A cube with side length 1 is sliced by a plane that passes through two diagonally opposite vertices $A$ and $C$ and the midpoints $B$ and $D$ of two opposite edges not containing $A$ or $C$, as shown. What is the area of quadrilateral $ABCD$?

**QUICK STATS:**

**MAA AMC GRADE LEVEL**
This question is appropriate for the junior high-school grade levels.

**MATHEMATICAL TOPICS**
Geometry: Solids, Pythagorean Theorem

**COMMON CORE STATE STANDARDS**

**G.GMD.4** Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

**MATHEMATICAL PRACTICE STANDARDS**

**MP1** Make sense of problems and persevere in solving them.

**MP2** Reason abstractly and quantitatively.

**MP3** Construct viable arguments and critique the reasoning of others.

**PROBLEM SOLVING STRATEGY**

**ESSAY 5:** SOLVE A SMALLER VERSION OF THE SAME PROBLEM

**SOURCE:** This is question #21 from the 2008 MAA AMC 10A Competition.
THE PROBLEM-SOLVING PROCESS:

As always, one starts the problem-solving process with …

**STEP 1:** Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This question doesn’t feel too scary to me. I think I can see that there is a plane that slices the cube through all four of the points as described in the question:

(Hang on! I know through any three points there is a plane. So there is a plane that goes through the points $A$, $B$, and $D$, for sure, and there is a plane that goes through the points $B$, $D$, and $C$. I guess the question says that these two planes are the same. I think I can believe that.)

So I need to figure out the area of the cross-section I shaded. It looks like all four sides of that cross-section quadrilateral are the same length (“one side-length over and half a side-length up”) so that shape is a rhombus.

I do know there is some formula for the area of a rhombus that I learned about in geometry class. I can’t remember it now. Hmm.

Well, I can break the quadrilateral into the two triangles I first thought of, $\triangle ABD$ and $\triangle CBD$, and work out those areas instead. And by symmetry it looks like these are identical triangles. Let’s find the area of just one of them. (That’s a smaller and easier version of the problem, and it avoids knowing a special formula for the area of a rhombus!)

I know the side-lengths of this triangle.

\[
BD = \text{the diagonal of the square} = \sqrt{1^2 + 1^2} = \sqrt{2}.
\]

\[
AD = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}
\]

and $BD = AD$. This triangle has base $\sqrt{2}$ and height

\[
h = \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{3}}{2},
\]

and so its area is:

\[
\frac{1}{2} \cdot \sqrt{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{4}.
\]

The area of the rhombus is double this, namely $\frac{\sqrt{6}}{2}$.

**Aside:** Prove that the area of a rhombus is half the product of the lengths of its two diagonals.

**Extension:** Find a plane through the center of the cube that yields a cross-section area larger than the one described in this question. Which plane through the center of the cube yields the cross-section of largest area of all? (And what is that largest area?) Which plane though the center yields a cross-section of smallest area?

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