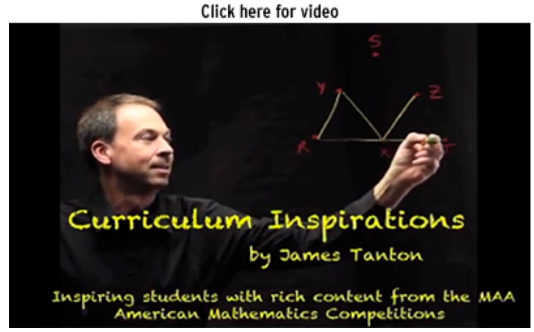
Curriculum Burst 89: Iterated Function Domain

By Dr. James Tanton, MAA Mathematician in Residence

> Let $f_1(x)=\sqrt{1-x}$, and for integers $n\geq 2$, let $f_n(x)=f_{n-1}\left(\sqrt{n^2-x}\right)$.
> If N is the largest value of n for which the domain of f_n is nonempty, the domain of f_N is $\{c\}$. What is $N+c$?

Click here for video



QUICK STATS:

MAA AMC GRADE LEVEL
This question is appropriate for the upper high-school grade levels.

MATHEMATICAL TOPICS
Functions: domain, compound functions. Iteration.

COMMON CORE STATE STANDARDS
F-IF.A2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F-IF.C8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

MATHEMATICAL PRACTICE STANDARDS
MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 7: **PERSEVERANCE IS KEY**

SOURCE: This is question # 21 from the 2011 MAA AMC 12A Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

Even though I don't like the look of this question, it does seem clear how I need to start. We're looking at the domains of functions, defined in some weird recursive way. Let's just plow our way through the details.

$f_1(x)=\sqrt{1-x}$ has domain all values of x with $x \le 1$.

$f_2(x)=f_1\left(\sqrt{4-x}\right)$. This has two aspects to consider when looking for allowable inputs. We need $x \le 4$ and we need $\sqrt{4-x} \le 1$. This second demand gives $x \ge 3$. The domain of f_2 is all values x with $3 \le x \le 4$.

$f_3(x)=f_2\left(\sqrt{9-x}\right)$. An allowable input must satisfy $x \le 9$ and $3 \le \sqrt{9-x} \le 4$. Squaring gives $9 \le 9-x \le 16$, and so $0 \le -x \le 7$, that is, $-7 \le x \le 0$ (which is consistent with the requirement that $x \le 9$ too). We have that the domain of f_3 is all values in the interval $[-7,0]$.

Is this going anywhere?

$f_4(x)=f_3\left(\sqrt{16-x}\right)$. An allowable input satisfies $x \le 16$ and $-7 \le \sqrt{16-x} \le 0$. Ooh! The radix symbol, $\sqrt{\ }$, by definition means the <u>non-negative</u> root. So $\sqrt{16-x}$ is not negative, but it can be zero. There is only one allowable input for the function f_4, namely, $x=16$, from setting $\sqrt{16-x}=0$.

Umm. What was the question?

> *If N is the largest value of n for which the domain of f_n is nonempty, the domain of f_N is $\{c\}$. What is $N+c$?*

So $N=4$ and $c=16$?

I feel like I should check the domain of f_5, just to make sure it is empty.

$f_5(x)=f_4\left(\sqrt{25-x}\right)$. An allowable input must satisfy $x \le 25$ and $\sqrt{25-x}=16$. This gives $25-x=256$, that is, $x=-231$, which works! The domain of f_5 is not empty. It is the set $\{-231\}$!

Okay, what about f_6? We have $f_6(x)=f_5\left(\sqrt{36-x}\right)$. For an allowable input we need $x \le 36$ and $\sqrt{36-x}=-231$. This is not going happen. The domain of f_6 is actually empty.

So we have $N=5$ and $c=-231$, giving $N+c=-226$.

Extension: This question uses the square numbers to create a sequence of functions as follows:

$$f_1(x)=\sqrt{1-x},$$
$$f_2=\sqrt{4-\sqrt{1-x}},$$
$$f_3(x)=\sqrt{9-\sqrt{4-\sqrt{1-x}}},$$

and so on.

Eventually these functions have empty domains.

Is there a sequence of numbers $a_1, a_2, a_{3,\dots}$ one can use instead of the square numbers so that each of the functions:

$$f_1(x)=\sqrt{a_1-x},$$
$$f_2(x)=\sqrt{a_2-\sqrt{a_1-x}},$$
$$f_3(x)=\sqrt{a_3-\sqrt{a_2-\sqrt{a_1-x}}},$$
$$\dots$$

has an allowed input?

Curriculum Inspirations is brought to you by the Mathematical Association of America and the MAA American Mathematics Competitions.

MAA acknowledges with gratitude the generous contributions of
the following donors to the Curriculum Inspirations Project:

The TBL and Akamai Foundations
for providing continuing support

The Mary P. Dolciani Halloran Foundation for providing seed
funding by supporting the Dolciani Visiting
Mathematician Program during fall 2012

MathWorks for its support at the Winner's Circle Level