Curriculum Burst 90: A Complex Compound Function

By Dr. James Tanton, MAA Mathematician in Residence

> Let $f(z)=\dfrac{z+a}{z+b}$ and $g(z)=f\left(f(z)\right)$, where a and b are complex numbers.
> Suppose that $|a|=1$ and $g\left(g(z)\right)=z$ for all z for which $g\left(g(z)\right)$ is defined.
> What is the difference between the largest and smallest possible values of $|b|$?

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QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the upper high-school grade levels.

MATHEMATICAL TOPICS

Functions: compound functions; domain. Complex Numbers.

COMMON CORE STATE STANDARDS

F-IF.A2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F-IF.C8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

N-CN.A Perform arithmetic operations with complex numbers.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 9: AVOID HARD WORK

SOURCE: This is question # 23 from the 2011 MAA AMC 12A Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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Oh heavens. This is scary! There are details about complex numbers a and b I am going to ignore for the moment because the thrust of the question seems to be about composition of compositions of functions. We have:

$$f(z)=\frac{z+a}{z+b}$$

$$g(z)=f(f(z))=\frac{\frac{z+a}{z+b}+a}{\frac{z+a}{z+b}+b}$$

and

$$z=g(g(z))=f(f(f(f(z)))).$$

There is no way I am going to work out the four-fold composition and set it equal to z!

Hmmm. What happens if I choose an easy value for z, say $z=0$? The last line gives $0=f(f(f(f(0))))$.

Now $f(0)=\frac{a}{b}$.

And $f(f(0))=f\left(\frac{a}{b}\right)=\frac{\frac{a}{b}+a}{\frac{a}{b}+b}=\frac{a+ab}{a+b^2}$.

And $f(f(f(0)))=f\left(\frac{a+ab}{a+b^2}\right)=\frac{\frac{a+ab}{a+b^2}+a}{\frac{a+ab}{a+b^2}+b}$,

which is starting to look dreadful!

Can I avoid some work? Can I avoid doing a four-fold composition?

What if I chose a value of z so that $f(z)=0$? I can see that $z=-a$ does that. Then we have:

$$-a=f(f(f(f(-a))))=f(f(f(0)))$$

This reads:

$$-a=\frac{\frac{a+ab}{a+b^2}+a}{\frac{a+ab}{a+b^2}+b}.$$

That is:

$$-a=\frac{a+ab+a(a+b^2)}{a+ab+b(a+b^2)}.$$

This gives:

$$b^3+b^2+(2a+1)b+(2a+1)=0$$

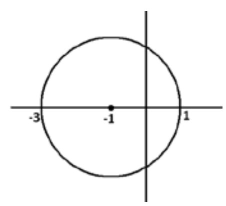
$$(b+1)(b^2+2a+1)=0$$

So either $b=-1$ or $b^2=-2a-1$. Okay. But I don't know what that means.

What's the question? We have $|a|=1$ and we are looking for the largest and smallest values of $|b|$.

Okay, $b=-1$ or $b^2=-2a-1$ where a can be any complex number on the unit circle centered about the origin.

Then "$-2a$" represents complex numbers on a circle of radius 2 centered about the origin, and "$-2a-1$" complex numbers on a circle of radius 2 centered about $(-1,0)$.



So either $b=-1$ and has $|b|=1$, or b^2 is a complex number on this circle. The furthest b^2 can be from the origin is 3 units, so the maximum possible value of $|b|$ is $\sqrt{3}$. The closest it can be again has $|b|=1$. The difference of these two values is $\sqrt{3}-1$. That's the answer!

> **Extension:** Does the function $f(z)=\frac{z+a}{z-1}$ really have four-fold composition equal to the identity function? Does the value of a matter?

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