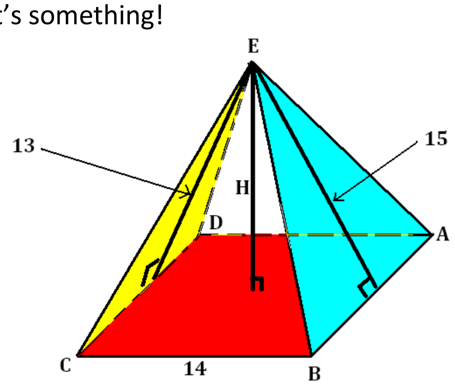
I am not sure how to do this, but I do have pieces of information I can play with to at least get started and so do something.

The area of the base is 196*:* This means the side-length of the square is 14 units.

The area of ΔABE *is* 105*:* If h is the height of this triangle, then we have $\frac{1}{2} \cdot 14 \cdot h = 105$. This gives $h = 15$.

The area of ΔCDE *is* 91*:* If g is the height of this triangle, then $\frac{1}{2} \cdot 14 \cdot g = 91$ gives $g = 13$.

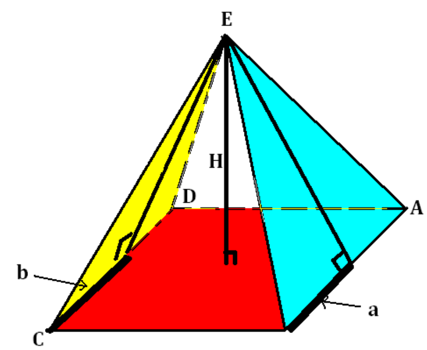
Okay. That's something!



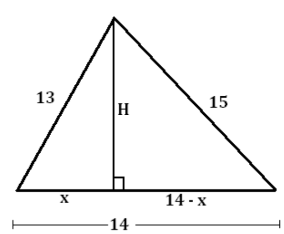
Can we work out height H from this?

We have three feet of three perpendicular lines in this picture. Are they collinear? Can I connect them and make a triangle with E at the top, the lengths 13 and 15 as two side lengths, and the line H as its altitude? That feels as though it would be very helpful if it is true.

Image the two colored faces and the base are hinged pieces of cardboard, which are folding up to make the pyramid. I can see in my mind's eye that the only way for the two apexes of the two triangles to meet at a common point E is for lengths a and b shown to be the same.



And then I can see that the feet of the three altitudes would have to align AND the distance between the feet of the triangular faces is the side-length of the square, 14.



Okay. We have a 13-14-15 triangle and H is the altitude of that triangle. Working out H is now an exercise from geometry class. With my labeling shown we have $x^2 + H^2 = 169$ and:

$$(14 - x)^2 + H^2 = 225$$

$$196 - 28x + 169 = 225$$

$$x = 5$$

and so $5^2 + H^2 = 13^2$ showing $H = 12$. The volume is $V = (1/3) \cdot 196 \cdot 12 = 784$. Phew!

Extension: Find a general formula for the volume of a square pyramid in terms of the area of its base and the areas of two opposite faces.

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