Curriculum Burst 91: The Biggest Circle

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> Consider all quadrilaterals $ABCD$ such that $AB = 14$, $BC = 9$, $CD = 7$, and $DA = 12$. What is the radius of the largest possible circle that fits inside or on the boundary of such a quadrilateral?

QUICK STATS:

Click here for video

MAA AMC GRADE LEVEL

This question is appropriate for the upper high-school grade levels.

MATHEMATICAL TOPICS

Geometry: Cyclic Quadrilaterals, Area formulas, Law of Cosines

COMMON CORE STATE STANDARDS

G-SRT.D Apply trigonometry to general triangles

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

PROBLEM SOLVING STRATEGY

ESSAY 3: **ENGAGE IN WISHFUL THINKING**

SOURCE: This is question # 24 from the 2011 MAA AMC 12A Competition.

THE PROBLEM-SOLVING PROCESS:

The best, and most appropriate, first step is always …

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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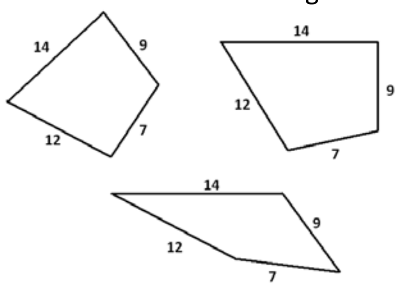
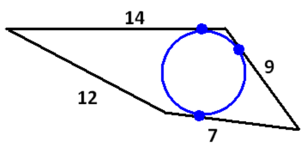
This question, at least, feels easy to understand. We're being asked to consider all possible quadrilaterals with given side lengths, which I imagine as four sticks of given lengths taped together in a ring. This ring of sticks won't be sturdy and will flop about to give different quadrilaterals with different interior angles.

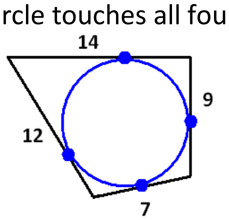


The biggest circle in a quadrilateral will touch at least three sides of the quadrilateral.

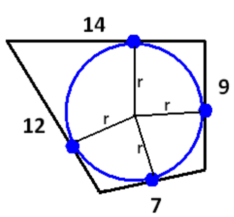


WISHFUL THINKING: Optimal solutions tend to be symmetrical. The largest inscribed circle probably touches all four sides of the quadrilateral, not just three.

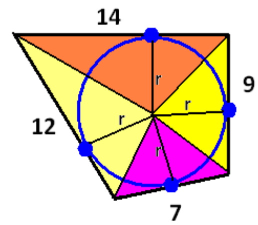
I don't know is this true, but let's go with it for now! Let's assume the biggest circle touches all four sides.



What is the radius of this circle? It seems appropriate to draw in some radii.



Is there any way to establish some connection between the side-lengths and the radius? I do recall using radii to compute the areas of polygons: each radius drawn is the height of a triangle with a side-length as base.

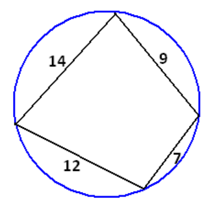


The area A of the quadrilateral is:

$$A = \frac{1}{2}\cdot 14\cdot r + \frac{1}{2}\cdot 9\cdot r + \frac{1}{2}\cdot 7\cdot r + \frac{1}{2}\cdot 12\cdot r = 21r\,.$$

So $r = \frac{1}{21}A$. We see that the largest radius sits with a quadrilateral of the largest area.

WISHFUL THINKING: Of all shapes of fixed perimeter the circle encloses largest area. So … the quadrilateral of largest area is probably the most circle-like one! And what is the most circle-like quadrilateral we can make with sides of lengths 14, 9, 7, and 12? <u>Answer</u>: A cyclic one! Let's assume we have that.



It's not part of the standard curriculum, but I happen to know a formula for the area of a cyclic quadrilateral: Brahmagupta's formula. We have:

$$A = \sqrt{(21-14)(21-9)(21-7)(21-12)}$$
$$= 42\sqrt{6}$$

So $r = 42\sqrt{6}\,/\,21 = 2\sqrt{6}$ is my wishful guess for the largest possible radius of an inscribed circle.

Now … Is any of this wishful thinking true?

Extension: Is it?

For a discussion of all the mathematics – with complete proofs – needed to fully justify the wishful thinking here (including a derivation of Brahmagupta's formula) see the COOL MATH ESSAY of April 2014 at www.jamestanton.com/?p=1072.

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